Problem set 5 solutions [40]

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Standard disclaimer: while every effort is made to ensure the correctness of these solutions, if you find anything that you suspect is an error, please email me at mfedderke@uchicago.edu.

All problem references to Arfken and Weber, 7th edition.

I. 11.8.11 [10]

Note: I was intentionally pedantic in grading this question. There are a number of potential pitfalls here and you need to be aware of them. Please read this solution carefully.

Note that the contour integral I am going to end up doing here is not the only way to solve the problem; you could also start by doing integration by parts, or something else correct, and might thus have different starting point for the application of the residue theorem to me here, but you still need to pay attention to all the same potential pitfalls in this question. Note that this is a feature of using complex analysis to evaluate real integrals: there is usually more than one valid way to do it.

Consider

\[ I \equiv \int_{-\infty}^{\infty} \frac{2}{\omega^2} (1 - \cos \omega t) \, d\omega \quad (1) \]

As it stands, the integral is not in a form appropriate for the application of the residue theorem, because it has no poles: \( \frac{2}{\omega^2} (1 - \cos \omega t) \sim \omega^0 \) around \( \omega = 0 \). We can however easily manipulate the integral to an equivalent one which does have the appropriate form:

\[ I = \int_{-\infty}^{\infty} \frac{2}{\omega^2} (1 - \cos \omega t) \, d\omega \quad (2) \]

\[ = \int_{-\infty}^{\infty} \frac{1}{\omega^2} (2 - e^{i\omega t} - e^{-i\omega t}) \, d\omega \quad (3) \]

\[ = \int_{-\infty}^{\infty} \frac{1}{\omega^2} (2 - e^{i\omega t} - e^{-i\omega t}) \, d\omega; \quad (4) \]

We would like to make the change of variable \( \omega \to -\omega \) in the integral over the last term, but this is not technically an allowed manipulation because to do so we would first have to split up the integrand per

\[ I = \int_{-\infty}^{\infty} \frac{1}{\omega^2} (2 - e^{i\omega t}) \, d\omega + \int_{-\infty}^{\infty} \frac{1}{\omega^2} (-e^{-i\omega t}) \, d\omega, \quad (5) \]

which is not allowed since neither integral is by itself well-defined. Let’s ignore this subtlety for the moment and press on to see why we would actually want to do this:

\[ I = \int_{-\infty}^{\infty} \frac{1}{\omega^2} (2 - e^{i\omega t}) \, d\omega + \int_{-\infty}^{\infty} \frac{1}{(\omega)^2} \left( -e^{-i(\omega)t} \right) d(-\omega) \quad (6) \]

\[ = \int_{-\infty}^{\infty} \frac{2}{\omega^2} (1 - e^{i\omega t}) \, d\omega, \quad (7) \]

where we have again ignored the problems of combining ill-defined integrals. If we now analytically continue the integrand, it is in a form appropriate to apply the residue theorem, because there is now a (simple!) pole at \( \omega = 0 \),
whose residue we can compute; we can then get the integral we want by considering an appropriate contour. But before we do that, let’s figure out exactly what we have to do to make this series of manipulations well-defined.

It is clear that the integral $I$ does exist, since the integrand is actually regular at $\omega = 0$ (if the removable discontinuity is appropriately removed) and the integrand is positive semi-definite and falls off faster than $1/|\omega|$ at large $|\omega|$. This means that the following limit exists:

$$I = \lim_{\epsilon \to 0} \frac{2}{\omega^2} \left( 1 - e^{i\omega t} - e^{-i\omega t} \right) \omega^2,$$

where we have explicitly excluded what is actually a removable discontinuity in the current function definition at $\omega = 0$. Then it is certainly true that the following limit also exists, and has the same value:

$$I = \lim_{\epsilon \to 0} \frac{2}{\omega^2} \left( 1 - e^{i\omega t} - e^{-i\omega t} \right) \omega^2,$$

which is just the statement that if an integral exists, it is equal to its principal value. But now we can perform our manipulations on the integral legally, since all the integrals of interest will be well-defined:

$$I = \lim_{\epsilon \to 0} \left[ \left( \int_{z = 0}^{\epsilon} + \int_{\epsilon}^{R} \right) \frac{1}{\omega^2} (2 - e^{i\omega t} - e^{-i\omega t}) d\omega \right]$$

$$= \lim_{\epsilon \to 0} \left[ \left( \int_{z = 0}^{\epsilon} + \int_{-R}^{\epsilon} \right) \frac{1}{\omega^2} (2 - e^{i\omega t}) d\omega + \left( \int_{-\epsilon}^{-R} + \int_{-\epsilon}^{R} \right) \frac{1}{\omega^2} (e^{-i\omega t}) d\omega \right]$$

$$= \lim_{\epsilon \to 0} \left[ \left( \int_{z = 0}^{\epsilon} + \int_{-R}^{\epsilon} \right) \frac{1}{\omega^2} (2 - e^{i\omega t}) d\omega + \left( \int_{-\epsilon}^{-R} + \int_{-\epsilon}^{-R} \right) \frac{1}{\omega^2} (e^{i\omega t}) d\omega \right]$$

$$= \lim_{\epsilon \to 0} \left[ \left( \int_{z = 0}^{\epsilon} + \int_{-R}^{\epsilon} \right) \frac{2}{\omega^2} (1 - e^{i\omega t}) d\omega \right]$$

$$= \text{PV} \int_{-\infty}^{\infty} \frac{2}{\omega^2} (1 - e^{i\omega t}) d\omega.$$

So the missing ingredient was a ‘PV’ prescription. (Note that symmetrising the limits before making the change of variables was crucial here, otherwise the limits on the original integral and those on the flipped-sign integral would not agree (e.g., the highest upper limits would be $R_2$ and $R_1$, respectively) and we could not recombine these integrals into a single integral; this tells us that the PV prescription is essential for making sense of this result, and that this digression has thus not been idle pedantry.)

Anyway, now that we have this sharpened result, let’s evaluate the PV integral. It is straightforward to just read off the residue of the integrand at the simple pole at $z = 0$:

$$\frac{2}{z^2} (1 - e^{i\omega t}) = \frac{2}{z^2} (1 - 1 - izt + \cdots) = \frac{-2it}{z} + \cdots \Rightarrow \text{Res}(z = 0) = -2it.$$

**I will now assume that $t > 0$, and return to the case for $t < 0$ below.** Consider the contour $C$ which runs from $-R$ to $-\epsilon$ on the real axis, skirts below pole at $z = 0$ on a radius-$\epsilon$ half-circle $C_\epsilon$ centered at the origin, runs from $\epsilon$ to $R$ on the real axis, and finally closes via a radius-$R$ half-circle $C_R$ in the upper-half complex plane back to $-R$ on the real axis; see Fig. 1. (See comment below if you skirted above the pole.)

By Cauchy’s theorem, we have

$$\oint_{C} \frac{2}{z^2} (1 - e^{i\omega t}) dz = 2\pi i \text{Res}(z = 0) = 4\pi t.$$

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FIG. 1 Contour $C$ for 11.8.11. The whole contour is traced out in a CCW fashion. The small-radius half-circle is $C_{\epsilon}$; the large-radius half-circle is $C_R$.

The segment of the contour integral which lie on the real axis, combined, give the PV integral we seek, in the $\epsilon \to 0$ and $R \to \infty$ limit.

On $C_R$, we have

$$
\int_{C_R} \frac{2}{z^2} (1 - e^{izt}) \, dz = \int_0^{\pi} \frac{2}{R^2 e^{2i\theta}} \left(1 - e^{-iR \sin \theta + iR \cos \theta}\right) (iRe^{i\theta} \, d\theta) \sim R^{-1} \to 0,
$$

in the $R \to \infty$ limit (because $t > 0$, the exponential term damps).

Finally, consider the integral on $C_{\epsilon}$: heuristically, since this contour only sweeps out half a circle around a simple pole at $z = 0$, you can immediately write down that

$$
\int_{C_{\epsilon}} \frac{2}{z^2} (1 - e^{izt}) \, dz = \frac{1}{2} \times 2\pi i \times \text{Res}(z = 0) = 2\pi t,
$$

but let’s be a little more careful:

$$
\int_{C_{\epsilon}} \frac{2}{z^2} (1 - e^{izt}) \, dz = \int_0^{2\pi} \frac{2}{e^{2i\theta}} \left(1 - e^{it\epsilon e^{i\theta}}\right) (i\epsilon e^{i\theta} \, d\theta),
$$

as $\epsilon \to 0$, so our heuristic argument was correct for the simple pole.

Note that it was important for the $\int_{C_{\epsilon}}$ result that there is a simple pole at the origin of this function, which is only true because the numerator $\sim z$ for small $z$. If you tried to split this integral up and evaluate the integral of each term independently, you’d have double poles at $z = 0$ in each integrand, and the result of each integral would be separately divergent. If you are careful, as I was here, you see that the divergences cancel in the sum of the two terms’ results leaving only a finite result (note the coefficient of the $1/\epsilon$ term above is $1 - 1 = 0$, with one of each of these divergences coming from one of each of the original two terms in the integral). However, if you naively applied reasoning similar to my first, heuristic, argument you can go badly astray: for example, suppose I just tried to look at the first term and made the heuristic argument: $\text{PV} \int_{-\infty}^{\infty} \frac{2}{z^2} \, dz = \frac{2}{i\pi \text{Res}[z = 0]} = 0$ since the
residue of the double pole is zero. This is a manifestly incorrect answer: it claims that the integral of a $C^\infty$ smooth (except at $z = 0$), positive-definite function with support on the whole real axis, taken over the whole real axis, is zero, which is impossible. This particular PV integral actually does not exist / diverges. Of course, if you made the same error with the second term, you might serendipitously stumble into the right answer, but the reasoning would be completely wrong.

To understand what is going wrong with the heuristic argument consider the following: while $\oint dz\, z^{-n} = 0$, $n > 1$, $n \in \mathbb{Z}$ this is due to the angular integral being over the whole range $[0, 2\pi]$, which enforces a cancellation making the whole integral zero, independent of the radius of the circle, which actually enters the result as $1/\epsilon^{n-1}$. If you only trace out a fraction of the full circle instead, the integral must diverge on small circles since there is no such angular cancellation (in general) and the $1/\epsilon^{n-1}$ diverges as $\epsilon \to 0$ for $n \neq 1$. It is only for a simple pole, $n = 1$, that the radius actually falls out of the integral entirely allowing you to make correct heuristic arguments about what fraction of the circle the contour traverses. Please bear this in mind: it is extremely important!

Let’s put this all together now (using an obvious shorthand, with the $R$ and $\epsilon$ limits understood)

$$\oint C = \left( \text{PV} \int \right) + \int_{C_{\epsilon}} + \int_{C_{R}} \quad (24)$$

$$\Rightarrow 4\pi t = \left( \text{PV} \int \right) + 2\pi t + 0 \quad (25)$$

$$\Rightarrow \text{PV} \int_{-\infty}^{\infty} \frac{2}{\omega^2} (1 - e^{i\omega t}) \, d\omega = 2\pi t \quad (26)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{2}{\omega^2} (1 - \cos \omega t) \, d\omega = 2\pi t \quad (t > 0). \quad (27)$$

Note that nothing would stop us from skirting above the pole at $z = 0$ on a different $C_{\epsilon}'$; in this case, $C$ would enclose no pole, so $f_C = 0$, but $\int_{C_{\epsilon}'} = -2\pi t$ since the contour is CW, so we would get the same result when the dust settles. On the other hand, for $t > 0$, the large-radius contour must close in the upper half-plane in this derivation to avoid a divergent integral on $C_{R}$.

For $t < 0$, the same steps as above are applicable, except that the radius-$R$ contour must close in the lower-half complex plane to avoid the divergent exponential. This means that $f = 0$, whereas all the rest of the steps are the same, so the sign of the result changes:

$$\oint C = \left( \text{PV} \int \right) + \int_{C_{\epsilon}} + \int_{C_{R}} \quad (28)$$

$$\Rightarrow 0 = \left( \text{PV} \int \right) + 2\pi t + 0 \quad (29)$$

$$\Rightarrow \text{PV} \int_{-\infty}^{\infty} \frac{2}{\omega^2} (1 - e^{i\omega t}) \, d\omega = -2\pi t \quad (30)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{2}{\omega^2} (1 - \cos \omega t) \, d\omega = -2\pi t \quad (t < 0). \quad (31)$$

We can sum up the overall result as

$$\int_{-\infty}^{\infty} \frac{2}{\omega^2} (1 - \cos \omega t) \, d\omega = 2\pi |t| \quad (32)$$
II. 11.8.16 [10]

This is a simple application of the residue theorem. Once again, we know the integral exists and is thus equal to its PV, so

\[
I = \int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} \, dx = \text{PV} \int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} \, dx = \left( \oint_{C} - \int_{C_R} \right) \frac{z^2}{z^4 + 1} \, dz, \tag{34}
\]

where \( C_R \) is a radius-\( R \) (\( R > 1 \)) half-circle in the upper-half complex plane, centered at \( z = 0 \), and traversed in a CCW fashion, and \( C = [-R, R] \cup C_R \) is the contour obtained by running from \(-R\) to \(+R\) on the real axis, and then back to \(-R\) via \( C_R \).

But on \( C_R \), the integral goes like

\[
\int_{C_R} z^2 \, dz \sim \int_{0}^{\pi} \frac{R^2 e^{2i\theta}}{1 + R^4 e^{4i\theta}} (iRe^{i\theta} d\theta) \sim R^{-1} \rightarrow 0, \tag{35}
\]
as \( R \to \infty \), while \( C \) encloses the two poles in the upper-half complex plane at \( z = \frac{1}{\sqrt{2}} [\pm 1 + i] = e^{i\pi/4}, e^{3i\pi/4} \) (i.e., the two fourth-roots of \(-1\) which lie in the upper half-plane), so

\[
I = \oint_{C} \frac{z^2}{z^4 + 1} \, dz = 2\pi i \left[ \text{Res} \left( z = e^{i\pi/4} \right) + \text{Res} \left( z = e^{3i\pi/4} \right) \right] \tag{36}
\]

So all that remains is to find the residues; to do this, write

\[
\frac{z^2}{z^4 + 1} = \frac{z^2}{(z - e^{i\pi/4})(z - e^{3i\pi/4})(z - e^{5i\pi/4})(z - e^{7i\pi/4})}. \tag{37}
\]

Then reading off the residues is straightforward: simply drop the factor in the denominator responsible for the pole of interest and evaluate what remains at that pole; we obtain

\[
\text{Res} \left( z = e^{i\pi/4} \right) = -\frac{1}{4} e^{3i\pi/4} \quad \text{Res} \left( z = e^{3i\pi/4} \right) = -\frac{1}{4} e^{i\pi/4}. \tag{38}
\]

Therefore,

\[
I = 2\pi i \frac{-e^{3i\pi/4} - e^{i\pi/4}}{4} = \frac{\pi}{\sqrt{2}} \tag{39}
\]

III. 11.8.22 [10]

It is not stated to which set \( n \) belongs; for simplicity, I’m going to assume that \( n \in \mathbb{Z} \) and that \( n > 1 \). The former if an artificial restriction in the sense that the result does hold more generally, but you need to be more careful in the derivation to make sure that what you are doing makes sense; for instance, if \( n \in \mathbb{Q} - \mathbb{Z} \), you have to worry about branch cuts and whether the function is single-valued, etc.

Here we exploit the periodicity in the function:

\[
f(z) = \frac{1}{1 + z^n} \tag{40}
\]

\[
f(re^{i\theta}) = \frac{1}{1 + r^n e^{2i\theta}} \tag{41}
\]

\[
\Rightarrow f(re^{i\theta + 2\pi k/n}) = \frac{1}{1 + r^n e^{2i\theta + 2\pi k}} = f(re^{i\theta}); \quad (k \in \mathbb{Z}). \tag{42}
\]
Meanwhile, the roots of $1 + z^n$ are at $z = (-1)^{1/n} = e^{i\pi/n + 2\pi ik/n}$ for $k \in \mathbb{Z}$. Note that $k = 0, 1, ..., (n - 1)$ exhausts the poles if $n \in \mathbb{Z}$.

Now consider the contour $C$ traversed CCW, defined thus: a straight line segment $C_1$ along the real axis from 0 to $R$, followed by $C_R$, the length of the radius-$R$ arc in the upper half-plane subtended by an angle $2\pi/n$, finally returning to the origin from the end-point of the arc via a straight line, $C_2$ (i.e., the contour given in Arfken and Weber, Fig. 11.8.22).

$C$ encloses the pole at $z = e^{i\pi/n}$, and on the two straight segments of the path $f(z) = f(x) = (1 + x^n)^{-1}$ (by periodicity); however, on $C_2$, we have $z = re^{2\pi i/n}$ and $r$ runs from $R$ down to 0. On the curved segment, the integrand goes like $R(1 + R^n)^{-1}$ at large $R$, so as long as $n > 1$, $\int_{C_R}$ vanishes as $R \to \infty$.

Putting this all together, in the limit $R \to \infty$, we have

$$
\oint_C f(z) \, dz = \int_{C_1} f(x) \, dx + \int_{C_R} f(z) \, dz + \int_{C_2} f(z) \, dz \quad (43)
$$

$$
\Rightarrow 2\pi i \text{Res}(f; z = e^{i\pi/n}) = \int_0^\infty f(x) \, dx + \int_{C_2} f(z) \, dz \quad (44)
$$

$$
\Rightarrow 2\pi i \text{Res}(f; z = e^{i\pi/n}) = (1 - e^{2\pi i/n}) \int_0^\infty f(x) \, dx. \quad (45)
$$

So all we need is the residue:

$$
\text{Res}(f; z = e^{i\pi/n}) = \lim_{z \to e^{i\pi/n}} \frac{z - e^{i\pi/n}}{1 + z^n} \frac{1}{1 + z^n - 1} = \frac{1}{n} \exp \left[ i\pi \frac{1 - n}{n} \right] = -\frac{1}{n} e^{i\pi/n}, \quad (46)
$$

since $e^{-i\pi} = -1$, so

$$
(1 - e^{2\pi i/n}) \int_0^\infty f(x) \, dx = 2\pi i \text{Res}(f; z = e^{i\pi/n}) = 2\pi i \times \frac{-1}{n} e^{i\pi/n}
$$

$$
\Rightarrow \int_0^\infty f(x) \, dx = -\frac{\pi}{n} e^{i\pi/n} - \frac{2i}{n} e^{2i\pi/n} = \frac{\pi}{n} e^{i\pi/n} - e^{-i\pi/n} = \frac{\pi/n}{\sin(\pi/n)}; \quad (n = 2, 3, ...). \quad (47)
$$

The integral diverges if $n \leq 1$. However, I again stress that the restriction to integer $n$ was artificial, and done for simplicity; the result does hold under more general assumptions on $n$.

### IV. 11.8.27 [10]

I was pretty lenient in grading this question. Please review this solution carefully though.

This is a surprisingly tricky question, so it pays to be pedantic in setting things up. The contours and various angles we will use in this question are defined in Fig. 2.

We consider the integral

$$
I \equiv \int_{0}^{1} \frac{1}{(x^2 - x^3)^{1/3}} \, dx. \quad (49)
$$

Let us view the integrand as a complex function of $z$:

$$
f(z) = z^{-2/3}(1 - z)^{-1/3}. \quad (50)
$$

Before we even think about performing any integrals, let us be careful in noting that this function has two branch points: $z = 0$ and $z = 1$; these can be joined by a branch cut that runs along the real axis between 0 and 1: see Fig. 2. Now suppose that we define $z = re^{i\theta}$ and $z - 1 = \rho e^{i\phi}$ (obviously $\rho, \phi$ and $(r, \theta)$ are not independent, but
FIG. 2 Contour $C = \bigcup_{i=1}^{4} C_i$ is the inner CW contour, $C'$ is the outer CCW contour. The branch cut and branch points are indicated in orange, and the angles $\theta$ and $\phi$ are defined per this diagram. The radii of the contours $C_4$, $C_2$ and $C'$ and $\epsilon_4$, $\epsilon_2$ and $R$, respectively. Points A-F are used to show how the phase of the integrand changes around the contour $C$.

for our current purposes, we don’t need to know how they are related), and re-write the integrand $f(z)$ in terms of $r, \rho, \theta, \phi$:

$$f(z) = r^{-2/3} e^{-2i\theta/3} (-1)^{-1/3} \rho^{-1/3} e^{-i\phi/3} = r^{-2/3} \rho^{-1/3} \exp \left[ -i\frac{2}{3} \theta - i\frac{1}{3} \phi + i\delta \right], \quad (51)$$

where we wrote $(-1)^{1/3} = e^{i\delta}$ with $\delta = \pi/3 + (2\pi/3)n$ for $n \in \mathbb{Z}$; the choice of the value of $\delta$ defines the branch of the function we are dealing with. We would like to have the function $f(z)$ to be real when $z = +0 + i0$ (i.e., infinitesimally displaced away from $z = 0$ into the first quadrant). But for $z = +0 + i0$, we have $\theta = 0$ and $\phi = \pi$, so we must select $\delta = \pi/3$; that is, on the branch of the function we will choose to deal with,

$$f(z) = |z|^{-2/3} |z - 1|^{-1/3} \exp \left[ -i\frac{2}{3} \theta - i\frac{1}{3} \phi + i\frac{\pi}{3} \right]. \quad (52)$$

You may have made a different choice, but it must be consistent, so all your results may only be a common phase $e^{2\pi in/3}$ off from mine for some fixed common $n \in \mathbb{Z}$.

Now consider integrating this function around the contour $C = \bigcup_{i=1}^{4} C_i$, defined in Fig. 2. As the contour $C$ is traversed both $\theta$ and $\phi$ decrease as the contour is traversed in a CCW fashion, and the phase of $f$ varies

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accordingly; we give some sample phases at A-F in Table I.

From the foregoing considerations, in the limit where \( \epsilon_{2,4} \to 0 \), we can see that on the contour segment \( C_1 \), we have \( \arg[f(z)] = 0 \), so

\[
\int_{C_1} f(z)dz = \int_{C_1} |z|^{-2/3}|z-1|^{-1/3}dz \to \int_0^1 \frac{dx}{(x^2 - x^3)^{1/3}} = I,
\]

where I used that \( z = x \) here and \( |z-1| = (1-x) \) on this piece of contour. We chose the branch so that the overall argument here is zero. Meanwhile, on the contour segment \( C_3 \), we have \( \arg[f(z)] = 2\pi/3 \), so

\[
\int_{C_3} f(z)dz = \int_{C_3} |z|^{-2/3}|z-1|^{-1/3}e^{2\pi i/3}dz \to e^{2\pi i/3} \int_0^1 \frac{dx}{(x^2 - x^3)^{1/3}} = -e^{2\pi i/3}I.
\]

Meanwhile, on the circular contour segment \( C_2 \), we have \( r = 1 + \mathcal{O}(\epsilon_2) \) and \( \rho = \epsilon_2 \), with \( \theta = 0 \) and \( \phi \) decreasing from \( \pi \) to \( -\pi \), so the integral is (dropping sub-dominant powers of \( \epsilon_2 \))

\[
\int_{C_2} f(z)dz \sim \int_{\pi}^{-\pi} (i\epsilon_2e^{i\phi}d\phi)e^{-i\phi/3+i\pi/3}e^{-1/3} \sim \epsilon_2^{2/3} \to 0,
\]

the latter in the limit where \( \epsilon_2 \to 0 \). Similarly, on the circular contour segment \( C_4 \), we have \( r = \epsilon_4 \) and \( \rho = 1 + \mathcal{O}(\epsilon_4) \), with \( \phi = -\pi \) and \( \theta \) decreasing from \( 0 \) to \( -2\pi \), so the integral is (dropping sub-dominant powers of \( \epsilon_4 \))

\[
\int_{C_4} f(z)dz \sim \int_{0}^{2\pi} (i\epsilon_4e^{i\theta}d\theta)e^{2\pi i/3+2\pi i/3} \sim \epsilon_4^{1/3} \to 0,
\]

the latter in the limit where \( \epsilon_4 \to 0 \). So in the limit \( \epsilon_{2,4} \to 0 \), we have

\[
\oint_C f(z)dz = (1 - e^{2\pi i/3})I.
\]

Now suppose I construct a single contour \( \tilde{C} = C \cup P \cup C' \cup (-P) \) where I start some point on \( C \), move all the way around \( C \) and back to just before this point, then pass to \( C' \) by some path \( P \) (which does not cross or run near the branch cut), then traverse all of \( C' \) and finally return to \( C \) in the opposite direction along a path which I will call \(-P\) which is infinitesimally displaced from \( P \). Then Cauchy’s Theorem together with the observation that \( f(z) \) is analytic in the interior of \( \tilde{C} \) tells us that \( \oint_{\tilde{C}} = 0 = \oint_C + \oint_P + \oint_{C'} + \oint_{-P} \). On \( P \) and \(-P\), the argument of \( f \) must differ by the amount of phase \( f(z) \) picks up on \( C \) or \( C' \), but since \( P \) and \(-P\) are not running near branch cuts, the phase shift must be a multiple of \( 2\pi \) (this is almost by definition and construction: if it did not, there

| TABLE I | Phase of the integrand at selected points A-F as defined in Fig. 2. Note that a full traversal of the contour (change in\( \Delta\theta = \Delta\phi = -2\pi \)) yields \( \Delta\arg[f(z)] = +2\pi \) as well, which is important: \( f \) is single-valued. |
|---------|---------|---------|---------|
| Point   | \( \theta \) | \( \phi \) | \( \arg[f(z)] \) |
| A       | 0        | \( \pi \) | 0        |
| B       | 0        | \( \pi \) | 0        |
| C       | 0        | 0        | \( \pi/3 \) |
| D       | 0        | \( -\pi \) | \( 2\pi/3 \) |
| E       | 0        | \( -\pi \) | \( 2\pi/3 \) |
| F       | \( -\pi \) | \( -\pi \) | \( 4\pi/3 \) |
| A (after traversing \( C \) once) | \( -2\pi \) | \( -\pi \) | \( 2\pi \) |
would be a branch cut between $\mathcal{P}$ and $-\mathcal{P}$; this is confirmed by Table I. Therefore, the integrals over $\mathcal{P}$ and $-\mathcal{P}$ will cancel out in the limit where the infinitesimal displacement between them vanishes, and we have

$$\oint_{C} f(z)dz = -\oint_{C'} f(z)dz \Rightarrow \oint_{C'} f(z)dz = (e^{2\pi i/3} - 1)I = 2ie^{i\pi/3} \sin(\pi/3) I = i\sqrt{3}e^{i\pi/3} I. \quad (58)$$

Now consider that in the limit where the radius $R$ of the contour $C'$ goes to $\infty$, both $\rho \approx r \approx R \gg 1$ and $\phi \approx \theta$ with both increasing by $2\pi$ as $C'$ is traversed CCW. Therefore, on this large-radius circular contour, we have

$$\oint_{C'} f(z)dz \to \int_{0}^{2\pi} (iRe^{i\theta}d\theta)R^{-2/3}R^{-1/3}e^{-i\theta}e^{i\pi/3} = ie^{i\pi/3} \int_{0}^{2\pi} d\theta = 2\pi ie^{i\pi/3}. \quad (59)$$

[I note in passing that this result is independent of the specific start and end points (in angle $\theta$) for the large-$R$ contour, so there is no concern about where exactly I constructed the path $P$ joining $C$ and $C'$ above.]

So finally, in the triple limit $\epsilon_2 \to 0$, $\epsilon_4 \to 0$ and $R \to \infty$, we have

$$\oint_{C'} f(z)dz = 2\pi ie^{i\pi/3} = i\sqrt{3}e^{i\pi/3} I = 2\pi ie^{i\pi/3}, \quad (60)$$

whence we conclude that

$$I = \int_{0}^{1} \frac{1}{(x^2 - x^3)^{1/3}} dx = \frac{2\pi}{\sqrt{3}}. \quad (61)$$