Mass of chiral fermion.

Consider left-handed fermion $\psi_L$ charged under U(1) with charge $Q_L$, and $\psi_R$ with $Q_R$. And $Q_L \neq Q_R$ (called chiral fermion)

kinetic term can be fully gauge invariant

$$\psi_L^\dagger \left( \partial_\mu - i g Q_L A_\mu \right) \psi_L + \psi_R^\dagger \left( \partial_\mu - i g Q_R A_\mu \right) \psi_R$$

However, mass term $m (\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$ is not gauge invariant. Therefore, in this case, chiral fermion is massless.

However, chiral fermion could acquire mass after SSB, if the Higgs field $\phi(x)$ has charge $Q_\phi = Q_L - Q_R$. In this case

$$\phi \, \psi_L^\dagger \psi_R + \phi^* \psi_R^\dagger \psi_L$$

is gauge invariant and allowed. After SSB, $\langle \phi \rangle = \frac{\langle \phi \rangle}{\sqrt{2}}$. These terms becomes fermion mass terms with $m = \frac{\langle \phi \rangle}{\sqrt{2}}$.

This is exactly analogous to the case of fermion mass in the standard model.
Mass mixing and mass matrix

First, recall an analogy with coupled harmonic oscillator in classical mechanics

\[ L = \frac{1}{2} \dot{x}_1^2 + \frac{1}{2} \dot{x}_2^2 - \frac{1}{2} \omega_1^2 x_1^2 - \frac{1}{2} \omega_2^2 (x_2 - x_1)^2 \]

\[ = \frac{1}{2} \dot{x}_1^2 + \frac{1}{2} \dot{x}_2^2 - \frac{1}{2} (x_1, x_2) M^2 (x_1, x_2) \]

where

\[ M^2 = \begin{pmatrix} \omega_1^2 + \omega_2^2 & - \omega_2^2 \\ - \omega_2^2 & \omega_2^2 \end{pmatrix} \]

We can find the eigenmodes (normal modes) by diagonalizing

\[ M^2 \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = U \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

\[ U M^2 U^T = \begin{pmatrix} \tilde{\omega}_1^2 & 0 \\ 0 & \tilde{\omega}_2^2 \end{pmatrix} \]

\[ \rightarrow L = \frac{1}{2} \dot{x}_1'^2 + \frac{1}{2} \dot{x}_2'^2 - \frac{1}{2} (x_1', x_2') \begin{pmatrix} \tilde{\omega}_1^2 & 0 \\ 0 & \tilde{\omega}_2^2 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \]
Similarly, several different fields can have “coupled” mass matrix

$$\mathcal{L} = \sum_{i=1}^{n} \frac{1}{2} (\partial \phi_i)^2 + \frac{1}{2} \sum_{i,j} \phi_i M_{ij} \phi_j$$

mass matrix

In this case, we would follow the same strategy to find eigenvalues and eigenmodes.

physical masses

wave-function of free particle
Consider 2 spin-1 gauge bosons with corresponding \( \text{U}(1) \) gauge symmetries

\[
\begin{array}{c|cc}
& \text{U}(1)_a & \text{U}(1)_b \\
\Phi & q^a & q^b \\
\end{array}
\]

In addition, there is a complex scalar \( \Phi \) with the following charge assignment

Covariant derivative

\[
D_\mu \Phi = \left( \partial_\mu + i \bar{q}^a A_\mu + i \bar{q}^b A_\mu \right) \Phi
\]

where \( \bar{q}_a = q_a \bar{q}^a \quad \bar{q}_b = q_b \bar{q}^b \)

\[
L = (D_\mu \Phi)^\dagger (D^\mu \Phi) + \ldots
\]
After spontaneous symmetry breaking, $\Phi$ acquire vacuum expectation value

$$\langle \Phi \rangle = \frac{v}{\sqrt{2}}$$

To obtain mass spectrum, plug $\langle \Phi \rangle$ back into $\mathcal{L}$.

$$\mathcal{L} \rightarrow \frac{1}{2} (A_a, A_b) \begin{pmatrix} 3a^2 v^2 & 3a 3_b v^2 \\ 3a 3_b v^2 & 3_b^2 v^2 \end{pmatrix} \begin{pmatrix} A_a \\ A_b \end{pmatrix}$$

Diagonalizing gauge boson mass matrix, we find physical states $A_1, A_2$ as

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} A_a \\ A_b \end{pmatrix}$$

$$\cos \theta = \frac{3_b}{\sqrt{3_a^2 + 3_b^2}}$$
$$\sin \theta = \frac{3_a}{\sqrt{3_a^2 + 3_b^2}}$$

with physical masses

$$m_{A_1}^2 = 0, \quad m_{A_2}^2 = (3_a^2 + 3_b^2) v^2$$

That is, only one massive spin-1 vector after “Higgsing.” This is to be expected. After all, spin-1 particle must “eat” a goldstone boson to become massive. However, there is only one goldstone to eat in $\Phi$. Therefore, there can only be one massive spin-1.
To summarize, the mass spectrum after SSB is

massless \( A_1 \)

massive \( A_2 \) \( m_{A_2}^2 = (\bar{s}_a^2 + \bar{s}_b^2) v^2 \)

write \( \Phi = \frac{1}{\sqrt{2}} (\bar{s} + h) e^{i\frac{\lambda}{\sqrt{2}}} \)

h, addition scalar, the "Higgs boson"

Although the Standard Model has more complicated gauge symmetries, the pattern is very similar to this example with two U(1)s.
We can push the analogy with the SM further by considering coupling $A_1, A_2$ to fermion $\psi$, with charge under $U(1)_{a,b}$ as

\[
\begin{array}{c|cc}
& U(1)_a & U(1)_b \\
\hline
\psi & q^f_a & q^f_b \\
\end{array}
\]

\[
L = i \overline{\psi} \gamma^\mu \left( \gamma_\mu + ig^f_a A^a + ig^f_b A^b \right) \psi
\]

\[
g^f_a = g_a q^f_a, \quad g^f_b = g_b q^f_b
\]

Going to mass eigenstate $A_1, A_2$

\[
A^a = \cos \theta A_1 + \sin \theta A_2
\]

\[
A^b = -\sin \theta A_1 + \cos \theta A_2.
\]

\[
L = i \overline{\psi} \gamma^\mu \psi
\]

\[+ i \left( ig^f_a \cos \theta - ig^f_b \sin \theta \right) A^a_{\mu} \overline{\psi} \gamma^\mu \psi
\]

\[+ i \left( ig^f_a \sin \theta + ig^f_b \cos \theta \right) A^b_{\mu} \overline{\psi} \gamma^\mu \psi
\]

Again, this is very similar to the photon and $Z$-boson coupling the fermions in the Standard Model.
Pattern of general SSB.

Consider global symmetry group $G$, under which a set of scalar fields $\phi_a$ furnish an irreducible representation.

Under $G$, the scalar fields transform as

$$\phi^a \rightarrow \phi^a + \alpha \Delta^a(\phi)$$

where $\Delta^a(\phi)$ is a general linear function in $\phi$.

After SSB, a subset of scalar fields acquire VEVs $\langle \phi^a \rangle = \phi^a_0$. As a result, the global symmetry is broken down to its subgroup

$$G \xrightarrow{\phi} H.$$ 

Note that $\phi^a_0$s do not transform under the unbroken subgroup:

$$\kappa \Delta^a(\phi) = 0$$

$$\kappa \in H$$

To study the spectrum after SSB, Taylor expand the scalar potential around the ground state

$$V(\phi) = V(\phi_0) + \frac{1}{2} \delta \phi^a \delta \phi^b \frac{\partial^2 V}{\partial \phi^a \partial \phi^b} |_{\phi^a, b = \phi^a_0, b} + ...$$

where $\delta \phi^a = \phi^a - \phi^a_0$.

Note that there is no linear term, since the ground state satisfies

$$\frac{\partial V(\phi)}{\partial \phi^a} |_{\phi^a = \phi^a_0} = 0.$$
Since transformation $\phi \to \phi + \omega \Delta(\phi)$ is a symmetry, it leaves $V(\phi)$ unchanged

$$V(\phi^a) = V(\phi^a + \omega \Delta^a(\phi))$$

or equivalently

$$\Delta^a(\phi) \frac{\partial}{\partial \phi^a} V(\phi) = 0$$

Taking another derivative w.r.t. $\phi^b$, and setting all scalar fields to their VEVs, we obtain

$$\frac{\partial}{\partial \phi^b} \left( \frac{\partial V}{\partial \phi^a} \right) \phi_{ab} = \phi_{ab} + \Delta^a(\phi) \left( \frac{\partial^2 V}{\partial \phi^a \partial \phi^b} \right) \phi_{a0} \phi_{b0} = 0$$

0 since $\frac{\partial V}{\partial \phi^a} |_{\phi_{ab} = \phi_{a0}} = 0$

minimization condition

$$\Rightarrow \Delta^a(\phi) \left( \frac{\partial^2 V}{\partial \phi^a \partial \phi^b} \right) \phi_{ab} = \phi_{ab} = 0$$

Two solutions

i) $\Delta^a(\phi) = 0$ VEV of $\phi^a$ does not break symmetry.

i.e., symmetry transformation belongs to the unbroken subgroup $H$.

ii) $\Delta^a \frac{\partial^2 V}{\partial \phi^a \partial \phi^b} = 0$

$\Delta^a$ is an eigenvector $M^2_{ab} = \frac{\partial^2 V}{\partial \phi^a \partial \phi^b}$ with zero eigenvalue. From equation 10, we see that $M^2_{ab}$ is just the mass matrix. The existence of zero eigenvalues $\Rightarrow$ zero mass states (Goldstones), identified with the corresponding eigenvectors.
From this discussion, it's clear that for symmetry breaking $G \rightarrow H$, the number of Goldstones is the same as the number of symmetry transformation generators which is not in $H$, or in other words, in the coset $G/H$.

* A simple example

Consider 2 real scalar fields $\phi_1, \phi_2$ with an SO(2) symmetry (This is equivalent to one complex scalar $\phi = \phi_1 + i\phi_2$ with a U(1) symmetry)

The SO(2) transformation is

$$\phi' = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\alpha \rightarrow 0 = \phi + \alpha \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

or in the notation we used earlier

$$\Delta(\phi) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -\phi_2 \\ \phi_1 \end{pmatrix}$$

In the symmetry breaking phase, we can choose VEVs

$$\langle \phi_1 \rangle = \phi_1^0, \quad \langle \phi_2 \rangle = 0$$

Hence, the Goldstone is $\Delta(\phi)|_{\phi = \phi_0} = (0)$ along the $\phi_2$ direction, as expected.