Kinematics for searches at LHC
Reconstruction of sparticle masses
with calculation details

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1 Cascade decays

If we assume that Supersymmetry is discovered at the LHC, most likely from inclusive studies based on large missing energy, it will be very important to attempt a reconstruction of the masses of the sparticles involved.

If the squarks and/or gluinos are kinematically accessible at the LHC, they are expected to have large production rates. The cross sections for the production of a squark (excluding stop) or a gluino at the LHC are displayed in Figure 1. The nearly diagonal lines delimit three regions:

**Figure 1:** Regions of the $m_0$ versus $m_{1/2}$ plane showing the production cross-sections and with main squark and gluino decays.

- **Region 1:** in this region, the gluinos are heavier than any of the squarks. The decay chains of the produced sparticles are expected to be

  \[ \tilde{g} \rightarrow \tilde{q} \tilde{q} , \tilde{g} \rightarrow q\chi \]  

  A fairly model independent constraint is that, as the squark and gluino masses are determined by running down the RGEs and both are dominated by the contributions from $M_3$, the gluino mass cannot exceed the squark mass by more than $\sim 20\%$.

- **Region 2:** in this region some squarks are heavier, other are lighter than the gluino. Hence, rather complicated decay chains are possible, for instance

  \[ \tilde{q}_L \rightarrow \tilde{g} \tilde{q} , \tilde{g} \rightarrow \tilde{b} \tilde{b} , \tilde{b} \rightarrow b\chi \]  

  as the $\tilde{q}_L$ of the first two generations are expected to be among the heaviest squarks and the $\tilde{b}_1$ (and $\tilde{t}_1$) among the lightest.

- **Region 3:** in this region, the gluinos are lighter than any of the squarks. A typical decay chain is then

  \[ \tilde{g} \rightarrow \tilde{g} \tilde{q} , \tilde{g} \rightarrow q\tilde{q}\chi \]  

  where the gluino gives rise to a three-body decay mediated by a virtual squark.
Neutralinos are ~ of the areas depend on the assumptions (MSUGRA) and on the value of decay via the ~. The decay of the ~ will then provide an excellent signature for the events and can be used as a starting point to reconstruct all masses in the chain.

The main decay modes of the ~, and hence the signatures, are

\[
\tilde{\chi}_2^0 \to \tilde{\ell} \ell, \ h^0 \tilde{\chi}_1^0, \ Z^0 \tilde{\chi}_1^0, \ \ell^+ \ell^- \tilde{\chi}_1^0
\]

(4)

where the last decay is mediated by the exchange of an off-shell Z or \( \tilde{\ell} \). The first decay corresponds to a gauge interaction coupling of a Wino to a slepton-lepton pair and dominates if it is kinematically allowed. When this decay is kinematically forbidden and \( m_{1/2} \) is large enough, so that \( m(\tilde{\chi}_2^0) - m(\tilde{\chi}_1^0) \gg m(h^0) \), the next preferred decay is to \( h^0 \). This corresponds to a gaugino-higgsino transition and thus requires a non-zero higgsino component in at least one of the two neutralinos. If also this decay is kinematically forbidden, direct three-body decays take place. The corresponding regions in the eective mass distributions. This method involves the least model dependent assumptions. It has been pioneered by the ATLAS Collaboration, see e.g. [1, 2, 3, 4, 5].

If the decay chain is long enough, all masses can be extracted from the measurement of end points in the effective mass distributions. This method involves the least model dependent assumptions. It has been pioneered by the ATLAS Collaboration, see e.g. [1, 2, 3, 4, 5].

In addition to the decays via a ~, a large fraction of squark decays will proceed via a ~ decay, which may lead to

\[
\tilde{\chi}_1^\pm \to \tilde{\nu} \nu, \ W^{\pm} \tilde{\chi}_1^0, \ H^{\pm} \tilde{\chi}_1^0, \ \ell^+ \ell^- \tilde{\chi}_1^0
\]

(5)

where the last decay is mediated by the exchange of an off-shell W, \( \tilde{\nu} \) or \( \tilde{\ell} \). As they involve new particles (~ and \( \tilde{\nu} \)), additional masses can be determined. Especially interesting is the leptonic decay which, by giving access to \( M(\ell) \) and \( M(\tilde{\nu}) \) allows the model independent sum rule

\[
M(\tilde{\ell})^2 - M(\tilde{\nu})^2 = -M_W^2 \cos 2\beta
\]

(6)

to be used to extract the value of \( \tan \beta \). The localization of the chargino decay modes in the \( (m_0, m_{1/2}) \) plane is illustrated for a MSUGRA case in Figure 2 (right).

Figure 2: Regions of the \( m_0 \) versus \( m_{1/2} \) plane with main \( \chi_2^0 \) decays (left) and main decays of \( \chi_1^\pm \) (right).
Further constraints beyond the MSUGRA ones can be imposed, for example the compatibility with the measured relic density. These limit very severely the available parameter space. However, the lack of knowledge of the SUSY breaking mechanism encourages the future experiments to prepare themselves to cope with the broadest possible spectrum of situations. Rather than restricting oneself to a very constrained model, it will be important to understand how to detect departures from the SM in a large variety of topologies and to investigate how to reconstruct the sparticle masses and other SUSY parameters. Once the sparticle masses have been determined, it should even be possible to fully reconstruct (some of?) the events. Of course, there is more information available in the events than just the end points, e.g. momentum asymmetries of the decay leptons, branching ratios and total cross section measurements. These additional informations are not taken into account here.

In the following, we will present a rather complete overview of the values of the various end points of the mass distributions as a function of the sparticle masses, from which the latter can be determined. The problem of how to select and identify the leptons and jets and how to reduce the backgrounds is not addressed in this note.

A difficulty is that, depending on the ratios of the sparticle masses, different configurations and hence different formulae can relate the masses to the end points in three- and four-body effective mass distributions. It will be shown that the identification of the correct configuration (and corresponding formula) can in principle be based on correlations between different effective mass distributions. Moreover, the applicability of a given formula depends on well defined mass ratios and only some of them can be valid at any given mass point.

2 General formulae

2.1 Two-body decays

Most production and decay chains can be viewed as two-body decays or as a succession of two-body decays. Some useful two-body decay formulae are gathered here.

In the two-body decay of a particle $M$ into $m_1$ and $m_2$, the energy and momentum in the rest frame of $M$ are given by:

$$M = E_1^* + E_2^* \quad , \quad |p_1^*| = |p_2^*| = p^*$$

squared the first equation:

$$2E_1^*E_2^* = M^2 - m_1^2 - m_2^2 - 2p^{*2}$$

squared again and extracting $p^*$:

$$p^{*2} = \frac{1}{4M^2}[(M^2 - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2]$$

$$= \frac{1}{4M^2}[M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2]$$

$$= \frac{1}{4M^2}[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]$$

and the energy becomes:

$$E_1^{*2} = p^{*2} + m_1^2$$

$$= \frac{1}{4M^2}[M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 + 4M^2m_1^2]$$

$$= \frac{1}{4M^2}[M^2 + m_1^2 - m_2^2]^2$$

In summary:

$$E_1^* = \frac{M^2 + m_1^2 - m_2^2}{2M}$$

$$|p_1^*| = |p_2^*| = \frac{\left[(M^2 - (m_1 + m_2)^2) (M^2 - (m_1 - m_2)^2)\right]^{1/2}}{2M}$$

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The formulae are considerably simplified if one of the decay particles is massless, which is frequently a good approximation. Assuming that $m_2 = 0$ they become:

$$E_1^i = \frac{M^2 + m_1^2}{2M}, \quad E_2^i = \frac{M^2 - m_1^2}{2M}, \quad |p_1^i| = |p_2^i| = \frac{M^2 - m_1^2}{2M}$$  \hspace{1cm} (10)

### 2.2 Lorentz transformation

The transformation from the rest frame of $M$ to the lab system is defined by two kinematical variables of $M$, the velocity $\beta = q/E$ and $\gamma = E/M$, where $q$ and $E$ are the lab frame momentum and energy of $M$. The Lorenz transformation to the lab system is most easily decomposed into a longitudinal and a transverse part:

$$E_i = \gamma [E_i^* + \beta p^*\cos \theta_i^*]$$

$$p_{Ti} = p_i^* = p^* \sin \theta_i^*$$

$$p_{Li} = \gamma [\beta E_i^* + p^* \cos \theta_i^*]$$ \hspace{1cm} (11)

with $\theta_i^* = \theta_i^* + \pi$. The maximum and minimum values of $E_i$ and $p_{Li}$ are obtained for $\cos \theta_i^* = +1$ and $-1$ respectively. The energy $E_i$ of particle $i$ in the lab frame has a central value which is boosted w.r.t. its rest frame energy $E_i$ by a factor $\gamma$ and acquires a spread of magnitude $\frac{q}{M}p^*$. The minimum $p_{Li}$ can become negative if $\beta_i^* = p_i^*/E_i^* > \beta$.

### 2.3 Lorentz transformation, some basic formulae

Most of the chain decays can be viewed as a sequence of two-body decays. As the observable particles can to a good approximation be taken as massless and we are interested in computing their effective mass, the Lorentz transformations can be considerably simplified. Moreover, maxima or minima of effective masses correspond to collinear configurations, where the particle is emitted along or opposite to the direction of the Lorentz boost. In this case,

$$E_U = \gamma E_U^* \left(1 \pm \beta\right)$$ \hspace{1cm} (12)

For the transformation of the massless particle from the frame $R$ to its parent $X$, the Lorentz transformation parameters are

$$\beta = \frac{M_X^2 - M_R^2}{M_X^2 + M_R^2}, \quad \gamma = \frac{M_X^2 + M_R^2}{2M_X M_R}$$ \hspace{1cm} (13)

we get

$$\gamma(1 \pm \beta) = \frac{M_X^2 + M_R^2}{2M_X M_R} \pm \frac{M_X^2 - M_R^2}{2M_X M_R} = \frac{1}{2M_X M_R} \left[(M_X^2 + M_R^2) \pm (M_X^2 - M_R^2)\right]$$

so that the maximum (along the boost) and minimum (opposite the boost) energies are:

$$E_U^{max} = \frac{M_X}{M_R} E_U^*, \quad E_U^{min} = \frac{M_R}{M_X} E_U^*$$ \hspace{1cm} (14)

reflecting the fact that the boost along increases the energy and the boost opposite reduces the energy. Similarly, for the momentum

$$p_U = \gamma E_U^* \left(\beta \pm 1\right)$$ \hspace{1cm} (15)

leading to (sign is measured in the boost direction)

$$p_U^{max} = E_U^{max}, \quad p_U^{min} = -E_U^{min}$$ \hspace{1cm} (16)

These expressions will considerably simplify our later calculations.
2.4 Lorentz transformation and mass for a two-step decay

When Lorentz transforming a massive particle (with mass $M_0$)

\[ E'_0 = \gamma (E^*_0 + \beta p^*_0 \cos \theta^*) \]
\[ p'_0 = \gamma (\beta E^*_0 + p^*_0 \cos \theta^*) \]  

(17)

where we left the angular dependence, defining $\theta^*$ as the angle of particle $O$ with respect to the direction of $R$. The same simplification as above cannot be made, due to the non-zero mass. However, we will be interested in effective masses (upper and lower edges). Take for instance the effective mass of particle $V$ (massless) and $O$ (massive) from the reaction $X \rightarrow R + V$, followed by $R \rightarrow O + U$ with $U$ massless.

\[ M^2_{VO} = (E'_V + E^*_0)^2 - (p'_V + p^*_0)^2 = 0 + M^2_0 + 2E'_V E^*_0 - 2p'_V \cdot p^*_0 \]

defining the z-axis in the direction of $R$, as in the example of figure 3. For collinear configurations, we will need to evaluate expressions of the form

\[ E'_0 + p^*_{0,L} = \gamma (1 + \beta) (E^*_0 + p^*_0 \cos \theta^*) \]
\[ E'_0 - p^*_{0,L} = \gamma (1 - \beta) (E^*_0 - p^*_0 \cos \theta^*) \]  

(18)

where the primed quantities are in the rest frame of $X$.

To compute the effective mass, as in this case $p'_V = -E'_V \hat{z}$, we have

\[ M^2_{VO} = M^2_0 + 2E'_V (E^*_0 + p^*_{0,L}) \]  

(19)

where $p^*_{0,L}$ is signed. Now, using the above expression, the transformation is again simplified as in the massless case.

\[ E'_0 + p^*_{0,L} = \frac{M_X}{M_R} (E^*_0 + p^*_0 \cos \theta^*) \]  

(20)

The energy/momentum of the massive particle $O$ in the frame of $R$ is

\[ E'_V = \frac{M_X^2 - M_R^2}{2M_X}, \quad E^*_0 = \frac{M_X^2 + M_0^2}{2M_R}, \quad p^*_0 = \frac{M_R^2 - M_0^2}{2M_R} \]  

so that

\[ E^*_0 + p^*_0 = M_R, \quad E^*_0 - p^*_0 = \frac{M^2_0}{M_R} \]  

(21)

(22)

For the configuration of Figure 3

- and $\cos \theta^* = +1$:

\[ M^2_{VOa} = M^2_0 + \frac{M_X^2 - M_R^2}{M_X} \frac{M_X}{M_R} M_R \]
\[ = M^2_0 + M_X^2 - M_R^2 \]  

(23)

- and $\cos \theta^* = -1$:

\[ M^2_{VOb} = M^2_0 + \frac{M_X^2 - M_R^2}{M_X} \frac{M_X}{M_R} M_R \]
\[ = M^2_0 + (M_X^2 - M_R^2) \frac{M^2_0}{M_R^2} = \frac{M_X^2 M_0^2}{M_R^2} \]  

(24)

The value of $M_{VOb}$ is obviously always smaller than $M_{VOa}$.
2.5 Lorentz transformation and mass for a three-step decay

Now suppose that there is an additional decay step, e.g. $Q \rightarrow X + q$ (with $q$ massless) as illustrated in figure 4. We now take as $z$-axis the direction of $X$. In this case, some caution is needed with the signs.

![Figure 4: Configuration of a three-step two-body decay.](image)

Then

$$ E_0 = \gamma_X (E'_0 + \beta_X p'_0 \cos \theta') $$

$$ p_0 = \gamma_X (\beta_X E'_0 + p'_0 \cos \theta') $$

where the $\theta'$ is the angle of $R$ with respect to the direction of $X$ and the two orientations of $R$ in the collinear decay of $X$ are shown in the figure. The quantities relevant to the calculation of the effective mass are

$$ E_0 + p_{0,L} = \gamma_X (1 + \beta_X) (E'_0 + p'_0 \cos \theta') $$

$$ E_0 - p_{0,L} = \gamma_X (1 - \beta_X) (E'_0 - p'_0 \cos \theta') $$

If $R$ is emitted parallel to the $z$-axis ($\cos \theta' = +1$)

$$ E_0 + p_{0,L} = \gamma_X (1 + \beta_X) (E'_0 + p'_0) $$

$$ E_0 - p_{0,L} = \gamma_X (1 - \beta_X) (E'_0 - p'_0) $$

If $R$ is emitted opposite to the $z$-axis ($\cos \theta' = -1$)

$$ E_0 + p_{0,L} = \gamma_X (1 + \beta_X) (E'_0 - p'_0) $$

$$ E_0 - p_{0,L} = \gamma_X (1 - \beta_X) (E'_0 + p'_0) $$

To illustrate the usage of the signs, let us recompute the mass of the system $OV$ of previous section. If $R$ is in the direction of $X$ (and hence of $\bar{z}$), we have $\bar{p}_V = -E'_V \bar{z}$ (as $V$ is massless) and thus

$$ M^2_{V,O} = M^2_0 + 2E'_V (E_0 + p_{0,L}) $$

Using the above equation

$$ E_0 + p_{0,L} = \frac{M_Q}{M_X} (E'_0 + p'_0) $$

Moreover, as $V$ is emitted opposite to the boost of $X$

$$ E_V = \frac{M_X}{M_Q} E'_V $$

and hence it reproduces the expression of previous section. If $R$ is in the direction opposite to $X$ (and hence to $\bar{z}$), we have $\bar{p}_V = +E'_V \bar{z}$ and this time

$$ M^2_{V,O} = M^2_0 + 2E'_V (E_0 - p_{0,L}) $$

in which case we have

$$ E_0 - p_{0,L} = \frac{M_X}{M_Q} (E'_0 + p'_0) $$

$$ E_V = \frac{M_Q}{M_X} E'_V $$
and we recover again the expression of previous section. It shows that this usage of signs is consistent, but illustrates that some caution is necessary.

Let us now come back to the calculation of the effective mass of \( q \) and \( O \) which is then (\( \vec{p}_q = -E_q \vec{z} \))

\[
M_{qO}^2 = M_0^2 + 2E_q(E_0 + p_{0,L})
\]  
(26)

For \( R \) along the \( z \)-axis

\[
E_0 + p_{0,L} = \frac{M_Q}{M_X}(E'_0 + p'_{0,L}) = \frac{M_Q}{M_X} \frac{M_X}{M_R} (E_0^* + p_0^* \cos \theta^*)
\]

or

\[
E_0 + p_{0,L} = \frac{M_Q}{M_R} (E_0^* + p_0^* \cos \theta^*)
\]  
(27)

For \( R \) in the direction opposite to the \( z \)-axis

\[
E_0 + p_{0,L} = \frac{M_Q}{M_X}(E'_0 - p'_{0,L}) = \frac{M_Q}{M_X} \frac{M_X}{M_R} (E_0^* - p_0^* \cos \theta^*)
\]

From this, the mass of the \( (q,O) \) system can now be computed in all the configurations. The energy of \( q \) is

\[
E_q = \frac{M_Q^2 - M_X^2}{2M_Q},
\]
(28)

and the values of the \( E_0^* \) and \( p_0^* \) combinations were given in (22).

For the configuration of Figure 4 (left)

- and \( \cos \theta^* = +1 \):

\[
M_{qOa}^2 = M_0^2 + \frac{M_Q^2 - M_X^2}{M_Q} \frac{M_Q}{M_R} M_R
= M_0^2 + M_Q^2 - M_X^2
\]  
(29)

- and \( \cos \theta^* = -1 \):

\[
M_{qOb}^2 = M_0^2 + \frac{M_Q^2 - M_X^2}{M_Q} \frac{M_Q}{M_R} \frac{M_R}{M_R}
= M_0^2 + (M_Q^2 - M_X^2) \frac{M_Q^2}{M_R^2} = M_0^2 \frac{M_Q^2 - M_X^2 + M_R^2}{M_R^2}
\]  
(30)

For the configuration of Figure 4 (right)

- and \( \cos \theta^* = +1 \):

\[
M_{qOc}^2 = M_0^2 + \frac{M_Q^2 - M_X^2}{M_Q} \frac{M_Q}{M_X} M_R M_0^2
= M_0^2 + (M_Q^2 - M_X^2) \frac{M_Q^2}{M_X^2} = \frac{M_Q^2 M_0^2}{M_X^2}
\]  
(31)

- and \( \cos \theta^* = -1 \):

\[
M_{qOd}^2 = M_0^2 + \frac{M_Q^2 - M_X^2}{M_Q} \frac{M_Q}{M_X} M_R M_0^2
= M_0^2 + (M_Q^2 - M_X^2) \frac{M_Q^2}{M_X^2} = \frac{M_Q^2 M_0^2 - M_X^2 + M_R^2}{M_X^2}
\]  
(32)

from which it follows that, independent of the sparticle masses, \( M_{qOc} \) is always the smallest and the following hierarchy is satisfied \( M_{qOc} \leq M_{qOd} \), \( M_{qOd} \leq M_{qOa} \) whereas the hierarchy between \( M_{qOa} \) and \( M_{qOb} \) depends on the mass ratios, with \( M_{qOd} \leq M_{qOb} \) if \( \frac{M_X}{M_R} \leq \frac{M_0}{M_R} \) or \( M_R^2 \leq M_X M_0 \).
2.6 Note on end point formulae

In the following, end points will be computed for collinear configurations. When the masses of quarks and leptons can be neglected, we will see that the upper end point takes the general form

\[ M_{\text{max}}^2 = M_1^2 (1 - \frac{M_2^2}{M_1^2}) (1 - \frac{M_3^2}{M_2^2}) \]  

(33)

where the \( M_i \) represent some sparticle masses (or systems) with \( M_1 \geq M_2 \geq M_3 \). This end point can have a maximum value when a given relation is satisfied among the masses. Let us fix \( M_1 \) and \( M_3 \) and vary \( M_2 \). The maximum is obtained for

\[ \frac{\partial M_{\text{max}}^2}{\partial M_2} = M_1^2 (1 - \frac{M_2^2}{M_1^2}) + M_2^2 (1 - \frac{M_3^2}{M_2^2}) (\frac{M_3^2}{M_2^2}) = 0 \]

from which

\[ \frac{M_2^2 - M_3^2}{M_2^2} = \frac{M_3^2}{M_2^2} (M_1^2 - M_2^2) \]

and the masses should satisfy the relation

\[ M_2^2 = M_1 M_3 \]  

(34)

For this value, the end upper point becomes

\[ M_{\text{max}}^2 = M_1^2 \frac{M_2^2 - M_1 M_3}{M_1^2} \frac{M_1 M_3 - M_2^2}{M_1 M_3} \]

\[ = (M_1 - M_3)^2 \]  

(35)

and corresponds to the case where the sparticle or system \( M_3 \) is put at rest in the frame of sparticle \( M_1 \).

Such an extremum exists only as a function of \( M_2 \). Else, \( M_{\text{max}}^2 \) is linear in \( M_1^2 \) and \( M_3^2 \) and shows no extremum.

3 Decays starting from a \( \tilde{\chi}_2^0 \)

Take a decay chain of the type:

\[ X \rightarrow V + R , \quad R \rightarrow U + O \]  

(36)

where \( O \) is a \( \tilde{\chi}_1^0 \) with mass \( M_0 \), \( R \) is an intermediate state with mass \( M_R \) and particles \( V \) and \( U \) are considered massless. Some examples are:

\[ \tilde{\chi}_2^0 \rightarrow f + \bar{f} , \quad \bar{f} \rightarrow f + \tilde{\chi}_1^0 \]  

(37)

\[ \tilde{q} \rightarrow q \tilde{\chi}_2^0 , \quad \tilde{\chi}_2^0 \rightarrow f + \bar{f} \]  

(38)

3.1 Direct 3-body decays

For a direct 3-body decay (e.g. through an o-shell \( \bar{f} \)) \( \tilde{\chi}_2^0 \rightarrow \tilde{\chi}_1^0 f \bar{f} \), the di-fermion effective mass has an upper edge

\[ M_{\text{max}}^\text{ff} = M_X - M_0 \]  

(39)

obtained when the \( \tilde{\chi}_1^0 \) is at rest in the frame of \( X = \tilde{\chi}_2^0 \) and the phase space of 2 particles in 3 has a maximum near its upper end, which makes this maximum mass very detectable. The mass distribution may, however, be distorted if the final state is dominated by the exchange of an off-shell particle.

Although this relation is rather well known, it may be worth giving its demonstration. Consider the \( \tilde{\chi}_1^0 \) recoiling against the \( ff \) system in the \( \tilde{\chi}_2^0 \) rest frame.

\[ M_X^2 = (E_0 + E_{ff})^2 - (p_0 + \vec{p}_{ff})^2 = M_0^2 + M_{ff}^2 + 2E_0E_{ff} - 2p_0 \cdot \vec{p}_{ff} \]
From which

\[ M_{ff}^2 = M_X^2 - M_0^2 - 2E_0 E_{ff} + 2\vec{p}_0 \cdot \vec{p}_{ff} \]

From momentum conservation, we have \( \vec{p}_0 = -\vec{p}_{ff} \). The mass \( M_{ff} \) is maximized when \( p_0 = p_{ff} = 0 \), so that

\[ M_{ff}^2 = M_X^2 - M_0^2 - 2M_0^2 M_{ff} \]

and finally

\[ M_X^2 = (M_{ff} + M_0)^2 \]

which gives relation (39).

When the \( f \bar{f} \) system is close to its maximum mass, both the \( f \bar{f} \) and the \( \chi_0^0 \) are nearly at rest in the system of \( X \). The Lorentz transformation to the laboratory system is then given by \( (M_{ff} \approx M_{ff}^{max}) \)

\[ \vec{p}_{ff} = \frac{E_X}{M_X} \vec{p}_X M_{ff} \]  

(40)

from which the laboratory momentum of \( X \) can be determined as a function of the masses

\[ \vec{p}_X = \frac{M_X}{M_{ff}} \vec{p}_{ff} = \left( 1 + \frac{M_0}{M_{ff}} \right) \vec{p}_{ff} \]  

(41)

Furthermore, the momentum of the \( \chi_0^0 \) is

\[ \vec{p}_0 = \frac{\vec{p}_X}{M_X} M_0 = \frac{M_0}{M_{ff}} \vec{p}_{ff} \]  

(42)

Hence, assuming the knowledge of the \( \chi_0^0 \) mass, the decay chain can be fully reconstructed. This was the basis of the approach adopted by the Catania group [11, 12].

It is, however, not possible to determine the full kinematics, including the masses \( M_X \) and \( M_0 \), from observables in a \( \chi_1^0 \) decay. This can be understood as follows: the variables describing the \( X = \chi_2^0 \) decay can be chosen as \( \vec{p}_X, M_X, M_0 \) and \( \theta^* \) (the decay angle in the rest frame of \( X \)), see Figure 5. The measured quantities are the lepton momenta \( \vec{p}_1 \) and \( \vec{p}_2 \) in the laboratory frame. The direction of \( \vec{p}_X \) can be determined by equalizing the \( p_{T1} \) and \( p_{T2} \) in the di-lepton plane, and is independent of the masses. But the di-lepton plane can be rotated around this direction without affecting the kinematics. Hence, we lose one measured variable out of the 6 in \( \vec{p}_1 \) and \( \vec{p}_2 \). It is thus not possible to determine all 6 unknowns above.

More explicitly, after determining the direction of \( \vec{p}_X \) as just said, the remaining quantities can be chosen as \( p_T, p_{L1} \) and \( p_{L2} \). Assuming that the \( \chi_1^0 \) is at rest in the frame of \( X = \chi_2^0 \), the following relations hold:

\[ p^* = \frac{(M_X - M_0)}{2} \]
\[ p_T = p^* \sin \theta^* \]
\[ p_{L1} = \gamma p^*(\beta + \cos \theta^*) \]
\[ p_{L2} = \gamma p^*(\beta - \cos \theta^*) \]
The value of $p_T$ allows to solve for the angle $\theta^*$ as a function of $p^*$. The longitudinal momenta can be combined to

$$p_{L1} + p_{L2} = p_{ff} = 2\beta \gamma p^* = \frac{p_X}{M_X} (M_X - M_0)$$

the same as relation (40), allowing the determination of $p_X$, and

$$p_{L1} - p_{L2} = 2\beta \gamma p^* \cos \theta^* = 2\gamma p^* \sqrt{1 - \frac{p_T^2}{p^*}}$$

As we have

$$\gamma^2 = \frac{E_X^2}{M_X^2} = \frac{p_X^2 + M_X^2}{M_X^2} = \frac{M_X^2}{M_{ff}^2} p_{ff}^2 + M_X^2 = \frac{E_{ff}^2}{M_{ff}^2}$$

we get an additional relation

$$p_{L1} - p_{L2} = \frac{E_{ff}}{M_{ff}} \sqrt{(M_X - M_0)^2 - 4p_T^2}$$

which shows that only a combination of masses can be determined, namely $(M_X - M_0)$, the one obtained from the end point. Hence, additional information is required to allow the individual masses to be determined.

Neglecting mismeasurements, the missing (transverse) momentum in the event is due to the 2 escaping $\tilde{\chi}^0_1$. By taking events with two decays $\tilde{\chi}^0_2 \rightarrow \tilde{\chi}^0_1 f f$, both with near maximum $M_{ff}$,

$$\vec{p}_{\text{miss}} = \vec{p}_{01} + \vec{p}_{02} = \left( \frac{\vec{p}_{ff1}}{M_{ff1}} + \frac{\vec{p}_{ff2}}{M_{ff2}} \right) M_0 \quad (43)$$

This equation, used in the transverse plane, allows in principle the mass of the $\tilde{\chi}^0_1$ to be determined from the missing and the di-fermion momenta. In practice, by requiring both di-lepton systems to be near the end point the remaining statistics may be insufficient.

### 3.2 Two-step sequential 2-body decays

#### 3.2.1 Effective mass calculation for a 2-body decay

An upper edge in the effective mass is also observable in case of a cascade decay, like above where $\tilde{f}$ would be on-shell. The effective mass can be computed in the rest frame of $X = \tilde{\chi}^0_2$. Choose the $z$-axis along the direction of the sfermion, as depicted on figure 6. The di-fermion effective mass is given by

![Figure 6: Kinematics of the $\tilde{\chi}^0_2$ decay via sfermions.](image)

$$M_{ff}^2 = (E_V + E_U)^2 - (-p_V + p_{LU})^2 - (0 + p_{TV})^2 = 2E_V E_U + 2p_V p_{LU} = 2E_V (E_U + p_{LU}) \quad (44)$$

where the indices $L$ and $T$ denote the longitudinal and transverse components. As particle $V$ is massless, the energy and momentum of $V$ and $R$ are given by the expressions (10) and similarly for particle $U$ in the rest frame of $R$

$$E^*_U = \left| p_{LU}^* \right| = \frac{M_R^2 - M_0^2}{2M_R}$$
The Lorenz transformation of the massless particle \( U \) to the rest frame of \( X \) becomes

\[
E_U = \gamma E'_U (1 + \beta \cos \theta^*)
\]
\[
p_{LU} = \gamma p'_{LU} (\beta + \cos \theta^*)
\]

from which

\[
E_U + p_{LU} = \gamma E'_U (1 + \beta)(1 + \cos \theta^*)
\]
\[
= \frac{M_X}{2M_R^2} (M_R^2 - M_0^2)(1 + \cos \theta^*)
\]

The effective mass then takes the simple expression

\[
M_{ff}^2 = 2 \frac{M_X^2 - M_R^2}{2M_X^2} \frac{M_X}{2M_R^2} (M_R^2 - M_0^2)(1 + \cos \theta^*)
\]

or

\[
M_{ff}^2 = 2 \frac{(M_X^2 - M_R^2)(M_R^2 - M_0^2)}{2M_R^2}(1 + \cos \theta^*)
\]

showing that the squared mass is linear in \( \cos \theta^* \).

3.2.2 Upper end point in \( M(\ell \bar{\ell}) \)

The upper end point of the dilepton (or difermion) mass distribution for the decay chain \( \tilde{\chi}_2^0 \rightarrow \tilde{\ell} + \bar{\tilde{\ell}} \), \( \tilde{\ell} \rightarrow \ell + \chi_1^0 \), is given by

\[
M_{max}^2 = M_X \sqrt{1 - \frac{M_R^2}{M_X^2}(1 - \frac{M_0^2}{M_R^2})}
\]

It corresponds to the configuration where the two fermions are emitted back-to-back in the rest frame of \( X = \tilde{\chi}_2^0 \), shown in Figure 7. The minimum value of the effective mass is obtained when the two fermions are emitted in the same direction and is zero. It should be noted that this expression for the effective mass remains valid independent of the production mode of the \( \tilde{\chi}_2^0 \), in particular when it is among the decay products of a heavier SUSY particle.

If it is assumed that particle \( R \) decays isotropically, which is expected for the decay of a sfermion, then relation (45) shows that the distribution in \( M_{ff}^2 \) should be flat and the differential decay width in \( M_{ff} \) should increase linearly with \( M_{ff} \).

\[
\frac{1}{\Gamma} \frac{d\Gamma}{dM_{ff}} dM_{ff} = 2 \frac{2}{(M_{max}^2)^2} M_{ff} dM_{ff}
\]

The upper edge of the distribution is thus expected to be well visible. However, as the minimum mass is zero, no end point can be observed at the lower edge. Provided the sharp edge of the distribution is not too much smeared out by the measurement uncertainties, this case should be well distinguishable from the phase space distribution of 2 particles out of 3, expected for the direct 3-body decay. In fact, the observation of such a triangular shape is a strong indication that the decay proceeded via a scalar \( (\ell) \) intermediate state.
Note that the $\tilde{\chi}_1^0$ can be put at rest in the rest frame of $X = \tilde{\chi}_2^0$, like for the 3-body decay, provided a given mass relation is fulfilled. The $\tilde{\chi}_1^0$ will be at rest if the $\beta$ of the Lorentz transformation equals the velocity of the $\tilde{\chi}_1^0$ in the rest frame of $R = l$. This requires

$$\frac{M_X^2 - M_R^2}{M_X^2 + M_R^2} = \frac{M_R^2 - M_0^2}{M_R^2 + M_0^2}$$

from which it follows that

$$-2(M_R^4 - M_R^2 M_0^2) = 0$$

leading to the mass relation

$$M_R = \sqrt{M_X M_0}$$

so that $M_R$ should be the geometric mean of $M_X$ and $M_0$. It is easily verified that in this case formula (46) yields a maximum mass $M_{ff}^{\text{max}} = M_X - M_0$, the same as for a 3-body decay. This situation corresponds to the largest possible value for $M_{ff}^{\text{max}}$. For smaller or larger values of $M_R$, the $\tilde{\chi}_1^0$ will have non-zero momentum along or opposite the direction of $R = \bar{l}$, respectively, and the maximum mass is given by (46), a value which is smaller than $M_X - M_0$.

3.2.3 The lower edge in $M(ll)$

When the two leptons (or fermions) are parallel, as illustrated in Figure 8, the effective mass is zero

![Figure 8: Configuration leading to the minimum of the mass for $(ll)$](image)

Figure 8: Configuration leading to the minimum of the mass for $(ll)$.

and this lower end point provides no information on the sparticle masses involved in the decay. Such events carry nevertheless important information which can be seen as follows. Using energy-momentum conservation, we have in the rest frame of $X = \tilde{\chi}_2^0$

$$M_R^2 = (E_{l2} + E_0)^2 - (\tilde{p}_{l2} + \tilde{p}_0)^2 = M_0^2 + 2E_{l2}(E_0 - p_0 \cos \theta_{20})$$

$$M_X^2 = (E_{l1} + E_{l2} + E_0)^2 - (\tilde{p}_{l1} + \tilde{p}_{l2} + \tilde{p}_0)^2$$

$$= (E_{l2} + E_0)^2 + 2E_{l1}(E_{l2} + E_0) - (\tilde{p}_{l2} + \tilde{p}_0)^2 - 2p_{l1}p_{l2}\cos \theta_{12} - 2p_{l1}p_0 \cos \theta_{10}$$

$$= M_R^2 + 2E_{l1}(E_{l2} - E_{l2}\cos \theta_{12} + E_0 - p_0 \cos \theta_{10})$$

from which we get the two relations

$$\frac{M_X^2 - M_R^2}{2E_{l2}} = E_0 - p_0 \cos \theta_{20}$$

$$\frac{M_X^2 - M_R^2}{2E_{l1}} = E_{l2}(1 - \cos \theta_{12}) + E_0 - p_0 \cos \theta_{10}$$

For parallel leptons, this yields

$$\frac{M_X^2 - M_R^2}{M_X^2 - M_R^2} = \frac{E_{l2}}{E_{l1}}$$

(49)

which provides a constraint on the sparticle masses from the ratio of the lepton energies. However, as there are few events around $M(ll) \approx 0$, it may help to include events with small $\theta_{12}$ and to extrapolate to $\theta_{12} = 0$ to determine more precisely the ratio $E_{l2}/E_{l1}$. Expanding the above equations gives

$$\frac{M_X^2 - M_R^2}{2E_{l2}} \approx \frac{M_X^2 - M_R^2}{2E_{l1}} + \frac{1}{2} E_{l2} \theta_{12}^2$$
or

\[
\frac{E_{12}}{E_{11}} \simeq \frac{M_R^2 - M_0^2}{M_X^2 - M_R^2} \frac{E_{12}^2}{M_{11}^2 - M_R^2} \theta_{12}^2
\]  

(50)

It shows that the ratio \( E_{12}/E_{11} \) is approached from below when \( \theta_{12} \to 0 \). For the ratio \( E_{11}/E_{12} \) it would be approached from above.

In real events, no distinction can be made between \( l_1 \) and \( l_2 \). We can then plot \( E_{\text{low}}/E_{\text{high}} \) as function of \( \theta_{12} \). If the value for \( \theta_{12} \to 0 \) is approached from below, the ratio measures \( E_{12}/E_{11} \), else it measures the inverse ratio. Hence, the approach allows the two cases to be distinguished.

The relation (49) allows for a simple geometrical interpretation. In the 3-dimensional space spanned by the masses (\( M_0, M_R, M_X \)) this constraint, rewritten in the form

\[
\frac{E_{12}}{E_{11}} M_X^2 + M_0^2 = (1 + \frac{E_{12}}{E_{11}}) M_R^2
\]

corresponds to a conical area around the axis \( M_R \). Moreover, it can be combined with relation (46) for the upper end point. Using (49) to replace \( M_R^2 - M_0^2 \) in (46) leads to

\[
M_{ll}^{\text{max}} = \frac{M_R^2}{M_R} \sqrt{\frac{E_{12}}{E_{11}}}
\]

and to replace \( M_X^2 - M_R^2 \) leads to

\[
M_{ll}^{\text{max}} = \frac{M_R^2 - M_0^2}{M_R} \sqrt{\frac{E_{11}}{E_{12}}}
\]

Hence the combination of the two constraints yields the parametric equations for \( M_0 \) and \( M_X \) as a function of \( M_R \)

\[
M_X^2 = M_R^2 + M_{ll}^{\text{max}} \sqrt{\frac{E_{11}}{E_{12}}} M_R
\]

\[
M_0^2 = M_R^2 - M_{ll}^{\text{max}} \sqrt{\frac{E_{11}}{E_{12}}} M_R
\]

4 Three-step sequential 2-body decays starting from a \( \tilde{q} \)

We next turn to the effective mass calculation for a three-step decay, like

\[
\tilde{q} \rightarrow q \chi_2^0, \chi_2^0 \rightarrow \tilde{l} + \tilde{l}, \tilde{l} \rightarrow l_2 + \chi_1^0
\]  

(51)

for which we want to compute the end points of the \((l_1q), (l_2q)\) and \((llq)\) mass distributions. The end point for \((ll)\) has been derived in section 3.

4.1 Overview of the collinear configurations

It is useful to start with an overview of the collinear configurations available for this decay sequence, except the one where all quarks and leptons are parallel, as this leads to all mass combinations being zero. This leaves three possible configurations. Also indicated are the end points which may give rise to the true end point which are listed as \( M_{ll}^{\text{max}}, M_{ll}^{\text{min}} \). The end point formulae and their condition of applicability are derived in the next sections.

\[
M_{ll}^{\text{max}}
\]

\[
M_{ll}^{\text{min}}
\]

18
\[
M_{\text{max}}; 2 l_2 q_{M_{\text{max}}} = M_{\text{min}} \ll q_{M_{\text{max}}}; 3 l q_{M_{\text{max}}} = R = l_1 \chi_{0} l_1 = X = Q = q_{0} \ll M_{\text{max}}; 2 l_2 q_{M_{\text{max}}} = M_{\text{Max}}; 2 l_2 q_{M_{\text{max}}}
\]
4.2 Upper end point in $M(l_1q)$

The generic formula (46) can also be applied to the second reaction listed in (38). For the decay chain
\( \tilde{q} \rightarrow q\tilde{\chi}_2^0, \tilde{\chi}_2^0 \rightarrow f + f \) the configuration leading to the maximum effective mass is shown in Figure 9, which is part of the configurations labelled ($q1$) and ($q2$) in section 4.1. The maximum mass is

\[
M_{l_1q_{\text{max}}} = M_Q \sqrt{1 - \frac{M_X^2}{M_Q^2}(1 - \frac{M_R^2}{M_X^2})}
\]

Similarly to the $ll$ case, the largest value of $M_{l_1q_{\text{max}}}$ is obtained when the $R = \tilde{l}$ is at rest in the $Q = q$ rest frame. This leads to $M_{l_1q_{\text{max}}} = M_Q - M_R$ and occurs when $M_X = \sqrt{M_Q M_R}$.

Given that the $\tilde{\chi}_2^0$ has spin 1/2 the ($l_1q$) mass distribution is affected by spin effects. However, the differences induced are between the ($l^+q$) and ($l^-q$) distributions and, if no sign selection is made, they cancel to yield a pure phase space distribution [6, 7]. Hence, in our case, we can again expect a triangular distribution for $M(l_1q)$,

\[
\frac{1}{\Gamma} \frac{d\Gamma}{dM_{l_1q}} = \frac{2}{(M_{l_1q_{\text{max}}})^2} M_{l_1q}
\]

smeread by the jet energy resolution.

4.3 End points in $M(l_2q)$

4.3.1 Upper end point in $M(l_2q)$

The maximum of the mass for ($l_2q$) is found for the configuration depicted as ($q3$) in section 4.1, which maximizes the energy of $l_2$ in the $q$ rest frame. The energy of $q$ in the rest frame of the $q$ is given by

\[
E_q = \frac{M_Q^2 - M_X^2}{2M_Q}
\]

and the energy of $l_2$ is obtained after a double Lorentz transformation using equation (14)

\[
E_{l_2} = \frac{M_Q M_X}{M_X} \frac{M_R^2 - M_0^2}{2M_R} = \frac{M_Q M_R^2 - M_0^2}{M_R}
\]

From (44), the effective mass is then

\[
(M_{l_2q_{\text{max}}})^2 = 4E_q E_{l_2}
\]

which gives

\[
M_{l_2q_{\text{max}}} = M_Q \sqrt{(1 - \frac{M_X^2}{M_Q^2})(1 - \frac{M_0^2}{M_R^2})}
\]

Using the results of section 2.6, it follows that this end point reaches its largest value for $M_X = \frac{M_0 M_Q}{M_R}$ where it amounts to $M_{l_2q_{\text{max}}} = M_Q - \frac{M_X M_0}{M_R}$ the fraction being the mass of the ($l_1q$) system.

This statement can also be derived explicitly as follows. The absolute maximum that $M_{l_2q}$ can reach for fixed mass of ($l_1, \tilde{\chi}_1^0$) is obtained when the system is put at rest in the rest frame of $Q = \tilde{q}$. This
condition can be computed as follows. In the rest frame of \( X = \chi_2^0 \), the momentum/energy of \( l_2 \) is given by

\[
p'_{l_2} = E'_{l_2} = \frac{M_X}{M_R} E^*_{l_2} = \frac{M_X}{M_R} \left( \frac{M_R^2 - M_0^2}{2M_R} \right)
\]

Then, the momentum and energy of the system \((l_1, \chi_1^0)\) in the rest frame of \( X = \chi_2^0 \) are

\[
p'_{l_{10}} = p'_{l_2} = \frac{M_X}{M_R} \left( \frac{M_R^2 - M_0^2}{2M_R} \right)
\]

\[
E'_{l_{10}} = M_X - E'_{l_2} = \frac{M_X}{2M_R} \left( 2M_R^2 - M_R^2 + M_0^2 \right) = \frac{M_X}{M_R} \left( \frac{M_R^2 + M_0^2}{2M_R} \right)
\]

Hence, the value of the velocity is

\[
\beta_{l_{10}} = \frac{M_R^2 - M_0^2}{M_R + M_0^2}
\]

which should be equal to (larger than) the velocity of \( X \)

\[
\beta_X = \frac{M_Q^2 - M_X^2}{M_Q^2 + M_X^2}
\]

in the rest frame of \( Q = \tilde{q} \). This condition is fulfilled if

\[
(M_R^2 - M_0^2)(M_Q^2 + M_X^2) \geq (M_R^2 + M_0^2)(M_Q^2 - M_X^2)
\]

\[-M_Q^2M_0^2 + M_R^2M_X^2 \geq 0
\]

Hence, if

\[
\frac{M_X}{M_Q} \geq \frac{M_0}{M_R} \tag{58}
\]

the system \((l_1, \chi_1^0)\) can be put at rest in the \( Q = \tilde{q} \) rest frame. The absolute maximum of the \( l_{2q} \) mass is then

\[
M_{l_{2q}}^{\text{max}} = M_Q \left( 1 - \frac{M_X}{M_Q^2} \right) = M_Q - \frac{M_0M_X}{M_R} \tag{59}
\]

Now the mass of the \((l_1, \chi_1^0)\) system, with the two particles running in parallel, is

\[
M_{l_{10}}^2 = E_{l_{10}}^2 - p_{l_{10}}^2 = \left( \frac{M_X}{2M_R^2} \right)^2 \left[ (M_R^2 + M_0^2)^2 - (M_R^2 - M_0^2)^2 \right]
\]

or

\[
M_{l_{10}} = \frac{M_X}{2M_R} \sqrt{2M_RM_0} = \frac{M_0M_X}{M_R} \tag{60}
\]

and the above condition can be rewritten as

\[
M_X^2 \geq M_Q M_{l_{10}} \tag{61}
\]

showing an absolute maximum for fixed \( M_{l_{10}} \). It gives an explicit example of what was mentioned in section 2.6.

One may wonder whether the end point formula still gives the maximum when the \((l_1, \chi_1^0)\) system is sent backwards in the \( Q = \tilde{q} \) frame. That it is the case is seen by computing the \( l_2 \) energy in the \( X = \chi_2^0 \) rest frame as a function of \( M_{l_{10}} \)

\[
E'_{l_2} = \frac{M_X^2 - M_{l_{10}}^2}{2M_X}
\]

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Lorentz transformed to the $Q = \tilde{q}$ frame the energy becomes

$$E_{l2} = \frac{M_Q}{M_X} E'_{l2}$$

which shows that the largest $l2$ energy, and hence $M_{l2q}^{max}$, is indeed obtained for the smallest $M_{l1q}$, i.e. the collinear configuration used above.

It is also interesting to check in which conditions the end point of $M(l1q)$ is higher or lower than the one in $M(l2q)$. From the comparison of the two end point formulae, it appears that for example $M(l2q)^{max} \geq M(l1q)^{max}$ occurs when

$$\frac{M_0^2}{M_R^2} < \frac{M_0^2}{M_X^2}$$

which gives

$$M_R^2 > M_X M_0$$

Hence, depending on the relative values of the masses of the $\tilde{l}$, the $\chi_2^0$ and the $\chi_1^0$ either $M_{l2q}^{max}$ or $M_{l1q}^{max}$ can be larger than the other.

The differential decay width in $M_{l2q}$ has been computed in [10] to be, neglecting spin correlations

$$\frac{1}{\Gamma} \frac{d\Gamma}{dM_{l2q}} = \frac{4M_{l2q}}{(M_{l2q}^{max})^2(1 - \frac{M_R^2}{M_X^2})} \ln \frac{M_R}{M_X} \frac{M_{l2q}^{max}}{M_{l2q}^2}$$

for $0 \leq M_{l2q} \leq \frac{M_R}{M_X} M_{l2q}^{max}$

and

$$\frac{1}{\Gamma} \frac{d\Gamma}{dM_{l2q}} = \frac{4M_{l2q}}{(M_{l2q}^{max})^2(1 - \frac{M_R^2}{M_X^2})} \ln \frac{M_X}{M_R} \frac{M_{l2q}^{max}}{M_{l2q}^2}$$

for $\frac{M_R}{M_X} M_{l2q}^{max} \leq M_{l2q} \leq M_{l2q}^{max}$

4.3.2 Distinguishing upper end points of $M(l1q)$ from $M(l2q)$

The extraction of the masses from the end points has sometimes been done by assigning the end points of the largest and the lowest $M(lq)$ combination to $M_{l1q}^{max}$ and $M_{l2q}^{max}$ (e.g. in early ATLAS papers). This, however, introduces a model dependent assumption. An alternative is to plot the high and low $(lq)$ mass combination separately, which is experimentally clear but makes the interpretation in terms of masses more involved. It is the purpose of this section to suggest another possible approach.

It is worth observing that, from the comparison of the configuration labelled $(q3)$ in section 4.1 and figure 7, it appears that the maximum mass for the $l2q$ system is reached for a configuration maximizing also the mass of the $(ll)$ system. On the contrary, the maximum $(l1q)$ mass is reached for any $M(ll)$. This would allow to distinguish the two $(lq)$ end points, by plotting, for example, $M(ll)$ versus $M(lq)$ for both combinations of the $(lq)$ system, as illustrated in Figure 10. The end point $M_{l2q}^{max}$ should become more prominent by selecting large masses of $(ll)$, which allows it to be identified without introducing model dependent assumptions.

4.3.3 Secondary end point in $M(l2q)$

Another, secondary, $(l2q)$ upper end point can be obtained by flipping the direction of the decay products of the $\chi_2^0$, in which case the configuration is the same as the one labelled $(q1)$ in section 4.1.

The lepton energy in the rest frame of the $\tilde{q}$ is

$$E_{l2} = \frac{M_Q}{M_X} \frac{M_R M_{l2q}^2 - M_0^2}{2M_R}$$

(64)

This leads to the following maximum mass

$$(M_{l2q}^{max})^2 = 4E_{l2} E_{l2} = 4 \frac{M_Q^2}{2M_Q} \frac{M_R}{M_X} \frac{M_{l2q}^2}{2M_R} - \frac{M_0^2}{M_R^2} = \frac{(M_Q^2 - M_R^2)(M_{l2q}^2 - M_0^2)}{M_X^2}$$

and thus

$$M_{l2q}^{max} = \frac{M_R}{M_X} M_{l2q}^{max}$$

(65)
This end point can never be larger than the one of (57). Whether it will be visible is an open question, as it is located in a region where there are few events, namely at small $M(ll)$.

It can nevertheless be asked under which conditions this secondary end point is larger than $M_{llq}^{\text{max}}$. This requires

$$\frac{(M_Q^2 - M_{\tilde{\chi}_1^0}^2)(M_R^2 - M_{\tilde{\chi}_2^0}^2)}{M_{\tilde{\chi}_1^0}^2} > \frac{(M_Q^2 - M_{\tilde{\chi}_1^0}^2)(M_R^2 - M_{\tilde{\chi}_2^0}^2)}{M_{\tilde{\chi}_2^0}^2}$$

hence

$$M_R^2 - M_{\tilde{\chi}_1^0}^2 > M_R^2 - M_{\tilde{\chi}_2^0}^2$$

or

$$2M_R^2 > M_{\tilde{\chi}_1^0}^2 + M_{\tilde{\chi}_2^0}^2$$

(66)

Therefore, when the true ($lq$) end point is given by $M_{llq}^{\text{max}}$, this secondary end point can be located between $M_{llq}^{\text{max}}$ and $M_{l2q}^{\text{max}}$. An enrichment of events contributing to this end point can, in principle, be obtained by selecting low $M(ll)$.

### 4.4 End points in $M(llq)$

In the ATLAS paper from the Cambridge group [3], three possible end point configurations are computed and the largest of them is kept as upper end point. This, of course, assumes an a priori knowledge of the sparticle masses, which will not be the case with real data. In the following, we will derive the formulae for the end points, establish the conditions on sparticle masses under which they apply and propose a model independent way of identifying which of them is observed in real data.

#### 4.4.1 Absolute maximum of $M(llq)$

The configuration relevant for the ($llq$) topology is shown in Figure 11. The largest value of $M_{llq}^{\text{max}}$ is reached when the $\tilde{\chi}_1^0$ is at rest in the $Q = \tilde{q}$ rest frame and is given by

$$M_{llq}^{\text{max}} = M_Q - M_0$$

(67)

It needs to be verified under which conditions this situation is kinematically possible. For a general 3-body final state of total mass $M$, the energy-momentum conservation equations are:

$$M = E_q + E_1 + E_2$$
where the labels $q, 1, 2$ refer to the quark, $l1$ and $l2$. Taking the direction of $q$ opposite to the $z$-axis and $y$ perpendicular to $z$

$$-p_{qz} = p_{1z} + p_{2z}$$

$$p_{1y} = -p_{2y}$$

with $p_{qy} = 0$. If all fermion masses are neglected, the angles $\theta_1$ and $\theta_2$ ($0 \leq \theta_i \leq 2\pi$) of $l1$ and $l2$ with respect to the $z$-axis can be obtained from the particle energies using momentum conservation:

$$\sin \theta_1 E_1 = -\sin \theta_2 E_2$$

$$E_q = \cos \theta_1 E_1 + \cos \theta_2 E_2$$

yielding the relations

$$\sin \theta_2 = -\sin \frac{\theta_1}{E_2}$$

$$\cos \theta_2 = \frac{E_q - \cos \theta_1 E_1}{E_2}$$

From this

$$1 = \sin^2 \theta_2 + \cos^2 \theta_2 = \sin^2 \theta_1 + \frac{E_q^2 - 2\cos \theta_1 E_q E_1 + \cos^2 \theta_1 E_1^2}{E_2^2}$$

$$= \frac{1}{E_2^2} (E_q^2 + E_1^2 - 2\cos \theta_1 E_q E_1)$$

and

$$\cos \theta_1 = \frac{E_q^2 + E_1^2 - E_2^2}{2E_q E_1} \quad (68)$$

Moreover

$$\cos \theta_2 = \frac{1}{E_2} (E_q - \frac{E_q^2 + E_1^2 - E_2^2}{2E_q E_1} E_1) = \frac{2E_q^2 - E_q^2 - E_1^2 + E_2^2}{2E_q E_2}$$

which gives

$$\cos \theta_2 = \frac{E_q^2 + E_1^2 - E_2^2}{2E_q E_2} \quad (69)$$

To be physical, they should satisfy $|\cos \theta_i| \leq 1$. Taking the example of $E_2$,

$$E_2^2 = E_q^2 + E_1^2 - 2E_q E_1 \cos \theta_1$$

is maximal for $\cos \theta_1 = -1$, which implies $\cos \theta_2 = +1$, i.e. a configuration where the particle for which energy is maximized goes opposite to all the other particles in the final state (parallel) and the neutralino is at rest. It also has a minimum, obtained for $\cos \theta_1 = +1$, with the two leptons parallel to each other and opposite to the quark. Thus

$$(E_q - E_1)^2 \leq E_2^2 \leq (E_q + E_1)^2$$
and similarly for \( \cos \theta_2 \) by replacing \( E_1 \leftrightarrow E_2 \). Together, they simply express the fact that by momentum conservation the three momentum vectors form a triangle, i.e.

\[
E_q \leq E_1 + E_2, \quad E_1 \leq E_q + E_2, \quad E_2 \leq E_q + E_1
\]

Using the energy conservation equation, they can be rewritten as

\[
M \geq 2E_q, \quad M \geq 2E_1, \quad M \geq 2E_2
\]

which have to be simultaneously satisfied. These relations are completely general for a 3-body final state of massless particles. Note that, from the \( \cos \theta \) dependences above, it is seen that the equalities in these relations, which determine the boundaries on the energies, each correspond to a specific collinear configuration.

The condition (70) can also be derived in a simpler (but less explicit) way. The absolute maximum effective mass of a system of particles is obtained when its constituents are sent as two back-to-back jets (say \( a \) and \( b \)) in the rest frame of the parent \( A \) and when the remaining particles (say \( c \)) are at rest. Then the system \((a, b)\) is also at rest and energy-momentum conservation can be expressed as

\[
M_A = M_c + M_{ab}, \quad \vec{p}_a = -\vec{p}_b
\]

then also

\[
M_{ab}^2 = 2(E_aE_b + p_a p_b) = 2(E_aE_b + p_a^2)
\]

If \( a \) and/or \( b \) consists of several (assumed massless) particles, we may have \( M_a \neq 0 \) and/or \( M_b \neq 0 \) in a non-collinear configuration. Take \( M_a = 0 \) and \( M_b \neq 0 \), then

\[
M_{ab}^2 = 2E_a(E_b + p_b) \geq 2E_a(2p_b) = 4E_a p_a = 4E_a^2
\]

from which

\[
M_{ab} \geq 2E_a \quad \text{or} \quad M_A - M_c \geq 2E_a
\]

where the equal sign is obtained for \( M_b = 0 \), i.e. a single particle or a system of parallel massless particles. As "\( a \)" may be any of the particles included in the calculation of the mass, the same condition applies to all of them, hence (70) is reproduced.

As there are 5 unknown kinematical quantities and only 3 constraints, the problem remains under-constrained at this stage. For the above \((llq)\) final state resulting from sequential 2-body decays and with the \( \chi_1^0 \) at rest in the \( Q = \bar{q} \) frame

\[
M = M_Q - M_0
\]

Moreover, the energy of the quark is fixed by (54) and the mass constraint from the decay \( \bar{l} \rightarrow l2 + \chi_1^0 \) gives in the rest frame of \( Q = \bar{q} \)

\[
M_R^2 = (E_2 + E_0)^2 - (\vec{p}_2 + \vec{p}_0)^2 = (E_2 + M_0)^2 - E_2^2 = M_0^2 + 2M_0E_2
\]

from which

\[
E_2 = \frac{M_R^2 - M_0^2}{2M_0}
\]

Note that this derivation is independent of the reference frame and is, therefore, valid in any frame where the \( \chi_1^0 \) is at rest. Note also that this energy is not the same as the one obtained by Lorentz transforming the energy of \( l2 \) from the \( R = \bar{l} \) rest frame to the squark rest frame in the configuration labelled \((q3)\), which would be \( E_2 = \frac{M_Q^2 - M_0^2}{2M_R} \). The difference is due to the fact that for the latter energy, no requirement on the \( \chi_1^0 \) at rest is made. However, both expressions for the energy lead to the same condition as below and hence either of them can be used.

With these two additional relations, the kinematics is now fully determined. But to obtain a physical solution, the masses involved have still to satisfy the above inequalities. The inequality for \( E_q \) yields

\[
M_Q - M_0 \geq 2 \frac{M_0^2 - M_R^2}{2M_Q}
\]
from which $M_X^2 \geq M_Q M_0$. From the inequality for $E_2$ it follows that $M_R^2 \leq M_Q M_0$. Together, to allow a $\tilde{\chi}_1^0$ to be put at rest in the $Q = \bar{q}$ frame, the sparticle masses have to satisfy the conditions

$$\frac{M_R^2}{M_0} \leq M_Q \leq \frac{M_X^2}{M_0}$$

with, by definition $M_Q \geq M_X$. If the left-hand inequality is not fulfilled, the $\tilde{\chi}_1^0$ has a negative momentum along the $z$-axis ($E_q$ too large); if the right-hand one is not, the $\tilde{\chi}_1^0$ has a positive momentum along the $z$-axis ($E_2$ too large). The upper edge of this interval corresponds to the collinear configuration labelled ($q_1$), the lower edge to the one labelled ($q_3$).

The inequality on $E_{l1}$ can be derived in a Lorentz invariant way from

$$M_X^2 = (E_1 + E_2 + E_0)^2 - (\vec{p}_1 + \vec{p}_2 + \vec{p}_0)^2$$

$$= (E_1 + E_2 + M_0)^2 - (E_1^2 + E_2^2 + 2E_1E_2\cos\theta_{12})$$

$$= M_0^2 + 2M_0 E_2 + (2M_0 + 4E_2)E_1$$

where we have used $\cos\theta_{12} = -1$ as for the configuration labelled ($q_2$). Therefore, introducing the above expression (72) for $E_2$

$$E_1 = \frac{M_X^2 - M_0^2 - 2M_0 E_2}{2M_0 + 4E_2}$$

$$= \frac{M_X^2 - M_0^2 - M_R^2 + M_R^2}{2M_0 + 2\frac{M_X^2 - M_0^2}{M_0}}$$

from which

$$E_1 = \frac{M_R M_X^2 - M_R^2}{2M_R}$$

Finally, the inequality on $E_{l1}$ yields

$$M_Q - M_0 \geq 2\frac{M_R M_X^2 - M_R^2}{2M_R}$$

from which

$$M_Q \geq \frac{M_X^2 M_0}{M_R^2}$$

Alternatively, Lorentz transforming from the $X = \tilde{\chi}_1^0$ rest frame leads to

$$M_Q - M_0 \geq 2\frac{M_Q M_X^2 - M_R^2}{2M_X}$$

from which the same inequality is derived as above. This will determine the lower edge of the non-collinear region if $M_R^2 \leq M_X M_0$. This edge corresponds to the collinear configuration labelled ($q_2$). It is seen that the lower edge of the interval (75) is always below the upper edge of interval (73) but can be either below or above the lower edge of interval (73). The two intervals will, therefore, always overlap. As all three energy inequalities have to be fulfilled, the final range where the non-collinear configuration is allowed with a $\tilde{\chi}_1^0$ at rest will be the logical AND of these two ranges.

### 4.4.2 First collinear end point in $M(llq)$

An upper end point can be obtained for a collinear configuration where the $\tilde{\chi}_1^0$ is emitted opposite to the Lorentz boost, as shown for the configuration labelled ($q_1$) in section 4.1. In this configuration the di-lepton system forms a zero mass object recoiling in the $R = \bar{l}$ rest frame against the $\tilde{\chi}_1^0$.

The maximum effective mass is obtained from

$$(M_{llq}^{max})^2 = (E_{llq})^2 - (p_{llq})^2 = 2E_q E_{ll} - 2\vec{p}_q \vec{p}_{ll}$$

$$= 4E_q E_{ll}$$

(76)
where the energies are expressed in the $\tilde q$ rest frame. The signs of the momenta follow equation (16). The quark energy is given in (54). The lepton energies are, using the simple Lorentz transformation formulae (14),

$$E_l = \frac{M_Q M^2_X - M^2_l}{2M_X}$$

from which the maximum effective mass is

$$(M_{llq}^{max})^2 = 4 \frac{M_Q^2 - M_X^2}{2M_Q} \frac{M_Q}{2M_X} (M_X^2 - M^2_l)$$

or

$$M_{llq}^{max} = M_Q \sqrt{(1 - \frac{M_X^2}{M^2_Q})(1 - \frac{M^2_0}{M^2_X})}$$

The largest value of $M_{llq}^{max}$ is reached when the $\tilde\chi^0_1$ is at rest in the $Q = \tilde q$ rest frame. This requires the velocity $\beta_X$ of the system $X = \tilde\chi^0_2$ in the squark rest frame to be equal to the velocity $\beta^*_0$ of the $\tilde\chi^0_1$ in the rest frame of $X$. Now

$$\beta_X = \frac{M_Q^2 - M_X^2}{M_Q^2 + M_X^2}$$

and $\beta^*_0$ for the configuration labelled $(q1)$ corresponds to two parallel leptons, hence $M(ll) = 0$ so that

$$\beta^*_0 = \frac{M_X^2 - M_0^2}{M_X^2 + M_0^2}$$

The equality between the two velocities requires that

$$M_X^2 = M_Q M_0 \text{ or } \frac{M_X}{M_Q} = \frac{M_0}{M_X}$$

and in these conditions, the end point takes the value $M_{llq}^{max} = M_Q - M_0$, i.e. the absolute maximum of equation (67). Now, when

$$M_Q \geq \frac{M_X^2}{M_0}$$

the $\tilde\chi^0_1$ in the squark rest frame will have non-zero momentum in the direction of the $X = \tilde\chi^0_2$ and the end point given by formula (78) may be the true end point. If, however, $M_X^2 > M_Q M_0$ the $\tilde\chi^0_1$ in the squark rest frame will again have non-zero momentum but in the direction opposite to the $X = \tilde\chi^0_2$. In this case the configuration labelled $(q1)$ no longer corresponds to the true maximum. A configuration like the one shown in Figure 11 with a suitable angle between the leptons would allow to put the $\tilde\chi^0_1$ at rest in the squark rest frame. It was indeed found in section 4.4.1 that in the range where $M_X^2 \geq M_Q M_0$ formula (67) may apply (provided another condition is also satisfied).

The two cases can be distinguished without the knowledge of the sparticle masses involved by making a plot of $M(llq)$ versus $M(ll)$. If the maximum of $M(llq)$ is reached for $M(ll) \approx 0$, equation (78) should apply. If it is reached for $M(ll) > 0$, the maximum may correspond to equation (67) or another collinear end point derived below.

Note that in both cases, the result is independent of the mass of the $R = \tilde l$. It remains, therefore, also valid for a direct 3-body decay of $X = \tilde\chi^0_2$.

**4.4.3 Second collinear end point in $M(llq)$**

A second collinear maximum is obtained by flipping the direction of the slepton in the $\tilde\chi^0_2$ decay, the configuration labelled $(q3)$ in section 4.1. This leads to

$$(M_{llq}^{max,2})^2 = 2(E_1 E_{11} + E_2 E_{12} + E_{11} E_{12}) - 2( + E_1 E_{11} - E_2 E_{12} - E_{11} E_{12})$$

$$= 4E_{12}(E_q + E_{11})$$

$$= 4E_{12}(E_q + 0)$$

(81)
with
\[ E_{11} = \frac{M_X M_X^2 - M_R^2}{2M_X} \] (82)
and
\[ E_{12} = \frac{M_Q M_X M_R^2 - M_0^2}{M_X M_R^2} = \frac{M_Q M_R^2 - M_0^2}{2M_R} \] (83)
and this maximum is
\[ (M_{llq}^{max,2})^2 = \frac{M_Q M_R^2 - M_0^2}{2M_R} \left( \frac{M_Q^2 - M_X^2}{2M_Q} + \frac{M_X M_R^2 - M_R^2}{2M_X} \right) \]

or
\[ M_{llq}^{max} = M_Q \sqrt{\left( 1 - \frac{M_R^2}{M_Q^2} \right) \left( 1 - \frac{M_0^2}{M_R^2} \right)} \] (84)

which is the second formula presented in the Cambridge paper. In this case the largest value is reached for \( M_R = M_Q M_0 \) and leads again to \( M_{llq}^{max,2} = M_Q - M_0 \). This end point is, however, not independent of the previous ones, as it can be easily verified that
\[ (M_{llq}^{max,2})^2 = (M_{llq}^{max})^2 + (M_{llq}^{max})^2 \] (85)

For this maximum to be larger than \( M_{llq}^{max} \), equation (78), the condition is
\[ M_X^2 (M_R^2 M_0^2 - M_R^2 M_Q^2) > M_R^2 (M_Q^2 M_R^2 - M_Q^2 M_0^2 - M_X^2 + M_R^2 M_0^2) \]
\[ M_R^2 (M_Q^2 M_R^2 + M_R^2 M_0^2) < M_Q^2 M_R^2 M_0^2 + M_X^2 M_R^2 \]
\[ M_Q^2 M_R^2 (M_R^2 - M_R^2) + M_X^2 M_R^2 (M_R^2 - M_R^2) < 0 \]
or
\[ M_Q \leq \frac{M_X M_R}{M_0} \] (86)

As, by definition, \( M_R < M_X \) this second end point is larger than the one of (78) in the region where \( M_Q M_0 < M_X \), i.e. when \( M_{llq}^{max} \) does not give the true end point, which is instead determined by the absolute maximum. When the absolute maximum can no longer be reached, in the range where \( M_Q M_0 \leq M_R^2 \), the condition (86) is automatically fulfilled. In this region of masses, the secondary end point of (84) may give the true maximum. However, the range (86) may be below the lower edge (75) of the non-collinear region. In this case \( M_{llq}^{max} \) may still be larger than \( M_{llq}^{max,2} \), but as we will see next, this is a region where \( M_{llq}^{max,3} \) dominates.

As the maximum (84) corresponds to a configuration where the \( M(ll) \) is maximal, it could be identified by selecting large \( (ll) \) masses.

4.4.4 Third collinear end point in \( M(llq) \)

However, rather than flipping the direction of the lepton from the \( \tilde{\chi}_R^0 \) decay, we can flip the direction of the \( l \) decay products, a configuration like the one labelled \((q2)\) in section 4.1. This would also lead to a configuration where \( M(llq) \) is possibly maximum. In this case
\[ (M_{llq}^{max,3})^2 = 2(E_q E_{l1} + E_q E_{l2} + E_{l1} E_{l2}) - 2(-E_q E_{l1} + E_q E_{l2} - E_{l1} E_{l2}) \]
\[ = 4E_{l1}(E_q + E_{l2}) \] (87)

As they correspond to the same kinematical configuration, we have
\[ (M_{llq}^{max,3})^2 = (M_{llq}^{max})^2 + (M_{llq}^{max})^2 \] (88)
and is therefore not independent of the previous end points. The end point of $M_{llq}^{max,3}$ could be obtained from this sum, but it is as easy to compute it directly.

$$E_{11} = M_Q \frac{M_X^2 - M_R^2}{2M_X}$$

$$E_{12} = \frac{M_X M_R (M_R^2 - M_0^2)}{2M_R}$$

so that the effective mass is

$$(M_{llq}^{max,3})^2 = 4 \frac{M_Q M_X^2 - M_R^2}{M_X} \left( \frac{M_Q^2 - M_X^2}{2M_Q} + \frac{M_X M_R (M_R^2 - M_0^2)}{2M_R} \right)$$

$$= 4 \frac{M_Q M_X^2 - M_R^2}{M_X} \frac{1}{2M_Q} (M_Q^2 - M_X^2) M_R^2 M_0^2$$

$$= \frac{M_X^2 - M_R^2}{M_X M_R} (M_Q^2 M_R^2 - M_X^2 M_0^2) = (M_X^2 - M_R^2) \left( \frac{M_Q^2}{M_X} \frac{M_R^2}{M_X} \right)$$

or

$$M_{llq}^{max,3} = M_Q \sqrt{(1 - \frac{M_R^2}{M_X})(1 - \frac{(M_X M_0)^2}{(M_Q M_R)^2})}$$

(90)

which is the third formula presented in the Cambridge paper. To find the largest value reachable for this end point, the formula can be rewritten in the standard form of Section 2.6.

This maximum could be larger than $M_{llq}^{max}$ provided

$$(M_X^2 - M_R^2)(M_Q^2 M_R^2 - M_X^2 M_0^2) \geq M_R^2 (M_Q^2 - M_X^2) (M_X^2 - M_0^2)$$

$$M_Q^2 M_R^2 M_0^2 = M_X^2 M_0^2 + M_R^2 M_0^2 - M_X^2 M_R^2 \geq M_X^2 M_R^2 M_0^2 - M_Z^2 M_R^2 M_0^2 - M_X^2 M_R^2 M_0^2$$

$$M_Q^2 M_R^2 (M_R^2 - M_0^2) + M_X^2 (M_0^2 - M_R^2) \leq 0$$

and thus

$$M_Q \leq \frac{M_X^2}{M_R}$$

(91)

As $M_X^2 / M_R$ is larger than $M_X^2 / M_0$, it cannot be larger than $M_{llq}^{max}$, formula (78), in the region $M_Q \geq M_X^2 / M_0$ where the latter applies. It may, however, be larger than $M_{llq}^{max}$ in the range $M_Q \leq M_X^2 / M_0$, provided this interval is not empty.

On the other hand, for the secondary maximum $M_{llq}^{max,3}$ to be larger than $M_{llq}^{max,2}$ would require

$$\frac{M_X^2 - M_R^2}{M_X M_R^2} (M_Q^2 M_R^2 - M_X^2 M_0^2) \geq \left( \frac{M_Q^2 - M_X^2}{M_R^2} \right) (M_R^2 - M_0^2)$$

$$M_Q^2 M_R^2 M_0^2 = M_X^2 M_0^2 + M_R^2 M_0^2 - M_X^2 M_R^2 \geq M_X^2 M_R^2 M_0^2 - M_Z^2 M_R^2 M_0^2 - M_X^2 M_R^2 M_0^2$$

$$M_Q^2 (M_R^2 - M_0^2) + M_X^2 (M_0^2 - M_R^2) \leq 0$$

and as $M_Q > M_X$, the condition for $M_{llq}^{max,3} \geq M_{llq}^{max,2}$ is

$$M_R^2 \leq M_X M_0$$

(92)

which is the same as the condition ensuring that $M_{llq}^{max} \leq M_{llq}^{max}$.

Hence the conditions for the true end point outside the non-collinear range can be classified according to the relations
• \( \frac{M_X}{M_Q} \leq \frac{M_X}{M_R} \frac{M_R}{M_Q} \) for which the true end point is given by \( M_{llq}^{\text{max.}1} \).

• \( M_X^2 \geq M_X M_0 \) for which the true end point is given by \( M_{llq}^{\text{max.}2} \) over the whole range \( M_X \leq M_Q \leq \frac{M_R^2}{M_R} \) or \( \frac{M_N}{M_R} \leq \frac{M_X}{M_Q} \).

• \( M_X^2 \leq M_X M_0 \) for which the true end point is given by \( M_{llq}^{\text{max.}3} \) over the whole range \( M_X \leq M_Q \leq \frac{M_R^2}{M_R} \) or \( \frac{M_N}{M_R} \leq \frac{M_X}{M_Q} \).

Also this end point could in principle be distinguished from (78) by selecting large values of \( M(ll) \). However to distinguish \( M_{llq}^{\text{max.}2} \) from \( M_{llq}^{\text{max.}3} \), it is necessary to compare them to \( M(llq) \). If it corresponds to a maximum in \((llq)\), the formula (90) should be used; if it corresponds to a maximum in \((l2q)\), formula (84) is the correct one.

4.4.5 Summary of the upper end points in \( M(llq) \)

To determine the relation between the upper end point in \( M(llq) \) and the sparticle masses several mass regions need to be considered. They are illustrated in Figure 12 which shows \((M_{llq}^{\text{max.}})^2\) as straight lines versus \( M_R^2 \).

The true end point is determined by collinear configurations in the following regions:

• if \( \frac{M_X}{M_Q} \leq \frac{M_X}{M_R} \frac{M_R}{M_Q} \), here the true end point is given by \( M_{llq}^{\text{max.}1} \) (78). It corresponds to the collinear configuration \((q1)\) of section 4.1 and is characterized by small values of \( M(ll) \).

• \( \frac{M_N}{M_R} \leq \frac{M_X}{M_Q} \frac{M_X}{M_Q} \): in this range, the true end point is given by \( M_{llq}^{\text{max.}2} \) (84) and corresponds to the collinear configuration \((q2)\) of section 4.1 which is characterized by near maximum values of \( M(ll) \) and of \( M(llq) \). In this case \( M(l2q) \) gives the true end point of \( M(llq) \).

• \( \frac{M_N}{M_X} \leq \frac{M_N}{M_Q} \frac{M_X}{M_Q} \): in this range, the true end point is given by \( M_{llq}^{\text{max.}3} \) (90). It corresponds to the collinear configuration \((q3)\) of section 4.1 and is characterized by large values of \( M(ll) \) and of \( M(llq) \). In this case \( M(llq) \) gives the true end point of \( M(llq) \).

Else, the true end point is found in a non-collinear configuration:

• \( \max(M_X, \frac{M_0^2}{M_R}, \frac{M_0^2}{M_R}) \leq M_Q \leq \frac{M_0^2}{M_R} \). in this region the upper end point is given by formula (67) and the configuration is the non-collinear one of Figure 11. It is characterized by a clustering around an intermediate value of \( M(ll) \). Its lower edge is \( M_Q = \frac{M_0^2}{M_R} \) if \( M_R^2 \geq M_X M_0 \) and is \( M_Q = \frac{M_0^2}{M_R} \) if \( M_R^2 \leq M_X M_0 \).

Figure 12: Example of the mass regions leading to maxima for \( M(llq) \).
Hence, the correlation between \( M(llq) \) near its upper edge and \( M(ll) \), \( M(llq) \) and \( M(llq) \) allows the type of end point (and the formula to be used) to be identified in a model independent way.

Note that not all combinations of end point solutions for \((ll)\) and \((llq)\) can simultaneously be true end points. The regions of masses where a given end point dominates all the others are given above. The combinations which can simultaneously lead to true end points are either \( M(llq)_{\text{max},1} \), \( M(llq)_{\text{max},3} \) or a non-collinear configuration; or \( M(llq) \) with \( M(llq)_{\text{max},1} \), \( M(llq)_{\text{max},2} \) or a non-collinear configuration.

4.4.6 End point in \( M(llq) \), lower edge

The minimum value of \( M(llq) \) is zero in general. But it is possible to obtain a non-zero end point by selecting events where \( M(ll) \) is non-zero. For example, taking events with \( \cos\theta^* > 0 \) in the rest frame of \( R = \tilde{l} \), i.e. with \( M(ll) > M(ll)_{\text{max}}/\sqrt{2} \), a lower end point is produced in \( M(llq) \). This corresponds to the kinematical configuration depicted in Figure 13.

![Figure 13: Configuration leading to a minimum of the mass for \((llq)\).](image)

To compute this case, it is best to transform the \( \chi_1^0 \) 4-vector to the squark rest frame. In the rest frame of \( R = \tilde{l} \), the \( \chi_1^0 \) energy and momentum are

\[
E_0^{**} = \frac{M_R^2 + M_Q^2}{2M_R}, \quad p_0^{**} = \frac{M_R^2 - M_Q^2}{2M_R}
\]

(93)

Transforming to the rest frame of \( X = \chi_2^0 \) the configuration with \( \cos\theta^* = 0 \), i.e. with \( p_{0L}^* = 0 \) gives

\[
E_0^* = \gamma_R(E_0^{**} + 0) = \frac{M_X^2 + M_R^2}{2M_X M_R} \frac{M_R^2 + M_Q^2}{2M_R}
\]

\[
p_{0L}^* = \gamma_R(\beta_R E_0^{**} + 0) = \frac{M_X^2 - M_R^2}{2M_X M_R} \frac{M_R^2 + M_Q^2}{2M_R}
\]

\[p_0^T = p_{0T}^* = \frac{M_R^2 - M_Q^2}{2M_R} \]

\[
p_0^* = \frac{1}{2M_R} \frac{1}{2M_X M_R} \sqrt{(M_X^2 - M_R^2)^2(M_R^2 + M_Q^2)^2 + 4M_X^2 M_R^2 (M_R^2 - M_Q^2)^2}
\]

(94)

Now the kinematical configuration leading to the minimal \((llq)\) mass corresponds to a \( \chi_1^0 \) emitted along the direction of the \( \chi_2^0 \) (to maximize the boost), as shown in Figure 14. The Lorentz transformation of the \( \chi_1^0 \) to the squark rest frame gives:

\[
E_0 = \gamma_X (E_0^* + \beta_X p_0^* ) = \frac{1}{2M_R M_X} [(M_R^2 + M_X^2) E_0^* + (M_Q^2 - M_X^2) p_0^*]
\]

(95)
Noting that $E_{lq} = M_Q - E_0$ and $p_{llq} = p_0$, the effective mass is obtained
\[
M_{llq}^2 = E_{llq}^2 - p_{llq}^2 = M_Q^2 + M_0^2 - 2M_Q E_0
\]
\[
= M_Q^2 + M_0^2 - \frac{1}{M_X}[(M_Q^2 + M_0^2)E_0^* + (M_Q - M_0)p_0^*]
\]
or
\[
M_{llq}^2 = M_Q^2 + M_0^2 - \frac{1}{4M^2 X M_R^2} [(M_Q^2 + M_0^2)(M_Q^2 + M_R^2)(M_R^2 + M_0^2)
\]
\[
+ (M_R^2 - M_X^2) \sqrt{(M_R^2 - M_X^2)^2 (M_R^2 + M_0^2)^2 + 4M^2 X M_R^2 (M_R^2 - M_0^2)^2} ]
\] (96)

This lower end point was also used by [2] for the determination of the sparticle masses.

## 4.5 End point for the sum $M(l1q) + M(l2q)$

It has been first proposed in [13] to also use the end point in the sum $M(l1q) + M(l2q)$ to constrain the sparticle masses. From the comparison of the two configurations it is obvious that the maximum of $M(l1q) + M(l2q)$ cannot correspond to the sum of the two maxima. Then, from inspecting figure 11, it follows that the maximum of the sum is obtained when the $\tilde{\chi}^0_1$ has its minimal energy in the $Q = \tilde{q}$ rest frame and the two leptons are opposite to the quark. This is the configuration labelled $ql$ which maximizes $M(ql)$. The $M(l1q)$ reaches its maximum and the value of $M(l2q)$ for this configuration is the one we computed for the secondary $M(l2q)$ configuration in Section 4.3.3,

\[
M_{l2q} = \frac{M_R}{M_X} M_{l2q}^{max}
\]

where the factor $M_R/M_X$ is due to the flip of direction of $R = \tilde{l}$ compared to the configuration where $M(l2q)$ is maximal.

The maximum of the sum of masses is then
\[
(M_{l1q} + M_{l2q})^{max} = M_{l1q}^{max} + \frac{M_R}{M_X} M_{l2q}^{max}
\]
\[
= M_Q \sqrt{1 - \frac{M_X^2}{M_Q^2}} \left[ \sqrt{1 - \frac{M_R^2}{M_X^2}} + \frac{M_R}{M_X} \sqrt{1 - \frac{M_0^2}{M_R^2}} \right]
\] (97)

which can be used as an additional constraint. On the other hand, the above formula is misleading and the end point does not contain information on $M(l2q)$. This can be seen from
\[
\frac{M_R^2}{M_X^2} (1 - \frac{M_0^2}{M_R^2}) = \frac{M_R^2}{M_X^2} - \frac{M_0^2}{M_X^2} = (1 - \frac{M_0^2}{M_X^2}) - (1 - \frac{M_R^2}{M_X^2})
\]

and hence
\[
(M_{l1q} + M_{l2q})^{max} = M_{l1q}^{max} + \sqrt{(M_{l1q}^{max})^2 - (M_{l1q}^{max})^2}
\] (98)

Mathematically, the end point for the sum $M(l1q) + M(l2q)$ is of course related to the other end points as there are only 4 masses to determine. But experimentally, it can be used as an additional constraint on the masses which can be included in a least squares fit, for example.

Rather than using the sum of effective masses, it would equally be possible to use their product, which is maximized for the same configuration. Some incentive for applying this method is based on the observation that for this configuration equation (76) shows, using (44), that
\[
(M_{l1q}^{max})^2 = (M_{l1q}^2 + M_{l2q}^2)^{max}
\]

Hence the really new information in the sum of effective masses resides in the double product. As they belong to the same configuration,
\[
(M_{l1q} + M_{l2q})^{2,max} = (M_{l1q}^2 + M_{l2q}^2)^{max} + 2(M_{l1q} \cdot M_{l2q})^{max} = (M_{l1q}^{max})^2 + 2 (M_{l1q} \cdot M_{l2q})^{max}
\]
\[
= ((M_{l1q} + M_{l2q})^{max})^2 = (M_{l1q}^{max})^2 + (M_{l1q}^{max})^2 - (M_{l1q}^{max})^2 + 2M_{l1q}^{max} \sqrt{(M_{l1q}^{max})^2 - (M_{l1q}^{max})^2}
\]

Therefore, an alternative could be to use as constraint
\[
(M_{l1q} \cdot M_{l2q})^{max} = M_{l1q}^{max} \sqrt{(M_{l1q}^{max})^2 - (M_{l1q}^{max})^2}
\] (100)

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4.6 End points for $M(llq)$ as a function of $M(ll)$

As an alternative and a generalization of the search for a minimum $M(llq)$ of a section above, we can fix the value of $M(ll)$ and compute the lower and upper end points of the $M(llq)$ distribution. The relevant configurations leading to an upper and lower edge are shown in figure 15. In the $X = \chi^0_2$ rest frame, the

![Figure 15: Configuration to compute the extrema of the mass $M(llq)$ as a function of $M(ll)$](image)

\[ E_{0}^{*} = \frac{M_{2}^{2} + M_{0}^{2} - M_{ll}^{2}}{2M_{X}} \]

\[ p_{0}^{*} = \frac{\sqrt{(M_{2}^{2} - M_{0}^{2} - M_{ll}^{2})^{2} - 4M_{0}^{2}M_{ll}^{2}}}{2M_{X}} \]

The Lorentz transformation to the rest frame of $Q = \bar{q}$ is given by

\[ E_{0} = \gamma_{X}(E_{0}^{*} \pm \beta_{X}p_{0}^{*}) = \frac{1}{2M_{Q}M_{X}}[(M_{2}^{2} + M_{2}^{2})E_{0}^{*} \pm (M_{Q}^{2} - M_{X}^{2})p_{0}^{*}] \]

where the ± signs correspond to the configurations with the neutralino opposite or along the direction of the quark. The $M(llq)$ is then, following the calculation of equation (96)

\[
M_{llq}^{2} = E_{0}^{2} - p_{0}^{2} = (M_{Q} - E_{0}^{2})^{2} - p_{0}^{2} = M_{Q}^{2} + M_{0}^{2} - 2M_{Q}E_{0} \\
= M_{Q}^{2} + M_{0}^{2} - \frac{1}{2M_{X}} [(M_{Q}^{2} + M_{X}^{2})E_{0}^{*} \pm (M_{Q}^{2} - M_{X}^{2})p_{0}^{*}] \\
= M_{Q}^{2} + M_{0}^{2} - \frac{1}{2M_{X}} [(M_{Q}^{2} + M_{X}^{2})(M_{Q}^{2} - M_{Q}^{2}) \pm (M_{Q}^{2} - M_{X}^{2})\sqrt{(M_{Q}^{2} - M_{Q}^{2} - M_{Q}^{2})^{2} - 4M_{0}^{2}M_{ll}^{2}}] \\
= M_{Q}^{2} + M_{0}^{2} - \frac{1}{2M_{X}} [(M_{Q}^{2} + M_{X}^{2})(M_{X}^{2} + M_{0}^{2}) \pm \frac{1}{2M_{X}}M_{Q}^{2}M_{X}^{2}M_{ll}^{2} \pm \frac{1}{2M_{X}}M_{Q}^{2}M_{X}^{2}M_{ll}^{2} \pm \frac{1}{2M_{X}}M_{Q}^{2}M_{X}^{2}M_{ll}^{2} \pm \frac{1}{2M_{X}}M_{Q}^{2}M_{X}^{2}M_{ll}^{2}]
\]

or

\[
M_{llq}^{2} = M_{Q}^{2} + M_{0}^{2} - \frac{1}{2M_{X}^{2}}(1 + \frac{M_{X}^{2}}{M_{Q}^{2}})(1 + \frac{M_{0}^{2}}{M_{X}^{2}}) \pm \frac{M_{Q}^{2}}{2M_{X}^{2}}M_{ll}^{2} \pm \frac{M_{Q}^{2}}{2M_{X}^{2}}M_{ll}^{2} \pm \frac{M_{Q}^{2}}{2M_{X}^{2}}M_{ll}^{2} \pm \frac{M_{Q}^{2}}{2M_{X}^{2}}M_{ll}^{2} \pm \frac{M_{Q}^{2}}{2M_{X}^{2}}M_{ll}^{2}
\]

(103)

It shows that by varying the interval in $M_{ll}$ different constraints are obtained on the masses, some of which have already been computed explicitly above. Note that this expression is again independent of the mass $M_{R}$ of the slepton. The variation of $M(llq)$ as a function of $M_{ll}$ is illustrated in figure 16 for benchmark point B’ ($m_{Q} = 558$, $m_{\chi} = 180$, $m_{\chi} = 117$, $m_{\chi} = 95$ GeV). For this example, the true maximum lies at $M_{ll} = 0$ and corresponds to the configuration leading to formula (78).

As a check, let us compute the end points for the case $M(ll) = 0$. The $M(llq)$ formula takes then the simpler form

\[
M_{llq}^{2} = M_{Q}^{2} + M_{0}^{2} - \frac{1}{2M_{X}^{2}}[(M_{Q}^{2} + M_{X}^{2})(M_{X}^{2} + M_{0}^{2}) \pm (M_{Q}^{2} - M_{X}^{2})(M_{X}^{2} - M_{0}^{2})]
\]

(104)
from which the upper and lower end points should be obtained taking the − and + sign respectively.

\[(M_{llq}^{\text{max}})^2 = M_Q^2 + M_0^2 - \frac{1}{M_X^2}(M_Q^2 M_0^2 + M_X^2) = \frac{1}{M_X^2}(M_Q^2 - M_X^2)(M_X^2 - M_0^2)\]

and

\[(M_{llq}^{\text{min}})^2 = M_Q^2 + M_0^2 - \frac{1}{M_X^2}(M_Q^2 M_X^2 + M_X^2 M_0^2) = 0\]

which reproduces the values of the $M(\ell \ell q)$ upper end point (78) obtained above. It is also possible, after some simple but tedious algebra, to derive from the above general formula (103) the $M(\ell \ell q)$ end points corresponding the $M(\ell \ell)^{\text{max}}$ given in (84) and (90). However, these formulae hold only when $M(\ell \ell)$ is strictly at its maximum. In practice, a band near the maximum $M(\ell \ell)$ has to be selected. Then, as seen from the example in Figure 16, the variation of $M(\ell \ell q)$, especially its lower edge, may be rather fast as a function of $M(\ell \ell)$, which makes the determination of the edge difficult.

### 4.7 Extraction of the sparticle masses from end points

In the decay of a squark considered above, there are 4 unknown masses: the $\tilde{q}$, $\tilde{\chi}_1^0$, $\tilde{l}$ and $\tilde{\chi}_1^0$. Their values can be extracted from the measurement of the mass upper end points for $(\ell \ell)$, $(\ell 1q)$, $(\ell 2q)$ and $(\ell lq)$ of collinear configurations listed in Section 4.1. To simplify the notation, let us define

\[ Q_{21} \equiv \left( \frac{M_{12q}^{\text{max}}}{M_{14q}^{\text{max}}} \right)^2, \quad Q_{q1} \equiv \left( \frac{M_{llq}^{\text{max}}}{M_{llq}^{\text{max}}} \right)^2 - 1, \quad Q_{ll} \equiv (M_{llq}^{\text{max}})^2 \]

chosen such that the dependence on the squark mass disappears. Expressed in terms of the remaining sparticles masses, they are:

\[ Q_{21} = \frac{M_X^2 M_R^2 - M_0^4}{M_X^2 M_R^2 - M_R^4}, \quad Q_{q1} = \frac{M_0^2 - M_X^2}{M_X^2 - M_R^2}, \quad Q_{ll} = \frac{1}{M_R^2}(M_X^2 - M_R^2)(M_R^2 - M_0^2) \]

The ratio $Q_{q1}$, obtained here from the $llq$ and $llq$ end points, can also be derived from the parallel dileptons energy ratio $E_{ll}/E_{ll}$ which is given by the same mass relation. From the first two equations we get

\[ Q_{21}/Q_{q1} = M_X^2/M_R^2 \]

which, injected into the ratios $Q_{ll}/Q_{q1}$ and $Q_{ll}/Q_{21}$ yields

\[ M_X^2 = \frac{Q_{ll}Q_{q1}}{(Q_{21} - Q_{q1})^2}, \quad M_R^2 = \frac{Q_{ll}Q_{21}}{(Q_{21} - Q_{q1})^2} \]
from which, taking $Q_{21} - Q_{q1}$ leads to

$$M_0^2 = M_{12}^2(1 - Q_{21} + Q_{q1})$$  \hfill (108)

Finally we can solve for the squark mass. This requires the introduction of a 4th variable, which we can choose as the ratio

$$Q_{1U} = \left( \frac{M_{1U}^{max}}{M_{1U}} \right)^2$$  \hfill (109)

It is given as a function of the sparticle masses by

$$Q_{1U} = \frac{M_R^2 M_Q^2 - M_X^2}{M_R^2 M_Q^2 - M_0^2}$$  \hfill (110)

Solving for $M_Q$ leads to

$$M_Q^2 = M_X^2 \left[ Q_{1U} \frac{M_R^2 - M_0^2}{M_R^2 M_Q^2 - M_0^2} + 1 \right]$$  \hfill (111)

As the end point for $(l1q)$ is not always easy to distinguish from $(l2q)$, e.g. when it is larger for $(l2q)$, an alternative way of computing $M_Q$ may be useful. An alternative choice for the 4th variable could be

$$Q_{q1} = \left( \frac{M_{q1}^{max}}{M_{q1}} \right)^2$$  \hfill (112)

which is given in terms of the sparticle masses by

$$Q_{q1} = \frac{M_R^2 M_Q^2 - M_X^2}{M_R^2 M_Q^2 - M_0^2}$$  \hfill (113)

and

$$M_Q^2 = M_X^2 \left[ Q_{q1} \frac{M_R^2 - M_0^2}{M_R^2 M_X^2 - M_0^2} + 1 \right]$$  \hfill (114)

These explicit solutions for the sparticle masses in terms of the end points show that these 4 end points are independent and suffice to determine all masses. They give a proof of principle.

If the mass relations are such that the $(l1q)$ end point is given by $M_{l1q}^{max,2}$ or $M_{l1q}^{max,3}$, this end point is not independent of the other ones and does not allow to determine the masses. In this case, it is possible to use the end point of $(M_{l1q} + M_{l2q})^{max}$ to extract $M_{l1q}^{max}$ and use the above formulation.

Finally, there could be a problem if all end points would simultaneously give only mass differences (which is sometimes claimed), i.e. would all reach their largest value for the same relation between the sparticle masses. It is easily verified that all the conditions required to have this situation can only be fulfilled if $M_Q = M_X = M_R = M_0$. On the other hand, it is possible for specific ratios between the sparticle masses that two of the end points become degenerate. For example, for $M_R^2 = M_X M_0$ the formulae of $M_{l1q}^{max}$ and $M_{l2q}^{max}$ become identical. But even so, the three remaining end points plus the mass relation are enough to determine all masses.

In practice, these analytical solutions will probably not be used, as some end points may not be well determined and there exists additional information which would be neglected. A least squares fit to all the information would be preferable and should lead to the most accurate determination of the masses.

4.8 Crude estimate of uncertainties

The uncertainties expected for the end points varies from case to case. The most accurate will certainly be the $(ll)$ end point, due to the good accuracy of the lepton energy measurement and the sharp edge of the mass distribution. The other end points involving quarks will be less precise due to the larger uncertainty on the jet energy and the shape of the mass distribution with a less sharp edge. The least
precise will be the lower end point $M_{ll}^{\min}$ which may be partly blurred by events with underestimated jet energy, e.g. due to neutrinos. Therefore, the latter was not used for estimating the masses above.

It was seen in Section 4.7 that the masses of $\tilde{\chi}_2^0$, $\tilde{\ell}$ and $\tilde{\chi}_1^0$ are determined from the ($ll$) end point and the two ratios $Q_{2l}$ and $Q_{q1}$. As the ratios may involve the same jet event by event in the numerator and denominator, the uncertainties on the jet energy may largely cancel. Hence, these particle masses can be expected to be determined with a good precision. However, the ratio $Q_{1l}$ or $Q_{ql}$ will directly reflect the uncertainties on the jet energy and the mass of the $\tilde{q}$ will thus be less precisely determined.

5 Four-step sequential 2-body decays starting from a $\tilde{g}$

In this case, the considered decay chain is

$$\tilde{g} \to q1\tilde{q}, \quad \tilde{q} \to q2\tilde{\chi}_2^0, \quad \tilde{\chi}_2^0 \to \tilde{l}1 + \tilde{l}, \quad \tilde{l} \to l2 + \tilde{\chi}_1^0$$  \hspace{1cm} (115)

The end point formulae for the decays starting from the squark have been presented previously and remain applicable after replacing $q$ by $q2$. Here, we will compute the end points for the mass combinations involving $q1$, namely $(q1q2)$, $(q1l1)$, $(q1l2)$, $(q1l1l2)$, $(q1q2l1)$, $(q1q2l2)$ and $(q1q2l1l2)$.

5.1 Overview of the collinear configurations

We start again with an overview of all collinear configurations available for this decay sequence, except the one where all quarks and leptons are parallel, as this leads to all mass combinations being zero. Also the end points which may give rise to the true end point are listed as $M^{\max,i}_{ll}$. The numbering of the configurations is defined according to the values of $M_{ll}$ and $M_{qq}$. The end point formulae and their condition of applicability are derived in the next sections.
(c1)

\[ X = \chi_0^2 \]
\[ Q = q \sim G = g \sim \]
\[ q_1 \quad q_2 \]
\[ R = l \sim l_1 \quad \chi_1 \quad l_2 \quad \]
\[ M_{\text{min}} \quad M_{\text{max}} \quad M_{l_1q_2}^{\text{min}} \quad M_{l_2q_2}^{\text{max}} \quad M_{l_1q_2}^{\text{min}} \quad M_{l_1q_1}^{\text{max}} \quad M_{l_2q_1}^{\text{max}} \quad M_{l_1qq}^{\text{max}} \quad M_{l_2qq}^{\text{max}} \quad M_{l_1qq}^{\text{max}}; \quad l_1 \quad q_2 \]

\[ M_{l_1q_2}^{\text{min}} \quad M_{l_2q_2}^{\text{max}} \quad M_{l_1q_1}^{\text{max}} \quad M_{l_2q_1}^{\text{max}} \quad M_{l_1qq}^{\text{max}}; \quad q_1 \quad q_2 \]

(d1)

\[ X = \chi_0^2 \]
\[ Q = q \sim G = g \sim \]
\[ q_1 \quad q_2 \]
\[ R = l \sim l_1 \quad \chi_1 \quad l_2 \quad \]
\[ M_{\text{min}} \quad M_{\text{max}} \quad M_{l_1q_2}^{\text{min}} \quad M_{l_2q_2}^{\text{max}} \quad M_{l_1q_2}^{\text{min}} \quad M_{l_1q_1}^{\text{max}} \quad M_{l_2q_1}^{\text{max}} \quad M_{l_1qq}^{\text{max}} \quad M_{l_2qq}^{\text{max}} \quad M_{l_1qq}^{\text{max}}; \quad l_1 \quad q_2 \]

(d2)

\[ X = \chi_0^2 \]
\[ Q = q \sim G = g \sim \]
\[ q_1 \quad q_2 \]
\[ R = l \sim l_1 \quad \chi_1 \quad l_2 \quad \]
\[ M_{\text{min}} \quad M_{\text{max}} \quad M_{l_1q_2}^{\text{min}} \quad M_{l_2q_2}^{\text{max}} \quad M_{l_1q_2}^{\text{min}} \quad M_{l_1q_1}^{\text{max}} \quad M_{l_2q_1}^{\text{max}} \quad M_{l_1qq}^{\text{max}} \quad M_{l_2qq}^{\text{max}} \quad M_{l_1qq}^{\text{max}}; \quad l_1 \quad q_2 \]
5.2 Upper end point in $M(q_1q_2)$

For the upper end point of $M(q_1q_2)$, the configuration is shown in figure 17, which appears in the

![Configuration leading to the maximum of the mass for (q1q2).]

Figure 17: Configuration leading to the maximum of the mass for $(q_1q_2)$.

The largest value of the mass is reached for $\left(\frac{Q}{2}\right)$ distribution is triangular for the same reasons as the $M(q1q2)$ distribution is triangular for the same reasons as the $M(l1q1)$ distribution.

Moreover, if gaugino mass universality is assumed, $M_G > M_q$, and amounts to $M_{max}^{q1q2} = M_G - M_X$.

Moreover, the $M(qq)$ distribution is triangular for the same reasons as the $M(lll)$ is. The end point is therefore clearly visible. It may, however, be rounded due to the variable jet mass and the jet energy resolution.

5.3 Upper end point in $M(l1q1)$

The configuration leading to the maximum $M(l1q1)$ is given in figure 18, which is part of the configurations

![Configuration leading to the maximum of the mass for (l1q1).]

Figure 18: Configuration leading to the maximum of the mass for $(l1q1)$.

The largest value for the end point is reached for $M_q = \frac{M_a M_b}{M_X}$ and amounts to $M_{max}^{l1q1} = M_G - \frac{M_a M_b}{M_X}$, where $M_a M_b / M_X$ represents the $(q2l)$ mass in the parallel configuration.

We can also look at which mass relations need to be fulfilled for the $M_{l1q1}^{max}$ end point to be larger than $M_{l1q2}^{max}$ or $M_{l2q2}^{max}$. For the first one, $M_{l1q1}^{max} \geq M_{l1q2}^{max}$

$$M_G^2 - M_Q^2 \geq M_Q^2 - M_X^2$$

Now, on the basis of the RGE which are dominated by $M_3$ for the gluino and the squark, we expect $M_G \lesssim 1.2 M_Q$. Moreover, if gaugino mass universality is assumed, $M_X \lesssim 0.3 M_G$. Therefore, $M_G^2 + M_X^2 \lesssim 1.09 M_G^2 \lesssim 1.6 M_Q^2$, showing that the above relation is not fulfilled. Within these assumptions it is thus expected that $M_{l1q1}^{max}$ is larger than $M_{l1q2}^{max}$.

For $M_{l1q1}^{max}$ to be larger than $M_{l1q2}^{max}$ requires

$$(M_G^2 - M_Q^2)(M_X^2 - M_R^2) \geq (M_G^2 - M_Q^2)(M_R^2 - M_Q^2)$$

$$M_G^2 - M_Q^2 \geq (M_G^2 - M_Q^2)\left(\frac{M_R^2}{M_X^2} - 1\right)$$

$$M_G^2 - M_Q^2 \geq (M_G^2 - M_R^2)$$

$$M_G^2 - M_Q^2 \geq (M_G^2 - M_R^2) + (M_G^2 - M_Q^2)\left(\frac{M_R^2}{M_X^2} - 1\right)$$

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Hence, the maximum mass becomes

\[ M_G^2 - M_Q^2 \geq (M_Q^2 - M_X^2) + (M_Q^2 - M_R^2) \frac{M_R^4 - M_Q^2 M_Q^2}{M_R (M_X^2 - M_R^2)} \]  

(119)

Note that, as this configuration is similar to the one leading to \( M_{12q}^{max} \), the second end point obtained by flipping the decay products of \( Q = \bar{q} \) will never be larger than the one above.

### 5.4 Upper end point in \( M(l2q) \)

The configuration for which \( M(q1l2) \) is maximal corresponds to the configuration \( d_2 \). The maximum \( M(l2q1) \) is given by

\[ M_{l2q1}^{max} = 4E_q E_{l2} \]  

(120)

where \( E_q \) and \( E_{l2} \) are defined in the rest frame of the gluino, for which

\[ E_q = \frac{M_Q^2 - M_Q^2}{2M_G} \]  

(121)

The lepton energy in the rest frame of \( R = \bar{i} \) is

\[ E_{l2} = \frac{M_R^2 - M_Q^2}{2M_R} \]  

and the Lorentz transformation to the gluino rest frame gives

\[ E_{l2} = \frac{M_G M_Q M_X - E_{l2}^*}{M_R^*} \]  

(122)

Hence, the maximum mass becomes

\[ M_{l2q1}^{max} = \frac{4M_G^2 - M_Q^2}{2M_G} M_G - M_R^2 \frac{M_R^2 - M_Q^2}{2M_R} = \frac{(M_Q^2 - M_Q^2) M_R^2 - M_Q^2}{M_R} \]  

or

\[ M_{l2q1}^{max} = M_G \sqrt{1 - \frac{M_Q^2}{M_G^2} (1 - \frac{M_Q^2}{M_R^2})} \]  

(123)

The largest value is reached when \( M_Q = \frac{M_Q M_R}{M_X} \) for which \( M_{l2q1}^{max} = M_G - \frac{M_Q M_R}{M_R} \).

This end point may be larger than \( M_{l1q1}^{max} \) provided

\[ \frac{M_Q}{M_R} \leq \frac{M_R}{M_X} \text{ or } \frac{M_R}{M_X} \geq M_Q M_0 \]  

(124)

which is the same condition as (62), ensuring that \( M_{l1q1}^{max} \geq M_{l2q2}^{max} \).

The condition to have \( M_{l2q1}^{max} \) larger than \( M_{l2q2}^{max} \) is

\[ M_Q^2 - M_Q^2 \geq M_Q^2 - M_X^2 \]  

(125)

i.e. the same as found for \( M_{l1q1}^{max} \geq M_{l1q2}^{max} \).

To have \( M_{l2q1}^{max} \) larger than \( M_{l1q2}^{max} \) requires

\[ \frac{(M_Q^2 - M_Q^2)(M_R^2 - M_Q^2)}{M_R^2} \geq \frac{(M_Q^2 - M_X^2)(M_X^2 - M_R^2)}{M_X^2} \]

\[ M_Q^2 - M_Q^2 \geq (M_Q^2 - M_X^2) \frac{M_R^2 M_X^2 - M_X^2}{M_X^2 M_R^2 - M_R^2} \]

\[ M_Q^2 - M_Q^2 \geq (M_Q^2 - M_X^2) + (M_Q^2 - M_X^2) \left( \frac{M_R^2 M_X^2 - M_X^2}{M_X^2 M_R^2 - M_R^2} - 1 \right) \]

or

\[ M_Q^2 - M_Q^2 \geq (M_Q^2 - M_X^2) + (M_Q^2 - M_X^2) \frac{M_X^2 M_Q^2 - M_X^2}{M_X^2 (M_R^2 - M_R^2)} \]  

(126)

Secondary end points could be computed by flipping the direction of the decay products of the squark or the \( \tilde{\chi}_1^0 \). But, like in the case of the \( (l2q) \) secondary end point, these will never be larger than the one considered above (it would increase the mass recoiling against the l2). Therefore, this discussion will be skipped here.
5.5 Conclusion on the $M(lq)$ end points

It may be useful to summarize the hierarchy of the $(lq)$ end points according to the relations between the sparticle masses. Two such relations appear in common to several cases. They are

$$M_R^2 \leq M_X M_0 \quad \text{for which} \quad M_{l1q2}^{\text{max}} \geq M_{l2q2}^{\text{max}} \quad \text{and} \quad M_{l1q1}^{\text{max}} \geq M_{l2q1}^{\text{max}}$$

$$M_R^2 - M_Q^2 \geq M_Q^2 - M_X^2 \quad \text{for which} \quad M_{l1q1}^{\text{max}} \geq M_{l1q2}^{\text{max}} \quad \text{and} \quad M_{l2q2}^{\text{max}} \geq M_{l2q1}^{\text{max}} \quad (127)$$

The hierarchies are summarized in Table 1. The ambiguities left by these relations can be solved using

$$M_R^2 < M_X M_0 \quad \text{1q1} > l_2 q_1 > l_1 q_2 > l_2 q_2 \quad l_1 q_2 > l_2 q_2 > l_1 q_1 > l_2 q_1$$

$$M_R^2 > M_X M_0 \quad l_2 q_1 > l_1 q_1 > l_2 q_2 > l_1 q_2 \quad l_2 q_2 > l_1 q_2 > l_2 q_1 > l_1 q_1$$

Table 1: Hierarchy between the $M(lq)$ end points, abbreviated by their particle contents.

(119) and (126). It is seen that, depending on the sparticle masses, any order can exist.

Note for later that if $M_R^2 - M_Q^2 < M_Q^2 - M_X^2$ the mass ratios have to fulfill the requirement $M_Q/M_G \geq M_X/M_Q$. In the opposite case $M_R^2 - M_Q^2 > M_Q^2 - M_X^2$ no definite ordering of the ratios is implied.

The configurations characterizing which is the true end point are the following:

- (11q1) is the true end point in the configurations (c1) and (d1). This end point is reached for low values of $M(qq)$ and all values of $M(ll)$.
- (12q1) is the true end point in the configuration (d2). This end point is reached for low values of $M(qq)$ but only near maximal $M(ll)$.
- (11q2) is the true end point in the configurations (a2), (b2), (c1) and (d1). This end point is reached for all values of $M(qq)$ and of $M(ll)$.
- (12q2) is the true end point in the configurations (b1) and (d2). This end point is reached for all values of $M(qq)$ but only near maximal $M(ll)$.

A summary of the main characteristics of the true $M(lq)$ end points is given in Table 2.

<table>
<thead>
<tr>
<th>Config</th>
<th>$M(ll)$</th>
<th>$M(qq)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{l1q1}^{\text{max}}$</td>
<td>c1 + d1</td>
<td>any</td>
</tr>
<tr>
<td>$M_{l2q1}^{\text{max}}$</td>
<td>d2</td>
<td>max</td>
</tr>
<tr>
<td>$M_{l1q2}^{\text{max}}$</td>
<td>a2 + b2 + c1 + d1</td>
<td>any</td>
</tr>
<tr>
<td>$M_{l2q2}^{\text{max}}$</td>
<td>b1 + d2</td>
<td>max</td>
</tr>
</tbody>
</table>

Table 2: Main characteristics when a given $(M_{l1q}^{\text{max}})$ gives the true end point.

5.6 End point for the sum of $M(l2q)^2$

It is seen from the configurations in section 5.1 that in one of them, namely (d2) both $M(l2q1)$ and $M(l2q2)$ take their maximum value. Also $M(ll)$ is maximal. As they occur in the same configuration, it is thus possible to compute the end point for the sum

$$[M(l2q1)^2 + M(l2q2)^2]^{\text{max}} = (M(l2q1)^{\text{max}})^2 + (M(l2q2)^{\text{max}})^2 = (M_R^2 - M_X^2)(1 - M_0^2/M_R^2) \quad (128)$$

This may bring valuable additional information or replace one of the $M(l2q)$ end points in case it is difficult to see.

This end point can be extracted by

1. selecting events near the maximum of $M(ll)$
2. computing for both lepton combinations the sum $M(lq1)^2 + M(lq2)^2$
3. keeping the largest sum to find $[M(l2q1)^2 + M(l2q2)^2]^{\text{max}}$
5.7 Upper end point in $M(llq1)$

5.7.1 Absolute maximum of $M(llq1)$

The absolute maximum of $M(llq1)$ is reached when the $(q2\tilde{\chi}_1^0)$ system is at rest in the $G = \tilde{g}$ frame and has its minimal mass $M_{20}$, i.e. with $q2$ and $\tilde{\chi}_1^0$ parallel. The mass $M(llq1)$ is then

$$M = M_G - M_{20}$$ (129)

The situation is more complicated in this case than in the others, because there is an intermediate step in the decay chain between $q2$ and $\tilde{\chi}_1^0$. Let us nevertheless start by following the lines already developed in Section 2.5, where it was shown that the smallest $(q2\tilde{\chi}_1^0)$ mass is obtained in the configuration of Figure 4 (right) with $cos\theta^* = +1$, the mass corresponds to $M_{q\Omega_c}$, or

$$M_{20} = \frac{M_Q M_0}{M_X}$$ (130)

Note that to reach this mass value, the slepton must also be emitted parallel to $q2$ and $\tilde{\chi}_1^0$, like in the collinear configuration (c1) (the configuration (a2) would also satisfy this requirement, but leads to $M_{llq1} = 0$).

The region of validity for the non-collinear configuration can be derived from energy-momentum conservation along the same lines as for the $(llq2)$ case in Section 4.4.1, but with the $(q2\tilde{\chi}_1^0)$ system at rest in the $G = \tilde{g}$ frame. To close the momentum triangle requires

$$M_G M_{20} \leq M_{Q}^2$$

The value of $E_{q1}$ is fixed by $M_G$ and $M_Q$, independent of the configuration. To maximize the value of $M$, it is thus possible to choose a configuration where $M_{20}$ is minimal, i.e. (c1). The condition on $E_{q1}$ then yields

$$M_G - M_{20} \geq \frac{M_Q^2 - M_{Q}^2}{M_G}$$

or $M_G M_{20} \leq M_{Q}^2$. After introducing the value of $M_{20}$, the condition is

$$M_G \leq \frac{M_Q M_X}{M_0}$$

For $E_{l2}$, the maximum value is reached when $l2$ is maximally boosted and the $(q2\tilde{\chi}_1^0)$ system is at rest in the $G = \tilde{g}$ frame, i.e. configuration (d2). In this case

$$E_{l2} = \frac{M_G M_R^2 - M_0^2}{2M_R}$$

and as in this case $M_{20} = M_{20b} = M_{q\Omega_b}$ of Section 2.5,

$$M_G - M_{20b} \geq \frac{M_G}{M_R} (M_R^2 - M_0^2)$$

from which $M_G M_0^2 \geq M_R^2 M_{20b}$. After introducing the value of $M_{20b}$, it gives the condition

$$M_G^2 \geq \frac{M_R^2}{M_0^2} (M_Q^2 - M_X^2 + M_R^2)$$

The two conditions taken together lead to the interval

$$\frac{M_R}{M_0} \sqrt{M_Q^2 - M_X^2 + M_R^2} \leq M_G \leq \frac{M_Q M_X}{M_0}$$ (132)

To derive the condition on $E_{l1}$, its maximum is obtained when it is opposite to $q2$, as in configuration (c1) or (d1), and is independent of the $R = \tilde{l}$ decay. The $l1$ energy is then

$$E_{l1} = \frac{M_G M_X^2 - M_R^2}{2M_X}$$
and
\[ M_G - M_{20} \geq \frac{M_G}{M_X} (M_X^2 - M_R^2) \]
which leads to \( M_G M_R^2 \geq M_X^2 M_{20} \). The strongest constraint is obtained for the largest value of \( M_{20} \), hence \( M_{20d} = M_{QOd} \) of Section 2.5 for configuration (d1). After replacing \( M_{20} \) by its value, this becomes
\[ M_G \geq \frac{M_X}{M_R} \sqrt{M_Q^2 M_R^2 - M_X^2 M_R^2 + M_X^2 M_0^2} \]  \( (133) \)
Rewriting this as
\[ M_G^2 \geq \frac{M_Q^2 M_X^2}{M_R^2} M_0^2 \frac{M_X^4}{M_R^4} (M_R^2 - M_0^2) \]
shows that it is always lower than the upper edge of the interval \( (132) \). It may be larger than the lower edge of interval \( (132) \), provided
\[ \frac{M_X^4}{M_R^4} (M_R^2 - M_0^2) + \frac{M_X^2 M_0^2}{M_R^2} \]
\( (134) \)
\[ \frac{M_X M_0}{M_R^2} \]

The difficulty in the case of \( M_{llq_1} \) is that the mass of the \( (q2\chi^0_1) \) system at each edge of the intervals takes a different value, rather than a constant as encountered in the other cases. It is, therefore, necessary to derive the expression which describes the variation of \( M_{llq_1} \) between the edges. It turns out that when this is done under the assumption that the system \( (q2\chi^0_1) \) is at rest in the \( G = \tilde{g} \) frame, the non-collinear maximum so obtained is nowhere larger than the collinear end points that will be derived below. Instead, the non-collinear \( M_{llq_1}^{max} \) between the edges corresponds to the configuration shown in Figure 19, where

Figure 19: Non-collinear configuration leading to the maximum of the mass for \( (llq_1) \).

The \( (q2\chi^0_1) \) system is not at rest.

The energy-momentum constraints then lead to the following formulae:

1. \( (llq_1) \) system:
\[ E_{q_1} = \frac{M_Q^2 - M_Q^2}{2M_G} = p_{q_1} \]  \( (135) \)
is fixed,
\[ \tilde{p}_{T,11} = -\tilde{p}_{T,12} \]  \( (136) \)
and
\[ M_{llq_1}^2 = (E_{q_1} + E_{ll})^2 - (\tilde{p}_{q_1} + \tilde{p}_{ll})^2 = M_{ll}^2 + 2E_{q_1}(E_{ll} + p_{ll}) \]  \( (137) \)
2. \((q^2\chi^0_1)\) system:

\[
E_{q^2} = \frac{M_Q M_Q^2 - M_X^2}{2M_Q} = \frac{M_Q^2 - M_X^2}{2M_G} = p_{q^2}
\]

from which

\[
E_{q_1} + E_{q_2} = \frac{M_Q^2 - M_X^2}{2M_G} = E_q
\]

3. overall energy-momentum conservation:

\[
M_G = E_{llq_1} + E_{q_20} = E_{ll} + E_{q_1} + E_{q_2} + E_0
\]

\[
0 = p_{llq_1} - p_{q_20} = p_{ll} - E_{q_1} - E_{q_2} + p_0
\]

where the index 0 refers to the \(\chi^0_1\) and the momentum signs are taken from Figure 19. From these equations we have also

\[
E_0 = \frac{M_Q^2 + M_X^2}{2M_G} - E_{ll}
\]

\[
p_0 = \frac{M_Q^2 - M_X^2}{2M_G} - p_{ll}
\]

We can eliminate \(E_0\) and \(p_0\) by

\[
M_0^2 = E_0^2 - p_0^2 = \left(\frac{M_Q^2 + M_X^2}{2M_G}\right)^2 - \left(\frac{M_Q^2 - M_X^2}{2M_G}\right)^2 - 2 \left(\frac{M_Q^2 + M_X^2}{2M_G}\right) E_{ll} + E_{ll}^2 + 2 \frac{M_Q^2 - M_X^2}{2M_G} p_{ll} - p_{ll}^2
\]

or

\[
0 = M_0^2 - M_X^2 + \frac{M_Q^2 + M_X^2}{M_G} E_{ll} - E_{ll}^2 - \frac{M_Q^2 - M_X^2}{M_G} p_{ll} + p_{ll}^2
\]

Note that \(M_Q\) nor \(M_X\) impose a new constraint on \((q^2\chi^0_1)\) nor \((ll/2)\), as they are trivially satisfied by the above equations. E.g. from \(M_Q\)

\[
M_Q^2 = (E_{q_2} + E_0 + E_{ll})^2 - (p_{q_2} + \tilde{p}_0 + \tilde{p}_{ll})^2
\]

\[
= (M_G - E_{q_1})^2 - (p_{q_1})^2 = \left(\frac{M_Q^2 + M_X^2}{2M_G}\right)^2 - \left(\frac{M_Q^2 - M_X^2}{2M_G}\right)^2 = M_Q^2
\]

From the above second order equation, we have

\[
M_{llq_1}^2 = M_0^2 - M_X^2 + \frac{M_Q^2 + M_X^2}{M_G} E_{ll} - \frac{M_Q^2 - M_X^2}{M_G} p_{ll}
\]

which can be used to simplify \(M_{llq_1}\)

\[
M_{llq_1}^2 = M_0^2 - M_X^2 + \left(\frac{M_Q^2 - M_X^2}{M_G}\right) E_{ll} + \left(\frac{M_Q^2 + M_X^2}{M_G} - \frac{M_Q^2 - M_X^2}{M_G}\right) p_{ll}
\]

or

\[
M_{llq_1}^2 = M_0^2 - M_X^2 + \frac{2M_Q^2 - M_X^2}{M_G} E_{ll} - \frac{M_Q^2 - M_X^2}{M_G} p_{ll}
\]

We can now maximize \(M_{llq_1}\) for, e.g., \(p_0\):

\[
\frac{\partial M_{llq_1}^2}{\partial p_0} = \frac{2M_Q^2 - M_X^2}{M_G} \frac{\partial E_{ll}}{\partial p_0} - \frac{M_Q^2 - M_X^2}{M_G} \frac{\partial p_{ll}}{\partial p_0}
\]

Using the energy-momentum conservation equations, we get

\[
\frac{\partial E_{ll}}{\partial p_0} = -\frac{\partial E_0}{\partial p_0} = -\frac{p_0}{E_0}
\]

\[
\frac{\partial p_{ll}}{\partial p_0} = -1
\]
so that

\[
\frac{\partial M_{04q1}^2}{\partial p_0} = - \frac{2M_Q^2 - M_0^2 + M_X^2}{M_G^2} p_0 + \frac{M_Q^2 - M_X^2}{M_G^2} = 0
\]

\[
E_0(M_Q^2 - M_X^2) = p_0(2M_Q^2 - M_0^2 + M_X^2)
\]

\[
(p_0^2 + M_0^2)(M_Q^2 - M_X^2)^2 = p_0^2(2M_Q^2 - M_0^2 + M_X^2)^2
\]

or

\[
p_0 = \frac{M_0(M_Q^2 - M_X^2)}{2M_G \sqrt{M_Q^2 - M_0^2 + M_X^2}}
\]

(144)

This value can be introduced in the energy-momentum equations to determine \(E_{ll}\) and \(p_{ll}\)

\[
E_{0ll}^2 = p_0^2 + M_0^2 = M_0^2 \left( \frac{(M_Q^2 - M_X^2)^2}{4M_G^2(M_Q^2 - M_0^2 + M_X^2)} + 1 \right) = M_0^2 \frac{(2M_Q^2 - M_0^2 + M_X^2)^2}{4M_G^2(M_Q^2 - M_0^2 + M_X^2)}
\]

\[
E_{ll} = M_G - E_q - E_0 = \frac{M_Q^2 + M_X^2}{2M_G} - \frac{M_0 (2M_Q^2 - M_0^2 + M_X^2)}{2M_G \sqrt{M_Q^2 - M_0^2 + M_X^2}}
\]

\[
p_{ll} = E_q - p_0 = \frac{M_Q^2 - M_X^2}{2M_G} - \frac{M_0 (M_Q^2 - M_X^2)}{2M_G \sqrt{M_Q^2 - M_0^2 + M_X^2}}
\]

Then

\[
M_{04q1}^2 = M_0^2 - M_X^2 + \frac{2M_Q^2 - M_0^2 + M_X^2}{2M_G^2} \left[ M_Q^2 + M_X^2 - M_0^2 \left( \frac{2M_Q^2 - M_0^2 + M_X^2}{M_Q^2 - M_0^2 + M_X^2} \right) \right]
\]

\[
= \frac{M_Q^2 - M_X^2}{2M_G^2} \left[ M_Q^2 - M_X^2 - \frac{M_0 (M_Q^2 - M_X^2)}{M_Q^2 - M_0^2 + M_X^2} \right]
\]

\[
= M_G^2 + M_0^2 - \frac{2M_Q^2 - M_0^2}{2M_G^2} - \frac{2M_0^2}{2M_G^2} \frac{2M_Q^2 - M_0^2 + M_X^2}{\sqrt{M_Q^2 - M_0^2 + M_X^2}}
\]

\[
+ \frac{M_Q^2 - M_X^2}{2M_G^2} \frac{2M_0^2}{\sqrt{M_Q^2 - M_0^2 + M_X^2}}
\]

\[
= M_G^2 - M_Q^2 + M_X^2 + M_0^2 - \frac{2M_Q^2 - M_0^2 + M_X^2}{\sqrt{M_Q^2 - M_0^2 + M_X^2}}
\]

or

\[
M_{04q1} = \sqrt{M_Q^2 - M_0^2 + M_X^2} - M_0
\]

(145)

We still need to compute the region in which this formula is applicable. It turns out that this region is considerably reduced, compared to the intervals found in (132) and (133). The simplest way to determine this region is to compute when the effective mass of formula (145) becomes equal to the ones for collinear configurations given in the subsequent sections. An equivalent way is to compute when the neutralino momentum \(p_0\) given from the momentum conservation in the collinear configurations equals the value found in equation (144).

The boundary for configuration (c1) is determined by

\[
(M_Q^2 - M_0^2) \frac{M_X^2 - M_0^2}{M_X^2} = M_G^2 - M_0^2 + M_X^2 + M_0^2 - 2M_0 \sqrt{M_Q^2 - M_0^2 + M_X^2}
\]
\[2M_R^2 M_0 \sqrt{M_0^2 - M^2_Q + M^2_R} = M^2_G - M^2_Q + M^2_X - (M^2_Q - M^2_R) M^2_0\]

\[4M_R^2 M_0^2 M^2_0 - 4M_X^2 M^2_0 (M^2_Q - M^2_X) = M^2_0 M^2_Q + 2M_R^2 M_0^2 [M^2_R - (M^2_Q - M^2_X) M^2_0] + [M^2_X - (M^2_Q - M^2_X) M^2_0]^2\]

or

\[M^2_0 = M^2_Q - M^2_X + \frac{M^4_R}{M^2_0} \quad \text{or} \quad M^2_0 - M^2_Q = \frac{M^2_X}{M^2_0} (M^2_R - M^2_X)\]  \(\text{(146)}\)

The boundary for configuration \((d2)\) is determined by

\[(M^2_0 - M^2_Q + M^2_X - M^2_R) \frac{M^2_R - M^2_0}{M^2_R} = M^2_0 - M^2_Q + M^2_X + M^2_R - 2M_0 \sqrt{M^2_0 - M^2_Q + M^2_X}\]

\[2M_R^2 M_0 \sqrt{M^2_0 - M^2_Q + M^2_X} = M^2_0 M^2_Q + M^2_R - (M^2_Q - M^2_X) M^2_0\]

\[4M_R^2 M_0^2 M^2_0 (M^2_Q - M^2_X) = M^2_0 M^2_Q + 2M_R^2 M_0^2 [M^2_R - (M^2_Q - M^2_X) M^2_0] + [M^2_X - (M^2_Q - M^2_X) M^2_0]^2\]

or

\[M^2_0 = M^2_Q - M^2_X + \frac{M^4_R}{M^2_0} \quad \text{or} \quad M^2_0 - M^2_Q = \frac{1}{M^2_0} (M^2_R - M^2_X M^2_0)\]  \(\text{(147)}\)

Finally, the boundary for configuration \((d1)\) is

\[(M^2_0 - M^2_Q + M^2_X - M^2_R) \frac{M^2_R - M^2_0}{M^2_R} = M^2_0 - M^2_Q + M^2_X + M^2_R - 2M_0 \sqrt{M^2_0 - M^2_Q + M^2_X}\]

\[2M_R^2 M_0 \sqrt{M^2_0 - M^2_Q + M^2_X} = M^2_0 M^2_Q + M^2_R - (M^2_Q - M^2_X) M^2_0\]

\[4M_R^2 M_0^2 M^2_0 (M^2_Q - M^2_X) = M^2_0 M^2_Q + 2M_R^2 M_0^2 [M^2_R - (M^2_Q - M^2_X) M^2_0] + [M^2_X - (M^2_Q - M^2_X) M^2_0]^2\]

or

\[M^2_0 = M^2_Q - M^2_X + \frac{M^4_R}{M^2_0} \quad \text{or} \quad M^2_0 - M^2_Q = \frac{M^2_X}{M^2_R} (M^2_R - M^2_X)\]  \(\text{(148)}\)

*Old calculation:* Another way to determine this region is to compute when the neutralino momentum \(p_0\) given from the momentum conservation equals the value found in equation \((144)\).

For the configuration \((c1)\), the energies are

\[E_{12} = \frac{M_G M_Q M_R M_R^2 - M_0^2}{M_Q M_X M_X - 2M_R} = \frac{M_G}{2M_X} (M^2_R - M^2_0)\]

\[E_{11} = \frac{M_G M_Q M_X^2 - M_R^2}{M_Q M_X^2 - 2M_R} = \frac{M_G}{2M_X} (M^2_X - M^2_R)\]

together with \((135)\) and \((138)\). This leads to

\[p_0 = E_{q1} + E_{q2} - E_{11} - E_{12} = \frac{M_0^2 - M^2_X}{2M_G} - \frac{M_G}{2M_X} (M^2_R - M^2_0)\]

\[= \frac{M_0^2 (M^2_Q - M^2_X)}{2M_G \sqrt{M^2_0 - M^2_Q + M^2_X}}\]

\[(M^2_0 - M^2_X) M^2_X - M^2_0 (M^2_X - M^2_R) = \frac{M^4_X M_0 (M^2_Q - M^2_X)}{\sqrt{M^2_0 - M^2_Q + M^2_X}}\]

\[M^2_0 M^2_Q - M^4_X = \frac{M^4_X M_0 (M^2_Q - M^2_X)}{\sqrt{M^2_0 - M^2_Q + M^2_X}}\]
The region of validity is below the value of the $M_G$ solution of this equation.

For the configuration (d1), the energies are the same, except

$$E_{t2} = \frac{M_G}{M_Q} \frac{M_X M_X M_R^2 - M_0^2}{2M_R} = \frac{M_X^2}{2M_G M_R} (M_R^2 - M_0^2)$$

This leads to

$$p_0 = E_{t1} + E_{t2} = \frac{M_G^2 - M_R^2}{2M_G} \frac{M_Q M_X M_X^2 - M_0^2}{M_R^2 - M_0^2} + \frac{M_X^2}{2M_G M_R} (M_R^2 - M_0^2)$$

or

$$M_G^2 - M_R^2 + M_X^2 = \frac{M_X^2 M_R^2 (M_Q^2 - M_R^2)}{(M_G^2 M_R^2 - M_X^2)^2}$$ (149)

The region of validity is above the value of the $M_G$ solution of this equation.

For the configuration (d2), the energies are the same, except

$$E_{t1} = \frac{M_G}{M_Q} \frac{M_X^2 M_R^2 - M_0^2}{2M_R} = \frac{M_G}{2M_R} (M_R^2 - M_0^2)$$

$$E_{t2} = \frac{M_G}{M_Q} \frac{M_X^2 M_R^2 - M_0^2}{2M_R} = \frac{M_G}{2M_R} (M_R^2 - M_0^2)$$

This leads to

$$p_0 = E_{t1} + E_{t2} + E_{t1} - E_{t2} = \frac{M_G^2 - M_R^2}{2M_G} + \frac{M_X^2 - M_R^2}{2M_G} + \frac{M_G}{2M_R^2} (M_R^2 - M_0^2)$$

or

$$M_G^2 - M_R^2 + M_X^2 = \frac{M_X^2 M_R^2 M_G (M_Q^2 - M_R^2)}{(M_G^2 M_R^2 - M_X^2)^2}$$ (150)

The region of validity is above the value of the $M_G$ solution of this equation.

For the configuration (d2), the energies are the same, except

or

$$M_G^2 - M_R^2 + M_X^2 = \frac{M_X^2 M_R^2 (M_Q^2 - M_R^2)}{(M_G^2 M_R^2 - M_X^2)^2}$$ (151)

The region of validity is above the value of the $M_G$ solution of this equation.
5.7.2 First collinear end point in $M(llq)$

The configuration leading to a maximum of $M(llq)$ is labelled (c1).

The di-lepton is a massless system recoiling against a $\tilde{\chi}^0$ in the $X = \tilde{\chi}^0$ rest frame. The $(llq)$ effective mass can be computed as

$$
(M_{llq}^{\text{max}})^2 = (E_{lq} + E_{ll})^2 - (\vec{p}_{lq} + \vec{p}_{ll})^2 = 2E_{qll}E_{ll} - 2\vec{p}_{qll}\vec{p}_{ll}
$$

$$
= 4E_{qll}E_{ll}
$$

(152)

With the di-lepton energy in the $G = \tilde{g}$ rest frame given by

$$
E_{ll} = \frac{M_G M_Q M_{X}^2 - M_{lq}^2}{2M_{Q} M_X} = \frac{M_G M_{Q}^2 - M_{lq}^2}{2M_X}
$$

(153)

the maximum effective mass is obtained as

$$
(M_{llq}^{\text{max}})^2 = 4\frac{M_G^2 - M_Q^2}{2M_G} M_{Q}^2 - M_{lq}^2 = \frac{1}{4} M_G^2 (1 - \frac{M_Q^2}{M_G^2})(1 - \frac{M_{lq}^2}{M_X^2})
$$

or

$$
M_{llq}^{\text{max},1} = M_G \sqrt{(1 - \frac{M_Q^2}{M_G^2})(1 - \frac{M_{lq}^2}{M_X^2})}
$$

(154)

As we saw in Section 2.6 the largest value for this end point is obtained when $M_Q M_X = M_G M_0$ for which the $(\tilde{\chi}^0 q2)$ system is at rest in the $G = \tilde{g}$ frame and the end point is given by (129) with a mass $M_0 = M_{Q0}$. It may give the true end point in the region above the upper bound of the non-collinear interval.

This configuration is characterized by low $M(ll)$, low $M(qq)$ and near maximum $M(llq2)$ and $M(llq1)$. As $M_0/M_X$ is always smaller than $M_0/M_R$ and than $M_R/M_X$, this end point is always larger than the one of $M_{llq1}^{\text{max}}$ and $M_{llq2}^{\text{max}}$. It can also be compared to the end point $M_{llq2}^{\text{max}}$. The condition for $M_{llq1}^{\text{max}} \geq M_{llq2}^{\text{max}}$ is

$$
(M_G^2 - M_Q^2) \frac{M_X^2 - M_0^2}{M_X^2} \geq (M_Q^2 - M_X^2) \frac{M_X^2 - M_0^2}{M_X^2}
$$

or

$$
M_G^2 - M_Q^2 \geq M_X^2 - M_0^2
$$

(155)

It is seen from Table 1 that this condition allows $M_{llq1}^{\text{max}}$ to give the true $(lq)$ end point, provided $M_R \leq M_X M_0$.

For $M_{llq1}^{\text{max}} \geq M_{llq2}^{\text{max}}$ it is needed that

$$
(M_G^2 - M_Q^2) \frac{M_X^2 - M_0^2}{M_X^2} \geq (M_Q^2 - M_R^2) \frac{M_R^2 - M_0^2}{M_R^2}
$$

or

$$
M_G^2 - M_Q^2 \geq (M_Q^2 - M_R^2) \frac{M_R^2 - M_0^2}{M_R^2}
$$

(156)

For $M_{llq1}^{\text{max}} \geq M_{llq2}^{\text{max}}$ we need to have

$$
(M_G^2 - M_Q^2) \frac{M_X^2 - M_0^2}{M_X^2} \geq \frac{M_R^2 - M_0^2}{M_R^2} \frac{M_X^2 - M_0^2}{M_X^2} = \frac{M_X^2 - M_0^2}{M_X^2 - M_0^2}
$$

or

$$
M_G^2 - M_Q^2 \geq \frac{M_R^2 - M_0^2}{M_R^2}
$$

(157)

For $M_{llq1}^{\text{max}} \geq M_{llq2}^{\text{max}}$ we need to have

$$
(M_G^2 - M_Q^2) \frac{M_X^2 - M_0^2}{M_X^2} \geq \frac{M_R^2 - M_0^2}{M_R^2} \frac{M_X^2 - M_0^2}{M_X^2} = \frac{M_X^2 - M_0^2}{M_X^2 - M_0^2}
$$

or

$$
M_G^2 - M_Q^2 \geq \frac{M_X^2 - M_0^2}{M_X^2 - M_0^2}
$$

(158)
5.7.3 Second collinear end point in $M(\ll q)$

A second collinear end point can be obtained in the configuration labelled $d_2$, in which case

$$
(M_{\ll q1}^{\text{max},2})^2 = (E_{\ll q1} + E_{\ll l1} + E_{\ll l2})^2 - (p_{\ll q1} + p_{\ll l1} + p_{\ll l2})^2
= 4E_{l2}(E_{\ll q1} + E_{\ll l1})
$$

As this configuration is the same as the one yielding the maximum masses $M_{\ll 2q1}^{\text{max}}$ and $M_{\ll l}^{\text{max}}$ the maximum effective mass is

$$
(M_{\ll q1}^{\text{max},2})^2 = (M_{\ll 2q1}^{\text{max}})^2 + (M_{\ll l}^{\text{max}})^2
= (M_Q^2 - M_Q^2 + M_X^2 - M_R^2)(1 - \frac{M_0^2}{M_R^2})
$$

To understand better the contents of this equation, we can look at the mass of the $(q2\ll_1^0)$ system, obtained in Section 2.5 for the configuration of Figure 4 (left) with $\cos \theta^* = -1$, where the mass was labelled $M_{qOb}$

$$
M_{20b}^2 = \frac{M_R^2}{M_R^2}(M_Q^2 - M_X^2 + M_R^2)
$$

Using this value, the maximum mass of $(llq1)$ can be rewritten as

$$
(M_{\ll q1}^{\text{max},2})^2 = (M_Q^2 - \frac{M_R^2}{M_0^2}M_{20b}^2)(1 - \frac{M_0^2}{M_R^2})
$$

To find the largest value of this end point, we can put it in the form used in Section 2.6, namely

$$
(M_{\ll q1}^{\text{max},2})^2 = M_G^2 (1 - \frac{1}{M_G^2} \frac{M_R^2}{M_0^2}M_{20b}^2)(1 - \frac{M_0^2}{M_R^2} M_R^2 M_{20b}^2)
$$

which shows clearly that the largest value is given an equation of the type (129), hence with the $(q2\ll_1^0)$ system at rest in the $G = \tilde{g}$ frame, but with a mass $M_{20} = M_{qOb} \geq M_{qOb}$. The condition for this to happen is

$$
M_{20b} = M_G \frac{M_R^2}{M_R^2}
$$

which can be rewritten as

$$
M_G^2 = \frac{M_R^2}{M_0^2}(M_Q^2 - M_X^2 + M_R^2)
$$

It touches the lower edge of the interval (132). This configuration may give the true end point in the region below the bound (147).

Note that the largest value is not the sum of the largest values of $M_{\ll 2q1}^{\text{max}}$ and $M_{\ll l}^{\text{max}}$. This is due to the fact that the mass condition which yields the largest value for $M_{\ll 2q1}^{\text{max}}$ is different from, and incompatible with, the condition for the largest value of $M_{\ll l}^{\text{max}}$. This is seen as follows. The largest $M_{\ll l}^{\text{max}}$ occurs when the $\ll_1^0$ is at rest in the frame of $X = \ll_2^0$, for which the requirement is that $M_R^2 = M_X M_0$. In this case the $(q2\ll_1^0)$ mass $M_{20b}$ would be

$$
M^2(q2\ll_1^0) = \frac{M_0}{M_X}(M_Q^2 - M_X^2 + M_X M_0)
$$

(which could also be derived explicitly from the $\ll_1^0$ at rest in the frame of $X = \ll_2^0$). This mass allows to fulfil the condition (163) provided

$$
\begin{align*}
M_Q^2 - M_X^2 &= M_Q^2 - M_X^2 + M_X M_0 \\
M_G^2 - M_X^2 &= M_X(M_Q^2 - M_X^2)
\end{align*}
$$

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Hence, the largest $M_{_{\text{max}}}^m$ is compatible with (a special case of) the mass and condition derived for the largest $M_{_{\text{max}}}^{m,2}$. On the other hand, the condition to obtain the largest $M_{_{\text{Lq1}}}^{m}$ is $M_Q = M_G M_0/M_R$, for which the above condition (163) can be rewritten as

\[
M_G^2 \frac{M_Q^2}{M_G^2} = M_R^2 + M_X^2 + M_R^2
\]

which would imply $M_X = M_R$, a non desirable feature. Thus, the condition for the largest $M_{_{\text{Lq1}}}^{m}$ is not compatible with the one for the largest $M_{_{\text{Lq1}}}^{m,2}$. This is the reason why the largest $M_{_{\text{Lq1}}}^{m,2}$ is not just the sum of the ones for $M_{_{\text{Lq1}}}^{m}$ and $M_{_{\text{Lq1}}}^{m}$ squared.

It is now possible to check in which case $M_{_{\text{Lq1}}}^{m,1} \geq M_{_{\text{Lq1}}}^{m,2}$. This requires

\[
\begin{align*}
M_{_G}^2 (M_{_G}^2 - M_{_Q}^2 + M_{_X}^2 - M_{_R}^2)(M_{_R}^2 - M_{_0}^2) &\leq M_{_G}^2 (M_{_G}^2 - M_{_Q}^2)(M_{_X}^2 - M_{_0}^2) \\
M_{_G}^2 (M_{_X}^2 - M_{_R}^2)(M_{_R}^2 - M_{_0}^2) &\leq (M_{_G}^2 - M_{_Q}^2)(M_{_X}^2 - M_{_R}^2) M_{_0}^2
\end{align*}
\]

or

\[
M_{_G}^2 - M_{_Q}^2 \geq M_{_G}^2 (M_{_R}^2 - M_{_0}^2) \tag{164}
\]

which is always satisfied in the validity region (155) for $M_{_{\text{Lq1}}}^{m,1}$. At the point where $M_{_{\text{Lq1}}}^{m,1}$ reaches its largest value, we have $M_{_G} = M_Q M_X/M_0$ and thus $M_{_G}^2 - M_{_Q}^2 = \frac{M_{_G}^2}{M_{_0}^2} (M_{_X}^2 - M_{_0}^2)$, which is larger than the upper bound of (164). Hence the region where $M_{_{\text{Lq1}}}^{m,2} \geq M_{_{\text{Lq1}}}^{m,1}$ is below the point where $M_{_{\text{Lq1}}}^{m,1}$ reaches its largest value.

On the other hand, for $M_{_{\text{Lq2}}}^{m,2} \geq M_{_{\text{Lq2}}}^{m,1}$, the condition can be derived from

\[
(M_{_{\text{Lq2}}}^{m,1})^2 \geq (M_{_{\text{Lq2}}}^{m,2})^2 - (M_{_{\text{Lq1}}}^{m,2})^2
\]

\[
(M_{_G}^2 - M_{_Q}^2) M_{_R}^2 - M_{_0}^2 < (M_{_G}^2 - M_{_Q}^2) M_{_X}^2 - M_{_0}^2 = (M_{_X}^2 - M_{_R}^2) M_{_0}^2
\]

\[
M_{_G}^2 - M_{_Q}^2 \geq (M_{_G}^2 - M_{_X}^2) \frac{M_{_R}^2 - M_{_0}^2}{M_{_X}^2} + (M_{_G}^2 - M_{_X}^2) \frac{M_{_R}^2 - M_{_0}^2}{M_{_X}^2}
\]

\[
M_{_G}^2 - M_{_Q}^2 \geq (M_{_G}^2 - M_{_X}^2) + (M_{_G}^2 - M_{_X}^2) \frac{M_{_R}^2 - M_{_0}^2}{M_{_X}^2} \frac{M_{_R}^2 - M_{_0}^2}{M_{_X}^2} - (M_{_G}^2 - M_{_X}^2) \frac{M_{_R}^2 - M_{_0}^2}{M_{_X}^2}
\]

\[
M_{_G}^2 - M_{_Q}^2 \geq (M_{_G}^2 - M_{_X}^2) + \frac{M_{_R}^2 - M_{_0}^2}{M_{_X}^2} \frac{M_{_R}^2 - M_{_0}^2}{M_{_X}^2} \frac{M_{_R}^2 - M_{_0}^2}{M_{_X}^2}
\]

or

\[
M_{_G}^2 - M_{_Q}^2 \geq (M_{_G}^2 - M_{_X}^2) + \frac{M_{_R}^2 - M_{_0}^2}{M_{_X}^2} \frac{M_{_R}^2 - M_{_0}^2}{M_{_X}^2} \frac{M_{_R}^2 - M_{_0}^2}{M_{_X}^2}
\]

(165)

The inequality $M_{_{\text{Lq1}}}^{m,2} \geq M_{_{\text{Lq2}}}^{m,2}$ requires $M_{_{\text{Lq1}}}^{m,2} \geq M_{_{\text{Lq2}}}^{m,2}$ and hence

\[
M_{_G}^2 - M_{_Q}^2 \geq M_{_G}^2 - M_{_X}^2 \tag{166}
\]

For $M_{_{\text{Lq1}}}^{m,2} \geq M_{_{\text{Lq2}}}^{m,3}$ we need to have

\[
(M_{_{\text{Lq1}}}^{m,2})^2 \geq (M_{_{\text{Lq2}}}^{m,2})^2
\]

which happens, as we saw in Section 5.4, for

\[
M_{_G}^2 - M_{_Q}^2 \geq (M_{_G}^2 - M_{_X}^2) + (M_{_G}^2 - M_{_X}^2) \frac{M_{_R}^2 - M_{_0}^2}{M_{_X}^2} \frac{M_{_R}^2 - M_{_0}^2}{M_{_X}^2}
\]

(167)
5.7.4 Third collinear end point in $M(llq1)$

A third collinear end point can be obtained in the configuration labelled (d1), in which case

$$(M_{llq1}^{\text{max},3})^2 = (E_{q1} + E_{l1} + E_{l2})^2 - (\vec{p}_{q1} + \vec{p}_{l1} + \vec{p}_{l2})^2$$

$$= 4E_{l1}(E_{q1} + E_{l2}) \quad \text{(168)}$$

As this configuration is the same as the one yielding the maximum masses $M_{llq1}^{\text{max}}$ and $M_{llq1}^{\text{max}}$, the maximum effective mass is

$$(M_{llq1}^{\text{max},3})^2 = (M_{llq1}^{\text{max}})^2 + (M_{llq1}^{\text{max}})^2 = [M_G^2 - M_Q^2 + \frac{M_X^2}{M_R^2} (M_R^2 - M_0^2)]\frac{M_X^2 - M_R^2}{M_X^2}$$

$$= (M_G^2 - M_Q^2 + M_X^2 - \frac{M_X^2}{M_R^2}) (1 - \frac{M_R^2}{M_X^2}) \quad \text{(169)}$$

This can also be computed directly from the energies

$$E_{l2} = \frac{M_Q M_X M_X M_R^2 - M_0^2}{2M_R} = \frac{M_X^2}{2M_R} (M_R^2 - M_0^2)$$

$$E_{l1} = \frac{M_Q M_X^2 - M_R^2}{2M_X} = \frac{M_G}{2M_X} (M_X^2 - M_R^2)$$

$$E_{q1} + E_{l2} = \frac{M_G^2 - M_Q^2}{2M_G} + \frac{M_X^2}{2M_R M_X^2} (M_R^2 - M_0^2)$$

$$= \frac{1}{2M_G M_R^2} (M_G^2 M_R^2 - M_Q^2 M_R^2 + M_X^2 M_R^2 - M_R^2 M_0^2)$$

so that

$$(M_{llq1}^{\text{max},3})^2 = \frac{M_G}{M_X^2} (M_X^2 - M_R^2) \frac{1}{M_G M_R^2} (M_G^2 M_R^2 - M_Q^2 M_R^2 + M_X^2 M_R^2 - M_R^2 M_0^2)$$

which gives the same formula.

To get a better insight into the contents of this equation, we can look at the mass of the $(q2\overline{q}^0)$ system, obtained in Section 2.5 for the configuration of Figure 4 (right) with $\cos \theta^* = -1$, where the mass was labelled $M_{gOd}$, or

$$M_{2od}^2 = \frac{1}{M_X^2} (M_Q^2 M_R^2 - M_X^2 M_R^2 + M_X^2 M_0^2) \quad \text{(170)}$$

from which

$$(M_{llq1}^{\text{max},3})^2 = (M_G^2 - \frac{M_R^2}{M_X^2} M_{2od}^2) (1 - \frac{M_R^2}{M_X^2}) \quad \text{(171)}$$

To find the largest value of this end point, we can put it in the form used in Section 2.6, namely

$$(M_{llq1}^{\text{max},3})^2 = M_G^2 (1 - \frac{1}{M_G^2} \frac{M_X^2}{M_R^2} M_{2od}^2) (1 - \frac{M_R^2}{M_X^2} \frac{M_{2od}^2}{M_R^2})$$

which shows that again the largest value is given an equation of the type (129), hence with the $(q2\overline{q}^0)$ system at rest in the $G = \hat{g}$ frame, but with a mass $M_{2od} = M_{gOd} \geq M_{gOd}$. The condition for this to happen is

$$M_{2od} = M_G \frac{M_X^2}{M_R^2}$$

Using the above value for the mass $M_{2od}$, this can be rewritten as

$$M_G^2 = \frac{M_X^2}{M_R^2} (M_Q^2 M_R^2 - M_X^2 M_R^2 + M_X^2 M_0^2) \quad \text{(172)}$$

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It touches the edge of the interval (133). This configuration may give the true end point in the region of $M_G$ below the bound (147).

Like we saw in the case of the largest value of $M_{lq1}^{\text{max,2}}$, in the present case, the largest value of $M_{lq1}^{\text{max,3}}$ is not the sum of squares of the ones of $M_{lq1}^{\text{max}}$ and $M_{lq1}^{\text{max}}$. The largest $M_{lq1}^{\text{max}}$ is obtained for $M_Q = M_G M_R / M_X$ which, inserted in (172) would lead to the requirement $M_R = M_0$. It is thus incompatible with the largest value above.

We can now look at the condition for which $M_{lq1}^{\text{max,1}} \geq M_{lq1}^{\text{max,3}}$

\[
(M_G^2 M_R^2 - M_Q^2 M_R^2 + M_X^2 M_R^2 - M_X^2 M_Q^2)(M_X^2 - M_R^2) \leq M_R^2 (M_G^2 - M_Q^2)(M_X^2 - M_Q^2)
\]

\[
-(M_G^2 - M_Q^2) M_R^2(M_R^2 - M_Q^2) \leq -(M_G^2 - M_Q^2) M_R^2 M_Q^2
\]

\[
-(M_G^2 - M_Q^2) M_R^2 (M_R^2 - M_Q^2) + M_X^2 (M_R^2 - M_Q^2) \leq 0
\]

or

\[
M_G^2 - M_Q^2 \geq \frac{M_X^2}{M_R^2} (M_X^2 - M_R^2) \tag{173}
\]

which is always satisfied in the validity region (155) for $M_{lq1}^{\text{max,1}}$. Like in the discussion of (164), also this upper bound is below the point where $M_{lq1}^{\text{max}}$ is largest.

We can also look in which region $M_{lq1}^{\text{max,1}} \geq M_{lq1}^{\text{max,2}}$. This requires

\[
(M_G^2 - M_Q^2 + M_X^2 - \frac{M_X^2 M_Q^2}{M_R^2})(1 - \frac{M_R^2}{M_X^2}) \geq (M_G^2 - M_Q^2 + M_X^2 - \frac{M_X^2 M_Q^2}{M_R^2})(1 - \frac{M_R^2}{M_X^2})
\]

\[
(M_G^2 - M_Q^2)(1 - \frac{M_R^2}{M_X^2}) + (1 - \frac{M_R^2}{M_X^2})(M_X^2 - M_R^2) \geq (M_G^2 - M_Q^2)(1 - \frac{M_R^2}{M_X^2}) + (M_X^2 - M_R^2)(1 - \frac{M_R^2}{M_X^2})
\]

\[
(M_G^2 - M_Q^2)\left(\frac{M_0^2}{M_R^2} - \frac{M_R^2}{M_X^2}\right) \geq 0
\]

or

\[
M_R^2 \leq M_X M_0 \tag{174}
\]

Hence, $M_{lq1}^{\text{max,3}}$ and $M_{lq1}^{\text{max,2}}$ cross at $M_G = M_Q$ and one is larger than the other in the whole range $M_G \geq M_Q$ depending on $M_X$, $M_R$, and $M_0$.

On the other hand, $M_{lq1}^{\text{max,3}} \geq M_{lq2}^{\text{max}}$ is equivalent to

\[
(M_G^2 - M_Q^2) \frac{M_X^2 - M_Q^2}{M_X^2 - M_R^2} \geq (M_Q^2 - M_X^2) \frac{M_X^2 - M_Q^2}{M_X^2 - M_R^2} - (M_X^2 - M_R^2) \frac{M_R^2 - M_Q^2}{M_R^2}
\]

\[
M_G^2 - M_Q^2 \geq (M_Q^2 - M_X^2) \frac{M_X^2 - M_Q^2}{M_X^2 - M_R^2} - \frac{M_X^2}{M_R^2} (M_R^2 - M_Q^2)
\]

\[
M_G^2 - M_Q^2 \geq (M_Q^2 - M_X^2)(1 + \frac{M_R^2}{M_X^2 - M_R^2}) - \frac{M_X^2}{M_R^2} (M_R^2 - M_Q^2)
\]

\[
M_G^2 - M_Q^2 \geq (M_Q^2 - M_X^2) + \frac{M_Q^2}{M_X^2 - M_R^2} (M_R^2 - M_Q^2)
\]

\[
M_G^2 - M_Q^2 \geq (M_Q^2 - M_X^2) + \frac{M_Q^2 M_R^2 - M_X^2 M_R^2 - M_Q^2}{(M_X^2 - M_R^2) M_R^2} (M_R^2 - M_Q^2)
\]

or

\[
M_G^2 - M_Q^2 \geq (M_Q^2 - M_X^2) + \frac{M_Q^2 M_R^2 - M_X^2 M_R^2 - M_Q^2}{M_X^2 - M_R^2} \frac{M_R^2 - M_Q^2}{M_X^2 - M_R^2} \tag{175}
\]

Next, the requirement $M_{lq1}^{\text{max,3}} \geq M_{lq1}^{\text{max,2}}$ is equivalent to $M_{lq1}^{\text{max,1}} \geq M_{lq2}^{\text{max}}$ and hence

\[
M_{lq1}^{\text{max,1}} \geq (M_{lq2}^{\text{max}})
\]

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\[ M_Q^2 - M_{\bar{Q}}^2 \geq (M_Q^2 - M_{\bar{Q}}^2) \frac{M_X^2 - M_R^2}{M_R^2 - M_X^2} \]
\[ M_Q^2 - M_{\bar{Q}}^2 \geq (M_Q^2 - M_{\bar{Q}}^2) + (M_Q^2 - M_{\bar{Q}}^2) \frac{M_X^2 - M_R^2}{M_R^2 - M_X^2} - 1 \]
\[ M_Q^2 - M_{\bar{Q}}^2 \geq (M_Q^2 - M_{\bar{Q}}^2) + (M_Q^2 - M_{\bar{Q}}^2) \frac{M_X^2 M_R^2 - M_{\bar{Q}}^2 M_R^2 - M_{\bar{Q}}^2 M_R^2 + M_{\bar{Q}}^2}{(M_X^2 - M_R^2) M_R^2} \]

or
\[ M_Q^2 - M_{\bar{Q}}^2 \geq (M_Q^2 - M_{\bar{Q}}^2) + (M_Q^2 - M_{\bar{Q}}^2) \frac{M_X^2 - M_R^2}{M_R^2 - M_X^2} \] (176)

For \( M_{lq1}^{max,3} \geq M_{lq2}^{max,3} \), we need to have
\[ (M_{lq1}^{max})^2 \geq (M_{lq2}^{max})^2 \]
which happens, as we saw in Section 5.3, for
\[ M_Q^2 - M_{\bar{Q}}^2 \geq M_Q^2 - M_X^2 \] (177)

### 5.7.5 Fourth collinear end point in \( M(llq) \)

A fourth collinear end point can be obtained in the configuration \((b1)\), with the \((q2, \chi^0_1)\) system with mass labelled \( M_{qQ6} \) in Section 2.5. In this case
\[ (M_{lq1}^{max,4})^2 = 4E_{l1}(E_{q1} + E_{l2}) \] (178)
With
\[ E_{l1} = \frac{M_G M_X M_X^2 - M_R^2}{M_Q M_Q} = \frac{M_G}{2M_Q} (M_X^2 - M_R^2) \]
\[ E_{l2} = \frac{M_Q M_Q M_X M_X^2 - M_R^2}{M_G M_X M_X^2} = \frac{M_Q}{2M_G M_R} (M_R^2 - M_0^2) \]
we get
\[ E_{q1} + E_{l2} = \frac{M_G^2 - M_Q^2}{2M_G} + \frac{M_Q^2}{2M_G M_R} (M_R^2 - M_0^2) = \frac{M_G M_R^2 - M_Q^2 M_0^2}{2M_G M_R} \]
so that
\[ (M_{lq1}^{max,4})^2 = \frac{M_G}{M_Q} (M_X^2 - M_R^2) \frac{M_G M_R^2 - M_Q^2 M_0^2}{2M_G M_R} \]
\[ = \frac{M_G}{M_Q} (M_X^2 - M_R^2)(1 - \frac{M_Q^2 M_0^2}{M_G M_R^2}) \]
or
\[ (M_{lq1}^{max,4})^2 = \frac{M_G^2 M_X^2}{M_Q^2} (1 - \frac{M_Q^2}{(M_G M_R)^2})(1 - \frac{M_R^2}{M_X^2}) \] (179)

Putting it in the canonical form
\[ (M_{lq1}^{max,4})^2 = \frac{M_G^2 M_X^2}{M_Q^2} (1 - \frac{M_G^2 M_X^2}{M_G M_Q - M_G M_X + M_Q M_0}) \]
shows that its largest value is
\[ M_{lq1}^{max,4} = \frac{M_G M_X}{M_Q} - M_0 = M_G - \frac{M_G M_Q - M_G M_X + M_Q M_0}{M_Q} \] (180)
which is reached when
\[
\frac{M_G M_X}{M_R} = \frac{M_Q M_X}{M_Q} M_0 \quad \text{or} \quad M_G = \frac{M_Q M_X M_0}{M_R} \tag{181}
\]

Note that this result does not reproduce the mass \( M_{qOd} \). This is due to the fact that the \((llq1)\) system cannot be put at rest in the \( G = \tilde{g} \) frame for this configuration. Indeed, the momentum of \((llq1)\) is given by
\[
p_{llq1} = E_{l1} + E_{l2} - E_1 = \frac{M_G}{2} \left[ (1 - \frac{M_Q^2 M_0^2}{M_R^2 M_R^2}) - \frac{M_G^2}{M_R^2} (1 - \frac{M_R^2}{M_X^2}) \right]
\]
which becomes, by virtue of (181)
\[
p_{llq1} = \frac{M_G}{2} \left[ (1 - \frac{M_Q^2 M_0^2}{M_R^2 M_R^2}) - \frac{M_G^2}{M_R^2} (1 - \frac{M_R^2}{M_X^2}) \right] = \frac{M_G}{2} (1 - \frac{M_Q^2}{M_Q^2}) (1 - \frac{M_R^2}{M_X^2})
\]
we always have \( M_{llq1}^{max,4} \leq M_{llq1}^{max,3} \) and hence this will never give the true end point.

### 5.7.6 Fifth collinear end point in \( M(l\bar{l}q1) \)

A fifth collinear end point can be obtained in the configuration \((b2)\), with the \((q2, \chi_1^0)\) system with mass labelled \( M_{qOd} \) in Section 2.5, in which case
\[
(M_{llq1}^{max,5})^2 = 4E_{l1}(E_{l1} + E_{l2}) \tag{182}
\]
Then
\[
E_{l1} = \frac{M_G M_Q M_X - M_R}{2M_X M_R} = \frac{M_G^2}{2M_G M_X^2} (M_X^2 - M_R^2)
\]
\[
E_{l2} = \frac{M_G M_X M_Q M_R - M_0^2}{2M_R^2} = \frac{M_G M_X}{2M_R^2} (M_X^2 - M_0^2)
\]
and we get
\[
E_{l1} + E_{l2} = \frac{M_G^2}{2M_R^2} + \frac{M_Q^2}{2M_G M_X^2} (M_X^2 - M_R^2) = \frac{M_G M_X^2 - M_Q M_R^2}{2M_R^2 M_X^2}
\]
so that
\[
(M_{llq1}^{max,5})^2 = \frac{M_G M_X^2}{2M_R^2 M_X^2} (M_X^2 - M_R^2)
\]
or
\[
(M_{llq1}^{max,5})^2 = \frac{M_G^2 M_Q^2}{M_Q^2} (1 - (M_Q M_R)^2) (1 - M_R^2)
\]
which is largest when
\[
\frac{M_Q^2 M_R^2}{M_Q^2} (1 - (M_Q M_R)^2) (1 - M_R^2)
\]
where it reaches
\[
M_{llq1}^{max,5} = \frac{M_G M_X}{M_Q} - M_0 = M_G - \frac{M_G M_Q - M_G M_X + M_Q M_0}{M_Q} \tag{184}
\]

\[
M_{llq1}^{max,5} = \frac{M_G M_X}{M_Q} - M_0 = M_G - \frac{M_G M_Q - M_G M_X + M_Q M_0}{M_Q} \tag{185}
\]
Note that also here the \((llq1)\) system cannot be put at rest in the \(G = \bar{g}\) frame for this configuration. In this case, the momentum of \((llq1)\) is given by

\[
p_{llq1} = E_{q1} + E_{l1} - E_{l2} = \frac{M_G}{2} \left[ (1 - \frac{M_G^2 M_R^2}{M_Q^2 M_K^2}) - \frac{M_Q^2}{M_R^2} (1 - \frac{M_Q^2}{M_R^2}) \right]
\]

which becomes, by virtue of (184)

\[
p_{llq1} = \frac{M_G}{2} \left[ (1 - \frac{M_G^2}{M_R^2}) - \frac{M_Q^2}{M_R^2} (1 - \frac{M_G^2}{M_R^2}) \right] = \frac{M_G}{2} \left( 1 - \frac{M_Q^2}{M_R^2} \right) (1 - \frac{M_G^2}{M_R^2})
\]

which is non-zero and in the direction of \(q1\).

Finally, as

\[
\frac{M_G^2 M_K^2}{M_Q^2 M_K^2} (1 - \frac{M_Q^2}{M_R^2}) = \frac{M_G^2 M_K^2}{M_Q^2 M_K^2} - M_R^2 = \frac{M_G^2}{M_Q^2} - \frac{M_G^2 (M_Q^2 - M_K^2)}{M_R^2} - M_R^2
\]

we always have \(M_{llq1}^{\text{max},5} < M_{llq1}^{\text{max},2}\) and hence this will never give the true end point.

### 5.7.7 Sixth collinear end point in \(M(llq1)\)

A sixth collinear end point can be obtained in the configuration \((a1)\), in which case

\[
(M_{llq1}^{\text{max},6})^2 = 4E_{q1}(E_{l1} + E_{l2})
\]

With

\[
E_{l1} = \frac{M_G}{M_Q} M_X, \quad E_{l2} = \frac{M_G}{M_Q} M_R M_R^2 - \frac{M_0^2}{M_Q}
\]

we get

\[
(M_{llq1}^{\text{max},6})^2 = \frac{M_G^2 - M_Q^2}{M_G}\frac{M_Q^2}{M_Q} (M_K^2 - M_0^2)
\]

or

\[
(M_{llq1}^{\text{max},6})^2 = \frac{M_G^2}{M_Q} M_R^2 (1 - \frac{M_Q^2}{M_G^2}) (1 - \frac{M_G^2}{M_X^2})
\]

As \(M_X \leq M_Q\), this end point is always smaller than \(M_{llq1}^{\text{max}}\) of (154). It can never lead to the true maximum.

### 5.7.8 Summary of the end points in \(M(llq1)\)

The validity regions for the various expressions of \(M_{llq1}^{\text{max}}\) are displayed in Figure 20 which shows \((M_{llq1}^{\text{max}})^2\) as straight lines versus \(M_G^2\). Some of the conditions under which the collinear configurations of \(M_{llq1}^{\text{max}}\) or \(M_{llq2}^{\text{max}}\) give the true end point are summarized in Table 3.

To identify the true end point in \(M_{llq1}^{\text{max}}\), several cases need to be considered:

- \(\text{max}(M_Q, \text{eq.}(147), \text{eq.}(148)) \leq M_G \leq \text{eq.}(146)\): in this region the configuration is non-collinear and the true end point is given by formula (145). It is characterized by low values of \(M(qq)\) and a clustering around an intermediate value of \(M(ll)\). Its lower edge is given by (147) if \(M_R^2 \geq M_X M_0\) and by (148) if \(M_R^2 \leq M_X M_0\).

- if \(M_G^2 - M_Q^2 \geq \frac{M_X^2}{M_G^2} (M_K^2 - M_0^2)\): the true end point is given by \(M_{llq1}^{\text{max},1}\). It corresponds to the collinear configuration labelled \((c1)\) and is characterized by small values of \(M(qq)\) and \(M(ll)\). As seen from the comparison with Table 1, the true \(M(llq)\) end point can be \(M(llq1)\) (in the same configuration) or \(M(l2q1)\) (in another configuration).
There are six collinear configurations which can lead to the true end point.

Hence, the correlation between \( M(\text{llq}) \) and \( M(\text{ll}) \) (and the formula to be used) to be identified in a model independent way.

5.7.9 Conclusion on the \( M(\text{llq}) \) end points

There are six collinear configurations which can lead to the true end point.

- if \( M_G^2 - M_Q^2 \leq \frac{M_R^4}{M_R^2} (M_R^2 - M_G^2) \): (implies \( M_R^2 \geq M_0 M_X \)) in this range, the true end point is given by \( M_{\text{llq}1}^{\text{max},2} \) and corresponds to the collinear configuration labelled (d2). It is characterized by low values of \( M(qq) \), maximal \( M(\text{ll}) \) as well as \( M(\text{ll}q1) \) and \( M(\text{ll}q2) \). As seen from the comparison with Table 1, the \( M(\text{ll}q1) \) combination gives in this case the true end point.

- if \( M_G^2 - M_Q^2 \geq \frac{M_R^4}{M_R^2} (M_R^2 - M_G^2) \): (implies \( M_R^2 \leq M_0 M_X \)) in this range, the true end point is given by \( M_{\text{llq}1}^{\text{max},3} \) and corresponds to the collinear configuration labelled (d1). It is characterized by low values of \( M(qq) \), maximal \( M(\text{ll}) \) as well as \( M(\text{ll}q1) \) and \( M(\text{ll}q2) \). As seen from the comparison with Table 1, the \( M(\text{ll}q1) \) combination gives in this case the true maximum.

Hence, the correlation between \( M(\text{ll}q1) \) near its upper edge and \( M(\text{ll}) \), \( M(qq) \) and \( M(\text{ll}q) \) allows the type of end point (and the formula to be used) to be identified in a model independent way.

Table 3: Conditions associated to a given \( M_{\text{llq}}^{\text{max}} \) to give the true end point.

- if \( M_G^2 - M_Q^2 \leq \frac{M_R^4}{M_R^2} (M_R^2 - M_G^2) \): (implies \( M_R^2 \geq M_0 M_X \)) in this range, the true end point is given by \( M_{\text{llq}1}^{\text{max},2} \) and corresponds to the collinear configuration labelled (d2). It is characterized by low values of \( M(qq) \), maximal \( M(\text{ll}) \) as well as \( M(\text{ll}q1) \) and \( M(\text{ll}q2) \). As seen from the comparison with Table 1, the \( M(\text{ll}q1) \) combination gives in this case the true end point.

- if \( M_G^2 - M_Q^2 \geq \frac{M_R^4}{M_R^2} (M_R^2 - M_G^2) \): (implies \( M_R^2 \leq M_0 M_X \)) in this range, the true end point is given by \( M_{\text{llq}1}^{\text{max},3} \) and corresponds to the collinear configuration labelled (d1). It is characterized by low values of \( M(qq) \), maximal \( M(\text{ll}) \) as well as \( M(\text{ll}q1) \) and \( M(\text{ll}q2) \). As seen from the comparison with Table 1, the \( M(\text{ll}q1) \) combination gives in this case the true maximum.

5.7.9 Conclusion on the \( M(\text{llq}) \) end points

There are six collinear configurations which can lead to the true end point.

- if the \( (\text{ll}q) \) end point is not correlated to a particular value of \( M(qq) \), it is given by a \( (\text{ll}q2) \) combination. The strategy outlined in Section 4.4.5 can be followed.

- if the \( (\text{ll}q) \) end point is correlated with low values of \( M(qq) \), it is given by a \( (\text{ll}q1) \) combination. The strategy outlined in Section 5.7.8 should then be followed.

A summary of the main characteristics of the true \( M(\text{llq}) \) end points is given in Table 4.
Table 4: Main characteristics when a given \( M_{lq}^{\text{max}} \) gives the true end point.

<table>
<thead>
<tr>
<th>Config</th>
<th>( M(ll) )</th>
<th>( M(qq) )</th>
<th>( M(lq) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_{lq1}^{\text{max}} )</td>
<td>( c_1 )</td>
<td>min</td>
<td>min ((l1q1) \text{ or } (l2q1) \text{ max} )</td>
</tr>
<tr>
<td>( M_{lq2}^{\text{max}2} )</td>
<td>d2</td>
<td>max</td>
<td>min ((l2q1) \text{ max is true} )</td>
</tr>
<tr>
<td>( M_{lq1}^{\text{max}3} )</td>
<td>d1</td>
<td>max</td>
<td>min ((l1q1) \text{ max is true} )</td>
</tr>
<tr>
<td>( M_{lq2}^{\text{max}2} )</td>
<td>a2 + c1</td>
<td>min</td>
<td>any ((l1q2) \text{ or } (l2q2) \text{ max} )</td>
</tr>
<tr>
<td>( M_{lq2}^{\text{max}3} )</td>
<td>b1 + d2</td>
<td>max</td>
<td>any ((l2q2) \text{ max is true} )</td>
</tr>
<tr>
<td>( M_{lq2}^{\text{max}} )</td>
<td>b2 + d1</td>
<td>max</td>
<td>any ((l1q2) \text{ max is true} )</td>
</tr>
</tbody>
</table>

5.8 Upper end point in \( M(l1qq) \)

The configurations leading to a maximum of \( M(l1qq) \) are similar to the ones for \( M(llq) \) discussed in section 4.4. The formulae of that section can be translated in a straightforward way.

5.8.1 Absolute maximum of \( M(l1qq) \)

The absolute maximum of \( M(l1qq) \) is reached when the slepton is at rest in the frame of \( G = \tilde{g} \) and amounts to

\[
M_{l1qq}^{\text{max}} = M_G - M_R
\]  
(189)

The conditions for the closure of the momentum triangle, which define the region of validity of this formula, then become

\[
\frac{M_X^2}{M_R} \leq M_T \leq \frac{M_Q^2}{M_R} 
\]  
(190)

and

\[
M_G M_X^2 \geq M_Q^2 M_R
\]  
(191)

5.8.2 First collinear end point in \( M(l1qq) \)

The maximum of \( M(l1qq) \) is reached for the configuration of figure 21, which is part of the configurations

\[
\frac{M^2_{G}}{M^2_{G}} \leq \frac{M^2_{Q}}{M^2_{R}} \leq \frac{M^2_{R}}{M^2_{G}}
\]  
(189)

Figure 21: Configuration leading to the maximum of the mass for \((l1qq)\).

\[
M_{l1qq}^{\text{max}} = M_G \sqrt{1 - \frac{M_Q^2}{M_T^2} (1 - \frac{M_R^2}{M_T^2})}
\]  
(192)

valid provided the slepton is not sent backwards in the \( G = \tilde{g} \) frame, i.e. \( M_Q^2 \leq M_G M_R \). Else the true end point is given by the non-collinear configuration and is \( M_{l1qq}^{\text{max}} = M_G - M_R \).

5.8.3 Second collinear end point in \( M(l1qq) \)

A second collinear end point, obtained by flipping the direction of the \( \tilde{\chi}^0_2 \) in the squark decay, is part of the configurations labelled \((c_1)\) and \((d_1)\). As they appear in the same configuration, it can be written as

\[
(M_{l1qq}^{\text{max}2})^2 = (M_{l1q1})^2 + (M_{l1q2})^2
\]  
(193)
It is given as a function of the sparticle masses, in analogy with formula (85), by

\[ M_{l1qq}^{\text{max},2} = M_G \sqrt{(1 - \frac{M_X^2}{M_G^2})(1 - \frac{M_R^2}{M_X^2})} \]  \hspace{1cm} (194)

Its region of validity is \( M_G M_R \leq M_X^2 \).

The condition under which \( M_{l1qq}^{\text{max},2} \geq M_{l1qq}^{\text{max}} \) is

\[
\frac{(M_{l1}^2 - M_X^2)(M_{l1}^2 - M_R^2)}{M_X^2} \quad \geq \quad \frac{(M_Q^2 - M_X^2)(M_Q^2 - M_R^2)}{M_Q^2}
\]

\[
(M_{l1}^2 - M_X^2)(M_{l1}^2 - M_R^2) \quad \geq \quad (M_Q^2 - M_X^2)(M_Q^2 - M_R^2)
\]

\[
-M_Q M_{l1}^2 M_R - M_Q M_X^2 \quad \geq \quad -M_Q M_{l1}^2 M_R - M_Q M_X^2
\]

\[
M_{l1}^2 M_R^2 (M_Q^2 - M_X^2) \quad \leq \quad M_Q M_X^2 (M_Q^2 - M_X^2)
\]

and so

\[ M_G \leq \frac{M_Q M_X}{M_R} \]  \hspace{1cm} (195)

which is always fulfilled below the lower edge of the condition (190).

### 5.8.4 Third collinear end point in \( M(l1qq) \)

A third collinear end point is obtained by flipping the direction of the \( \tilde{\chi}_Q^0 \) decay products and is part of the configurations labelled \((a_2)\) and \((b_2)\). As they appear in the same configuration, it can be written as

\[ (M_{l1qq}^{\text{max},3})^2 = (M_{l1qq}^{\text{max}})^2 + (M_{l1qq}^{\text{max},2})^2 \]  \hspace{1cm} (196)

It is given as a function of the sparticle masses, in analogy with formula (90), by

\[ M_{l1qq}^{\text{max},3} = M_G \sqrt{(1 - \frac{M_Q M_R}{M_X^2})^2(1 - \frac{M_X^2}{M_Q^2})} \]  \hspace{1cm} (197)

Its region of validity is \( M_G M_X^2 \leq M_Q^2 M_R \).

This maximum can be larger than \( M_{l1qq}^{\text{max}} \) provided

\[ M_G \leq \frac{M_Q}{M_X} \]  \hspace{1cm} (198)

which is always satisfied below the lower edge of the condition (191).

Also \( M_{l1qq}^{\text{max},3} \geq M_{l1qq}^{\text{max},2} \) when

\[ M_X^2 \leq M_Q M_R \]  \hspace{1cm} (199)

### 5.8.5 Summary of the end points in \( M(l1qq) \)

A summary of the validity regions for the end points of \( M(l1qq) \) is illustrated in figure 22.

It may be worth noting for later that if \( M(l1qq) \) is the true end point, the relation \( M_G \geq M_Q^2 / M_R \) implies

\[ M_{l1}^2 - M_{l1}^2 \geq \frac{M_Q^2}{M_R^2} - M_{l1}^2 \geq \frac{M_Q^2}{M_X^2} - M_{l1}^2 = \frac{M_Q^2}{M_X^2} (M_Q^2 - M_X^2) \geq M_Q^2 - M_X^2 \]

and hence the true \((lq)\) end point can be \((l1q)\) or \((l2q)\).

### 5.9 Upper end point in \( M(l2qq) \)

The maximum of \( M(l2qq) \) is obtained when the particles \((l1 \tilde{\chi}_Q^0)\) not included for the effective mass have their minimal mass (are parallel). The formulae can be obtained directly from the \((l1qq)\) case by replacing \( M_R \) by \( M_{110} = \frac{M_X M_6}{M_R} \).
Figure 22: Example of the mass regions leading to maxima for $M(l1qq)$.

5.9.1 Absolute maximum of $M(l2qq)$

The absolute maximum of $M(l2qq)$ is reached when the $(l1\bar{q}0)$ system is at rest in the frame of $G = \bar{g}$ and amounts to

$$M_{l2qq}^{\text{max}} = M_G - M_{l10} = M_G - \frac{M_X M_0}{M_R}$$  \hfill (200)

The conditions which define the region of validity of this formula then become

$$\frac{M_X^2}{M_{l10}} \leq M_G \leq \frac{M_Q^2}{M_{l10}}$$  \hfill (201)

or

$$\frac{M_R}{M_0} M_X \leq M_G \leq \frac{M_Q M_R}{M_X M_0} M_Q$$  \hfill (202)

and

$$M_G \geq \frac{M_Q^2}{M_X} M_{l10} = \frac{M_Q M_0}{M_X M_R} M_Q$$  \hfill (203)

5.9.2 First collinear end point in $M(l2qq)$

The maximum of $M(l2qq)$ corresponds to the configuration labelled $(b_2)$. It can be viewed as the recoil of a massless $(l2q2)$ system against $(l1\bar{q}0)$ in the $Q = \bar{q}$ rest frame.

It is similar to the one leading to equation (192). The maximum mass is then given by

$$M_{l2qq}^{\text{max}} = M_G \sqrt{1 - \frac{M_Q^2}{M_G^2}(1 - \frac{M_{l10}^2}{M_Q^2})}$$  \hfill (204)

or

$$M_{l2qq}^{\text{max}} = M_G \sqrt{1 - \frac{M_Q^2}{M_G^2}(1 - \frac{(M_X M_0)^2}{(M_Q M_R)^2})}$$  \hfill (205)

which is valid provided the $(l1\bar{q}0)$ system is not sent backwards in the $G = \bar{g}$ frame, i.e. $M_G \geq \frac{M_Q^2}{M_{l10}}$, above the upper edge of the range (202). Else the true end point is given by the non-collinear configuration above.

The condition yielding $M_{l2qq}^{\text{max}} \geq M_{l1qq}^{\text{max}}$ is easily derived from the comparison of (192) and (205). It requires

$$\frac{M_R}{M_Q} \geq \frac{M_X M_0}{M_Q M_R}$$
or

\[ M_R^2 \geq M_X M_0 \]

i.e. the same as the condition which guarantees \( M_{\ell 2q}^{\text{max}} \geq M_{\ell 1q}^{\text{max}} \).

For \( M_{\ell 2q}^{\text{max}} \geq M_{\ell 1qq}^{\text{max}} \), we need to have

\[
(M_Q^2 - M_Q^2)(1 - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2}) \geq (M_Q^2 - M_X^2)(1 - \frac{M_R^2}{M_X^2})
\]

or

\[
(M_Q^2 - M_X^2)(\frac{M_R^2}{M_Q^2 M_R^2} - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2}) \geq (M_Q^2 - M_X^2)(1 - \frac{M_R^2}{M_X^2})
\]

Alternatively, expressed purely as a function of mass ratios

\[
M_Q^2 (1 - \frac{M_R^2}{M_Q^2})(1 - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2}) \geq M_Q^2 (1 - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2})(1 - \frac{M_R^2}{M_X^2})
\]

\[
\frac{M_Q^2 M_R^2}{M_Q^2 M_R^2} - \frac{M_R^2}{M_Q^2 M_R^2} - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2} \geq \frac{M_Q^2 M_R^2}{M_Q^2 M_R^2} - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2} - \frac{M_R^2}{M_X^2}
\]

or

\[
\frac{M_Q^2 M_R^2}{M_Q^2 M_R^2} - \frac{M_R^2}{M_Q^2 M_R^2} - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2} \geq \frac{M_Q^2 M_R^2}{M_Q^2 M_R^2} - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2} - \frac{M_R^2}{M_X^2}
\]

\[
M_Q^2 (1 - \frac{M_R^2}{M_Q^2})(1 - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2}) \geq M_Q^2 (1 - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2})(1 - \frac{M_R^2}{M_X^2})
\]

or

\[
(M_Q^2 - M_X^2)(\frac{M_R^2}{M_Q^2 M_R^2} - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2}) \geq (M_Q^2 - M_X^2)(1 - \frac{M_R^2}{M_X^2})
\]

And for \( M_{\ell 2q}^{\text{max}} \geq M_{\ell 1qq}^{\text{max}} \), we need

\[
\frac{M_Q^2 - M_Q^2}{M_Q^2} (1 - \frac{M_R^2}{M_Q^2 M_R^2}) \geq \frac{M_Q^2 - M_X^2}{M_Q^2}(1 - \frac{M_R^2}{M_X^2})
\]

or

\[
(M_Q^2 - M_X^2)(\frac{M_R^2}{M_Q^2 M_R^2} - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2}) \geq (M_Q^2 - M_X^2)(1 - \frac{M_R^2}{M_X^2})
\]

Alternatively, expressed purely as a function of mass ratios

\[
M_Q^2 (1 - \frac{M_R^2}{M_Q^2})(1 - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2}) \geq M_Q^2 (1 - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2})(1 - \frac{M_R^2}{M_X^2})
\]

\[
\frac{M_Q^2 M_R^2}{M_Q^2 M_R^2} - \frac{M_R^2}{M_Q^2 M_R^2} - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2} \geq \frac{M_Q^2 M_R^2}{M_Q^2 M_R^2} - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2} - \frac{M_R^2}{M_X^2}
\]

or

\[
\frac{M_Q^2 M_R^2}{M_Q^2 M_R^2} - \frac{M_R^2}{M_Q^2 M_R^2} - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2} \geq \frac{M_Q^2 M_R^2}{M_Q^2 M_R^2} - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2} - \frac{M_R^2}{M_X^2}
\]

\[
M_Q^2 (1 - \frac{M_R^2}{M_Q^2})(1 - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2}) \geq M_Q^2 (1 - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2})(1 - \frac{M_R^2}{M_X^2})
\]

or

\[
(M_Q^2 - M_X^2)(\frac{M_R^2}{M_Q^2 M_R^2} - \frac{M_X^2 M_R^2}{M_Q^2 M_R^2}) \geq (M_Q^2 - M_X^2)(1 - \frac{M_R^2}{M_X^2})
\]

5.9.3 Second collinear end point in \( M(\ell 2qq) \)

A second collinear end point, obtained by flipping the direction of the \( \tilde{\chi}_2^0 \) in the squark decay, labelled \( (d_2) \). As they belong to the same configuration, we have

\[
(M_{\ell 2qq}^{\text{max}})^2 = (M_{\ell 2q}^{\text{max}})^2 + (M_{\ell 1q2}^{\text{max}})^2
\]

It is given in analogy with formula (194)

\[
M_{\ell 2qq}^{\text{max},2} = M_G \sqrt{(1 - \frac{M_X^2}{M_G^2})(1 - \frac{M_{110}^2}{M_X^2})}
\]
or

\[ M_{12qq}^{\text{max},2} = M_G \sqrt{1 - \frac{M_X^2}{M_G^2}(1 - \frac{M_0^2}{M_R^2})} \]  

(212)

It reaches the absolute maximum value for \( M_X M_R = M_G M_0 \). Its region of validity is \( M_X M_R \geq M_G M_0 \) or \( M_X^2 \geq M_G M_{110} = M_G \frac{M_X M_0}{M_R} \), below the lower edge of the range (202).

We still need to check whether \( M_{12qq}^{\text{max},2} \) may give the true end point.

To have \( M_{12qq}^{\text{max},2} \geq M_{11qq}^{\text{max}} \) requires, in analogy with (86),

\[
(M_G^2 - M_X^2) \frac{M_R^2 - M_0^2}{M_R^2} \geq (M_G^2 - M_Q^2) \frac{M_R^2 M_R^2 - M_Q^2 M_Q^2}{M_R^2 M_R^2}
\]

\[
(M_G^2 - M_X^2) (M_Q^2 M_R^2 - M_Q^2 M_R^2) \geq (M_G^2 - M_Q^2) (M_Q^2 M_R^2 - M_Q^2 M_Q^2)
\]

\[
-M_G^2 M_Q^2 M_R^2 - M_G^2 M_R^2 M_Q^2 \geq -M_G^2 M_Q^2 M_R^2 - M_Q^2 M_R^2 M_Q^2
\]

\[
M_G^2 (M_Q^2 - M_X^2) - M_Q^2 M_R^2 (M_Q^2 - M_X^2) \leq 0
\]

or

\[ M_G M_0 \leq M_Q M_R \]  

(213)

This region falls below the lower edge of the range (202), as \( M_X \) is smaller than \( M_Q \). Hence, \( M_{12qq}^{\text{max},2} \) may give the true end point of \( M_{12qq} \) in its range of validity.

For the inequality \( M_{12qq}^{\text{max},2} \geq M_{11qq}^{\text{max}} \) we need

\[
(M_G^2 - M_X^2)(1 - \frac{M_0^2}{M_R^2}) \geq (M_G^2 - M_Q^2)(1 - \frac{M_0^2}{M_R^2})
\]

\[
M_G^2 \left( \frac{M_Q^2}{M_R^2} - G \frac{M_0^2}{M_R^2} \right) + M_X^2 \left(1 - \frac{M_0^2}{M_R^2} \right) \leq M_Q^2 \left(1 - \frac{M_0^2}{M_R^2} \right)
\]

or

\[
(M_G^2 - M_Q^2) \left( \frac{M_0^2}{M_R^2} - \frac{M_R^2}{M_Q^2} \right) \leq (M_Q^2 - M_X^2) \left(1 - \frac{M_0^2}{M_R^2} \right)
\]

(214)

Alternatively, expressed purely as a function of mass ratios

\[
M_G^2 \left(1 - \frac{M_X^2}{M_G^2} \right) \frac{M_Q^2}{M_R^2} \left(1 - \frac{M_0^2}{M_R^2} \right) \geq M_G^2 \left(1 - \frac{M_Q^2}{M_G^2} \right) \frac{M_R^2}{M_Q^2} \left(1 - \frac{M_0^2}{M_R^2} \right)
\]

\[
\frac{M_G^2}{M_R^2} \frac{M_Q^2}{M_R^2} \frac{M_0^2}{M_R^2} \left(1 - \frac{M_0^2}{M_R^2} \right) \geq \frac{M_G^2}{M_Q^2} \frac{M_R^2}{M_Q^2} \frac{M_0^2}{M_R^2} \left(1 - \frac{M_0^2}{M_R^2} \right)
\]

or

\[
\frac{M_G^2}{M_R^2} \frac{M_Q^2}{M_R^2} \left( \frac{M_Q^2}{M_R^2} - \frac{M_0^2}{M_R^2} \right) + \frac{M_G^2}{M_Q^2} \left( \frac{M_Q^2}{M_R^2} - \frac{M_0^2}{M_R^2} \right) + \left( \frac{M_G^2}{M_R^2} - \frac{M_0^2}{M_R^2} \right) \geq 0
\]

(215)

For \( M_{12qq}^{\text{max},2} \geq M_{11qq}^{\text{max}} \)

\[
(M_G^2 - M_X^2) \frac{M_R^2 - M_0^2}{M_R^2} \geq (M_G^2 - M_X^2) \frac{M_R^2 - M_0^2}{M_X^2}
\]

hence

\[ M_R^2 \geq M_X M_0 \]  

(216)

i.e. the same condition as for \( M_{12qq}^{\text{max}} \geq M_{11qq}^{\text{max}} \).
Finally, for $M_{12qq}^{\text{max},2} \geq M_{12qq}^{\text{max},3}$

\[
(M_Q^2 - M_X^2)(1 - \frac{M_Q^2}{M_R^2}) \geq (M_Q^2 - M_X^2) \frac{M_Q^2 M_X^2 - M_Q^2 M_R^2}{M_Q^2 M_X^2}
\]

\[
(M_Q^2 - M_Q^2)(1 - \frac{M_Q^2}{M_R^2}) \geq (M_Q^2 - M_X^2) \left[ \frac{M_Q^2 M_X^2 - M_Q^2 M_R^2}{M_Q^2 M_X^2} - \frac{M_R^2 - M_Q^2}{M_R^2} \right]
\]

\[
(M_Q^2 - M_Q^2)(1 - \frac{M_Q^2}{M_R^2}) \geq \frac{M_Q^2 M_X^2 - M_Q^2 M_R^2}{M_Q^2 M_X^2}
\]

\[
(M_Q^2 - M_Q^2) \frac{M_Q^2 M_R^2 - M_Q^2 M_0^2}{M_Q^2 M_X^2} \geq (M_Q^2 - M_X^2) M_Q^2 M_R^2 - M_Q^2 (M_R^2 - M_X^2 M_0^2)
\]

or

\[
(M_Q^2 - M_Q^2) \left( \frac{M_Q^2}{M_R^2} - \frac{M_Q^2}{M_X^2} \right) \geq (M_Q^2 - M_X^2) \left( \frac{M_Q^2}{M_R^2} - \frac{M_R^2}{M_X^2} \right)
\] (217)

Alternatively, expressed purely as a function of mass ratios

\[
M_Q^2 (1 - \frac{M_X^2}{M_R^2}) (1 - \frac{M_Q^2}{M_G^2}) - M_Q^2 (1 - \frac{M_X^2}{M_R^2}) (1 - \frac{M_Q^2}{M_G^2}) \geq \frac{M_Q^2 M_X^2 - M_Q^2 M_R^2}{M_Q^2 M_X^2} - \frac{M_R^2 - M_Q^2}{M_R^2}
\]

or

\[
\frac{M_Q^2 M_X^2 - M_Q^2 M_R^2}{M_Q^2 M_X^2} + \frac{M_Q^2 M_X^2 - M_Q^2 M_R^2}{M_Q^2 M_X^2} + \frac{M_X^2 M_R^2 - M_X^2 M_Q^2}{M_Q^2 M_X^2} \geq 0
\] (218)

5.9.4 Third collinear end point in $M(12qq)$

A third collinear end point, obtained by flipping the direction of the $\tilde{\chi}^0_2$ decay products, labelled $b_1$. As they are in the same configuration, we have

\[
(M_{12qq}^{\text{max},3})^2 = (M_{12qq}^{\text{max},2})^2 + (M_{12qq}^{\text{max},2})^2
\] (219)

It is given, in analogy with formula (197), by

\[
M_{12qq}^{\text{max},3} = M_G \sqrt{1 - \frac{(M_Q M_{110})^2}{(M_G M_X)^2}(1 - \frac{M_X^2}{M_Q^2})}
\] (220)

or

\[
M_{12qq}^{\text{max},3} = M_G \sqrt{1 - \frac{(M_Q M_{110})^2}{(M_G M_X)^2}(1 - \frac{M_X^2}{M_Q^2})}
\] (221)

It reaches the absolute maximum value for $M_G M_X^2 = M_Q^2 M_{110} = M_Q^2 M_X^2 M_R^2$. Its region of validity is $M_G M_X^2 \leq M_Q^2 M_{110} = M_Q^2 \frac{M_X^2}{M_R^2}$. This maximum can be larger than $M_{12qq}^{\text{max},2}$ provided

\[
M_Q^2 \geq M_G M_X
\] (222)

Also $M_{12qq}^{\text{max},3} \geq M_{12qq}^{\text{max},2}$ when

\[
(M_Q^2 - M_X^2) \frac{M_Q^2 M_R^2 - M_Q^2 M_0^2}{M_Q^2 M_X^2} \geq (M_Q^2 - M_X^2) \frac{M_R^2 - M_Q^2}{M_R^2}
\]

\[
- M_Q^2 M_X^2 M_0^2 - M_Q^2 M_X^2 M_R^2 \geq - M_Q^2 M_X^2 M_0^2 - M_Q^2 M_X^2 M_R^2
\]

\[
(M_Q^2 - M_Q^2) M_Q^2 M_0^2 \geq (M_Q^2 - M_Q^2) M_Q^2 M_R^2
\]
Alternatively, expressed purely as a function of mass ratios

\[ M_G M_0 \geq M_X M_R \]

For \( M_{12qq}^{\text{max},3} \geq M_{11qq}^{\text{max}} \) it is required that

\[ M_G^2 \frac{M_Q^2 - M_X^2}{M_Q^2} (1 - \frac{M_Q^2 M_0^2}{M_Q^2 M_R^2}) \geq (M_G^2 - M_X^2)(1 - \frac{M_R^2}{M_Q^2}) \]

or

\[ (M_G^2 - M_X^2) \frac{M_Q^2}{M_Q^2} (1 - \frac{M_Q^2 M_0^2}{M_Q^2 M_R^2}) \leq (M_Q^2 - M_X^2)(1 - \frac{M_R^2}{M_Q^2}) \] (223)

Alternatively, expressed purely as a function of mass ratios

\[ M_G^2 (1 - \frac{M_X^2}{M_Q^2}) (1 - \frac{M_Q^2 M_0^2}{M_Q^2 M_R^2}) \geq M_G^2 (1 - \frac{M_X^2}{M_Q^2}) (1 - \frac{M_R^2}{M_Q^2}) \]

\[ M_G^2 \frac{M_Q^2 - M_X^2}{M_Q^2} (1 - \frac{M_Q^2 M_0^2}{M_Q^2 M_R^2}) \geq M_G^2 \frac{M_G^2}{M_Q^2} (1 - \frac{M_R^2}{M_Q^2}) \]

or

\[ M_G^2 \frac{M_Q^2}{M_Q^2} (1 - \frac{M_Q^2 M_0^2}{M_Q^2 M_R^2}) \leq M_G^2 (1 - \frac{M_Q^2}{M_X^2}) \left[ 1 - \frac{M_Q^2}{M_G^2} + \frac{M_Q^2}{M_X^2} \right] \] (224)

To have \( M_{12qq}^{\text{max},3} \geq M_{11qq}^{\text{max},2} \)

\[ M_G^2 \frac{M_Q^2 - M_X^2}{M_Q^2} (1 - \frac{M_Q^2 M_0^2}{M_Q^2 M_R^2}) \geq (M_G^2 - M_X^2)(1 - \frac{M_R^2}{M_X^2}) \]

or

\[ (M_G^2 - M_X^2) \frac{M_Q^2}{M_Q^2} (1 - \frac{M_Q^2 M_0^2}{M_Q^2 M_R^2}) \leq (M_Q^2 - M_X^2)(1 - \frac{M_R^2}{M_X^2}) \] (225)

Alternatively, expressed purely as a function of mass ratios

\[ M_G^2 (1 - \frac{M_X^2}{M_Q^2}) (1 - \frac{M_R^2}{M_Q^2}) \geq M_G^2 (1 - \frac{M_X^2}{M_Q^2}) (1 - \frac{M_R^2}{M_X^2}) \]

\[ M_G^2 \frac{M_Q^2 - M_X^2}{M_Q^2} (1 - \frac{M_Q^2 M_0^2}{M_Q^2 M_R^2}) \geq M_G^2 \frac{M_G^2}{M_Q^2} (1 - \frac{M_R^2}{M_X^2}) \]

or

\[ M_G^2 \frac{M_Q^2}{M_Q^2} (1 - \frac{M_Q^2 M_0^2}{M_Q^2 M_R^2}) \leq M_G^2 (1 - \frac{M_Q^2}{M_X^2}) \left[ 1 - \frac{M_Q^2}{M_G^2} + \frac{M_Q^2}{M_X^2} \right] \] (226)

Finally, for \( M_{12qq}^{\text{max},3} \geq M_{11qq}^{\text{max},3} \) it is required that

\[ \frac{M_R^2 M_Q^2 - M_X^2 M_0^2}{M_R^2} \geq \frac{M_Q^2 M_X^2 - M_Q^2 M_R^2}{M_X^2} \]

\[ -M_Q^2 M_X^2 M_0^2 \geq -M_Q^2 M_R^2 \]

or again

\[ M_R^2 \geq M_X M_0 \]

the same condition as the one ensuring that \( M_{12qq}^{\text{max}} \geq M_{11qq}^{\text{max}} \) and \( M_{12qq}^{\text{max},2} \geq M_{11qq}^{\text{max},2} \).

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5.9.5 Summary of the end points in $M(l2qq)$

A summary of the validity regions for the end points of $M(l2qq)$ is shown in figure 23.

It may be worth noting for later that if $M_{l2qq}^{\text{max},1}$ is the true end point, the relation $M_G \geq \frac{M_0}{M_\mathbb{X} M_\mathbb{M}} M_Q^2$ implies

$$M_G^2 - M_Q^2 \geq \frac{M_\mathbb{X}^2 M_Q^4}{M_0 M_\mathbb{M}^2} - M_Q^2 \geq \frac{M_\mathbb{X}^2}{M_\mathbb{M}} (M_G^2 - M_Q^2) \geq M_Q^2 - M_X^2$$

and hence the true $(lq)$ end point can be $(l1q1)$ or $(l2q1)$.

5.10 Conclusion on the $M(lqq)$ end points

There are six collinear configurations which can lead to a true end point. The identification and the conditions under which the true maximum of $M(lqq)$ is obtained can be summarized as follows. A general rule is that:

- for $M_R^2 \geq M_\mathbb{X} M_0$ we have $M_{l2qq}^{\text{max},i} \geq M_{l1qq}^{\text{max},i}$ (for same value of $i$) and the true end point is given by $M_{l2qq}^{\text{max},i}$.

- for $M_R^2 \leq M_\mathbb{X} M_0$ we have $M_{l1qq}^{\text{max},i} \geq M_{l2qq}^{\text{max},i}$ (for same value of $i$) and the true end point is given by $M_{l1qq}^{\text{max},i}$.

Within each of these two categories, the conditions under which a given $M_{lqq}^{\text{max},i}$ gives rise to the true end point are listed in Table 5.

| $M_{lqq}^{\text{max},1}$ | \begin{array}{c|c|c|c} M_X & M_R & M_Q \\ \hline M_R & M_X & M_Q \\ \hline M_Q & M_R & M_X \\ \end{array} | $M_{lqq}^{\text{max},2}$ | \begin{array}{c|c|c|c} M_X & M_R & M_Q \\ \hline M_R & M_X & M_Q \\ \hline M_Q & M_R & M_X \\ \end{array} | $M_{lqq}^{\text{max},3}$ | \begin{array}{c|c|c|c} M_X & M_R & M_Q \\ \hline M_R & M_X & M_Q \\ \hline M_Q & M_R & M_X \\ \end{array} |

Table 5: Conditions associated to a given ($M_{lqq}^{\text{max}}$) to give the true end point.

The identification of the true end point can be based on the following:

- if $\min(M_Q, \frac{M_\mathbb{X}^2 M_\mathbb{M}}{M_\mathbb{N}} M_Q^2 M_R^2) \leq M_G \leq \frac{M_\mathbb{X}^2}{M_\mathbb{N}}$ and the $M_{l1qq}$ end point corresponds to the absolute maximum, $M_{l1qq}^{\text{max}} = M_G - M_R$ in a non-collinear configuration. The first lower edge happens when

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure23.png}
\caption{Example of the mass regions leading to maxima for $M(l2qq)$.}
\end{figure}
\(M_X^2 \geq M_Q M_R\), the second one when \(M_X^2 < M_Q M_R\). If \(\min(M_Q, \frac{M_X M_R}{M_y}, \frac{M_C^2 M_0}{M_X M_R}) \leq \frac{M_G^2 M_0}{M_X M_R}\), the \(M_{12qq}\) end point corresponds to the absolute maximum \(M_{12qq}^{\max} = M_G - M_0\) in a non-collinear configuration. If the true end point is unrelated to \(M(\ll qq)\), it is given by \(M_{11qq}\). If it corresponds to a clustering at near maximum \(M(\lll)\), it is given by \(M_{12qq}\).

- if the true end point is given by \(M_{11qq}^{\max,1}\), the configuration is \((a1)\) or \((b1)\), characterized by any \(M(\lll)\), near maximum \(M(\ll qq)\) and low \(M(\ll 1q2)\). The conditions on masses are given in Table 5. As hinted to in Section 5.8.5 these conditions imply that the true \(\ll qq\) end point is from \((\ll 1q1)\) but, as it does not occur in the same configuration, there is no correlation between the true end points of \((\ll qq)\) and \((\ll qq)\).

- if the true end point is given by \(M_{11qq}^{\max,2}\), the configuration is \((c1)\) or \((d1)\), characterized by any \(M(\lll)\), low \(M(\ll qq)\) and maximum \(M(\ll 1q1)\) and \(M(\ll 1q2)\). The conditions on masses are given in Table 5. By comparing to Section 5.5, it is seen that either \(M(\ll 1q1)\) or \(M(\ll 1q2)\) is the true end point in the same configuration.

- if the true end point is given by \(M_{11qq}^{\max,3}\), the configuration is \((a2)\) or \((b2)\), characterized by any \(M(\lll)\), maximum \(M(\ll qq)\) and \(M(\ll 1q2)\). The conditions on masses are given in Table 5. By comparing to Section 5.5, it is seen that the maximum of \(M(\ll 1q2)\) may be the true end point in this case.

- if the true end point is given by \(M_{12qq}^{\max,1}\), the configuration is \((b2)\), characterized by maximum \(M(\lll)\), \(M(\ll qq)\) and \(M(\ll 1q2)\) and low \(M(\ll 2q2)\). The conditions on masses are given in Table 5 which imply that the maximum of \(M(\ll 2q1)\) is the true end point but, as it does not occur in the same configuration, there is no correlation between the true end points of \((\ll qq)\) and \((\ll qq)\).

- if the true end point is given by \(M_{12qq}^{\max,2}\), the configuration is \((d2)\), characterized by maximum \(M(\lll)\), low \(M(\ll qq)\), maximum \(M(\ll 1q2)\) and \(M(\ll 2q1)\). The conditions on masses are given in Table 5. By comparing to Section 5.5, it is seen that either \(M(\ll 1q1)\) or \(M(\ll 1q2)\) is the true end point.

- if the true end point is given by \(M_{12qq}^{\max,3}\), the configuration is \((b1)\), characterized by maximum \(M(\lll)\), \(M(\ll qq)\) and \(M(\ll 1q2)\). The conditions on masses are given in Table 5. By comparing to Section 5.5, it is seen that the maximum of \(M(\ll 2q2)\) may be the true end point in this case.

A summary of the main characteristics of the true \(M(\ll qq)\) end points is given in Table 6.

<table>
<thead>
<tr>
<th>Config</th>
<th>(M(\lll))</th>
<th>(M(\ll qq))</th>
<th>(M(\ll lqq))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M_{11qq}^{\max,1})</td>
<td>(a1 + b1)</td>
<td>any</td>
<td>max</td>
</tr>
<tr>
<td>(M_{11qq}^{\max,2})</td>
<td>(c1 + d1)</td>
<td>any</td>
<td>min</td>
</tr>
<tr>
<td>(M_{11qq}^{\max,3})</td>
<td>(a2 + b2)</td>
<td>any</td>
<td>max</td>
</tr>
<tr>
<td>(M_{12qq}^{\max,1})</td>
<td>(b2)</td>
<td>max</td>
<td>max</td>
</tr>
<tr>
<td>(M_{12qq}^{\max,2})</td>
<td>(d2)</td>
<td>max</td>
<td>min</td>
</tr>
<tr>
<td>(M_{12qq}^{\max,3})</td>
<td>(b1)</td>
<td>max</td>
<td>max</td>
</tr>
</tbody>
</table>

Table 6: Main characteristics when a given \(M_{lqq}^{\max}\) gives the true end point.

5.11 Upper end point in \(M(\llqq)\)

For the \((\llqq)\) system, there are 7 different collinear configurations. In the following, we will derive the end points for the 4 configurations where one of the particles included in the system recoils against all the others.

5.11.1 Absolute maximum of \(M(\llqq)\)

The absolute maximum of \(M(\llqq)\) is obtained when the \(\bar{\chi}^1_{\lambda}\) is at rest in the \(G = \bar{g}\) frame:

\[
M_{lqq}^{\max} = M_G - M_0
\]
and the energy conservation equation is

\[ M = E_{q_1} + E_{q_2} + E_{l_1} + E_{l_2} \]  

(228)

The conditions to be satisfied for the existence of such configuration can be derived in analogy with the \( M(llq) \) case of section 4.4.1. The requirement that the sum of the 4 momentum vectors close is

\[ E_{q_1} \leq E_{q_2} + E_{l_1} + E_{l_2} \cot \theta , \quad E_{q_2} \leq E_{q_1} + E_{l_1} + E_{l_2} \cot \theta , \quad E_{l_1} \leq E_{q_1} + E_{q_2} + E_{l_2} \cot \theta , \quad E_{l_2} \leq E_{q_1} + E_{q_2} + E_{l_1} \cot \theta \]

Using the energy conservation equation, they can be rewritten as

\[ M \geq 2E_{q_1} , \quad M \geq 2E_{q_2} , \quad M \geq 2E_{l_1} , \quad M \geq 2E_{l_2} \]  

(229)

They correspond respectively to the largest values achievable for the configurations \((a1), (a2), (d1), (d2)\).

Two of these conditions can easily be expressed in terms of the masses involved.

\[ E_{q_1} = \frac{M_G^2 - M_Q^2}{2M_G} \]

and the energy of the second lepton is the same as in the case of \( M(llq) \)

\[ E_{l_2} = \frac{M_R^2 - M_0^2}{2M_0} \]

The first condition leads to

\[ M_G - M_0 \geq \frac{M_Q^2 - M_X^2}{M_G} \]

from which \( M_G^2 \geq M_G M_0 \). The second one gives \( M_G M_0 \geq M_R^2 \). Together, to allow a \( \tilde{\chi}_1^0 \) to be put at rest in the \( G = \bar{g} \) frame, the sparticle masses have to satisfy the conditions

\[ \frac{M_R^2}{M_0} \leq M_G \leq \frac{M_Q^2}{M_0} \]  

(230)

If the left-hand inequality is not fulfilled, the \( \tilde{\chi}_1^0 \) has a negative momentum along the z-axis (\( E_{q_1} \) too large); if the right-hand one is not, the \( \tilde{\chi}_1^0 \) has a positive momentum along the z-axis (\( E_{l_2} \) too large).

The third inequality can be computed for \( E_{q_2} \) in the configuration \((a2)\):

\[ 2E_{q_2} = \frac{M_G}{M_G} \frac{M_Q^2 - M_X^2}{M_Q} \leq M_G - M_0 \]

\[ M_G M_Q^2 - M_G M_X^2 \leq M_G M_Q^2 - M_X^2 M_R \]

from which the third condition is found

\[ M_G \geq \frac{M_Q^2 M_0}{M_X} \]  

(231)

Altogether, combined with previous inequality, the lower range of \( M_G \) for (231) is always below the upper range of (230). Hence the two intervals overlap.

The fourth inequality for \( E_{l_1} \) can be derived from the energy computed in the \( (llq) \) case

\[ M_G - M_0 \geq \frac{M_0}{M_R} (M_X^2 - M_R^2) \]

from which the fourth condition is found

\[ M_G \geq \frac{M_X^2 M_0}{M_R^2} \]  

(232)

The lower range of \( M_G \) for (232) is always below the upper range of (230). Hence the two intervals also overlap.
5.11.2 First collinear end point in $M(llqq)$

The first collinear configuration for the maximum of $M(llqq)$ is shown as configuration (a1). In this case the effective mass is given by

$$(M_{llqq}^{\text{max}})^2 = 4E_{l1}E_{l0}$$

(233)

After the Lorentz transformation the energy $E_{l0}$ in the $G = \tilde{g}$ rest frame is

$$E_{l0} = \frac{M_G M_0^2 - M_{llq}^2}{2M_Q}$$

(234)

so that

$$(M_{llqq}^{\text{max}})^2 = 4\frac{M_G^2 - M_{llq}^2}{2M_G}$$

$$= (M_G^2 - M_{llq}^2) \frac{M_0^2}{M_Q^2}$$

Finally

$$M_{llqq}^{\text{max}} = M_G \sqrt{(1 - \frac{M_0^2}{M_G^2})(1 - \frac{M_0^2}{M_Q^2})}$$

(235)

Similarly to the case of $M(llq)^{\text{max}}$, equation (80), this formula applies in the region $M_Q^2 \leq M_G M_0$. Else, if $M_Q^2 \geq M_G M_0$ the absolute maximum may be used and the configuration is no longer collinear. In its region of validity, this end point may yield the true end point.

5.11.3 Second collinear end point in $M(llqq)$

A second end point can be derived for the configuration where the decay products of both the squark and the $\tilde{\chi}_0^0$ are flipped, like the one of (d2). As the $llqq$ system is massless, the effective mass can be computed from

$$(M_{llqq}^{\text{max},2})^2 = 4E_{l1qq}E_{l2}$$

(236)

As the configurations are the same, it can be rewritten as

$$(M_{llqq}^{\text{max},2})^2 = (M_{ll}^{\text{max}})^2 + (M_{l2qq}^{\text{max}})^2 = (M_{ll}^{\text{max}})^2 + (M_{l2q1}^{\text{max}})^2 + (M_{l2q2}^{\text{max}})^2$$

(237)

Now,

$$E_{l1qq} = \frac{M_R^2 - M_{l0}^2}{2M_R}$$

(238)

and

$$E_{l2} = \frac{M_G M_R^2 - M_{l0}^2}{2M_R}$$

(239)

Then

$$(M_{llqq}^{\text{max},2})^2 = (M_R^2 - M_{l0}^2) \frac{M_R^2 - M_{l0}^2}{M_R^2}$$

(240)

or

$$M_{llqq}^{\text{max},2} = M_G \sqrt{(1 - \frac{M_R^2}{M_G^2})(1 - \frac{M_0^2}{M_R^2})}$$

(241)

This expression is maximized for $M_R^2 = M_G M_0$ and also reaches the absolute maximum of (227). Its region of validity is for $M_R^2 \geq M_G M_0$, else the absolute maximum holds (but see also the next end points).
For $M_{llqq}^{\text{max,3}} \geq M_{llqq}^{\text{max}}$ we need

$$
(M_Q^2 - M_R^2) \frac{M_Q^2 - M_0^2}{M_R^2} \geq (M_Q^2 - M_0^2) \frac{M_Q^2 - M_0^2}{M_Q^2}
$$

or

$$
M_G M_0 \leq M_Q M_R
$$

(242)

5.11.4 Third collinear end point in $M(llqq)$

A third collinear configuration can be obtained with $q2$ recoiling against $q1, l_1, l_2$, the one shown as configuration $(o2)$. The two leptons being parallel, the easiest way to compute the mass is from

$$
(M_{llqq}^{\text{max,3}})^2 = 4E_{q2}(E_{q1} + E_l)
$$

(243)

As the configurations are the same, it can be written as

$$
(M_{llqq}^{\text{max,3}})^2 = (M_{qq}^{\text{max}})^2 + (M_{llq2}^{\text{max}})^2
$$

(244)

From this, we obtain

$$
(M_{llqq}^{\text{max,3}})^2 = \left( M_G^2 - M_Q^2 \right) \frac{M_Q^2 - M_X^2}{M_Q^2} + \left( M_Q^2 - M_X^2 \right) \frac{M_X^2 - M_0^2}{M_X^2}
$$

$$
= \left( M_X^2 - M_Q^2 \right) \frac{M_G^2 M_X^2 - M_Q^2 M_X^2 + M_Q^2 M_X^2 - M_X^2 M_Q^2}{M_Q^2 M_X^2}
$$

$$
= \frac{M_G^2}{M_Q^2} (M_Q^2 - M_X^2)(1 - \frac{M_X^2}{M_Q^2})
$$

(245)

or

$$
M_{llqq}^{\text{max,3}} = M_G \sqrt{1 - \frac{M_Q^2 M_0^2}{M_G^2 M_X^2}} (1 - \frac{M_X^2}{M_Q^2})
$$

(246)

To determine the condition under which this end point takes its largest value, let us put it in the "standard" form:

$$
M_{llqq}^{\text{max,3}} = M_G \sqrt{1 - \frac{M_Q^2 M_0^2}{M_G^2 M_X^2}} (1 - \frac{M_X^2}{M_Q^2})
$$

(246)

from which it is seen that the absolute maximum is reached when $\frac{M_G^2 M_X^2}{M_Q^2} = M_G M_0$ or $M_G M_X^2 = M_Q^2 M_0$, at the edge of interval (231). The validity range of this formula is thus $M_G M_X^2 \leq M_Q^2 M_0$, else the non-collinear configuration gives the true end point.

For $M_{llqq}^{\text{max,3}} \geq M_{llqq}^{\text{max}}$ we need

$$
(M_G^2 - M_Q^2) \frac{M_G^2 - M_0^2}{M_Q^2} \geq (M_G^2 - M_0^2) \frac{M_Q^2 - M_0^2}{M_Q^2}
$$

(247)

or

$$
M_G M_0 \leq M_Q M_R
$$

(248)
and so
\[ M_G M_X \leq M_Q^2 \quad \text{or} \quad M_G M_0 \leq \frac{M_0}{M_X} M_Q^2 \] (247)

\[ M_{llqq}^{\text{max}, 3} \geq M_{llqq}^{\text{max}, 2} \]

requires
\[ M_G^2 \frac{M_Q^2 - M_X^2}{M_Q^2} \geq M_R^2 - M_0^2 \]
\[ (M_Q^2 M_R^2 - M_X^2 M_R^2)(M_Q^2 M_X^2 - M_Q^2 M_R^2) \geq (M_Q^2 M_X^2 - M_X^2 M_R^2)(M_Q^2 M_R^2 - M_Q^2 M_0^2) \]
\[ -M_G^2 M_X^2 M_R^2 - M_Q^2 M_R^2 M_0^2 \geq -M_Q^2 M_R^2 M_0^2 M_0^2 - M_Q^2 M_R^2 M_0^2 \]
\[ (M_Q^2 M_R^2 - M_Q^2 M_0^2)(M_Q^2 M_0^2 - M_X^2 M_R^2) \geq 0 \]
as \( M_G \geq M_Q \) and \( M_X \geq M_R \), the first factor can be dropped and so
\[ M_Q M_0 \geq M_X M_R \] (248)

### 5.11.5 Fourth collinear end point in \( M(llqq) \)

A fourth collinear configuration can be obtained after flipping the directions of \( q2 \) and of \( l2 \), the one shown as configuration (d1). The two leptons being back-to-back and the two quarks parallel, the easiest way to compute the mass is from
\[ (M_{llqq}^{\text{max}, 4})^2 = 4E_{l1}(E_{qq} + E_{l2}) \] (249)

As the configurations are the same, it can be rewritten as
\[ (M_{llqq}^{\text{max}, 4})^2 = (M_{llqq}^{\text{max}, 2})^2 + (M_{llqq}^{\text{max}})^2 \] (250)

Then
\[ E_{l1} = \frac{M_G M_Q M_X^2 - M_R^2}{2M_X} = \frac{M_G M_X^2 - M_R^2}{2M_X} \] (251)

and
\[ E_{qq} = E_{q1} + E_{q2} = \frac{M_G^2 - M_Q^2}{2M_G} + \frac{M_Q M_X^2 - M_Q^2}{2M_Q} = \frac{M_G^2 - M_X^2}{2M_G} \] (252)

and
\[ E_{l2} = \frac{M_Q M_X^2 M_R^2 - M_0^2}{2M_R} = \frac{M_X M_R^2 M_0^2 - M_0^2}{2M_R} \] (253)

Then
\[ E_{qq} + E_{l2} = \frac{1}{2M_G M_R^2} \left[ M_R^2(M_G^2 - M_X^2) + M_X^2(M_R^2 - M_0^2) \right] \]
\[ = \frac{M_G^2 M_X^2 - M_X^2 M_R^2}{2M_G M_R^2} \]

From this
\[ (M_{llqq}^{\text{max}, 4})^2 = 4 \frac{M_G M_X^2 - M_R^2}{2M_X} \frac{M_R^2 M_0^2 - M_R^2 M_0^2}{2M_G M_R^2} \]
or
\[ M_{llqq}^{\text{max}, 4} = M_G \sqrt{(1 - \frac{(M_X M_0)^2}{(M_G M_R)^2})(1 - \frac{M_R^2}{M_X^2})} \] (254)
To determine the condition under which this end point takes its largest value, we put it in the "standard" form:

\[ M_{llqq}^{\text{max},4} = M_G \sqrt{1 - \frac{1}{M_G^2} \frac{M_R^2 M_{R}^2}{M_X^2} \left(1 - \frac{M_R^2}{M_G^2 M_R^2} M_X^2 \right)} \]  \hspace{1cm} (255)

from which it is seen that the absolute maximum is reached when \( \frac{M_R^2 M_{R}^2}{M_X^2} = M_G M_0 \) or \( \frac{M_R^2}{M_G^2 M_R^2} M_X^2 = M_G^2 M_0 \), at the edge of interval (232).

To get \( M_{llqq}^{\text{max},4} \geq M_{llqq}^{\text{max},0} \) requires

\[
\begin{align*}
&M_G^2 M_{R}^2 \frac{M_X^2 - M_R^2 M_{R}^2 - M_X^2 M_0^2}{M_X^2} \geq (M_G^2 - M_R^2) \frac{M_{R}^2 - M_0^2}{M_X^2} \\
&(M_G^2 M_X^2 - M_R^2 M_{R}^2)(M_G^2 M_{R}^2 - M_X^2 M_0^2) \geq (M_G^2 - M_R^2)(M_{R}^2 M_X^2 - M_X^2 M_0^2) \\
&-M_G^2 M_X^2 M_0^2 - M_G^2 M_{R}^2 M_R^2 \geq -M_G^2 M_X^2 M_0^2 - M_G^2 M_{R}^2 M_R^2 \\
&M_G^2 M_X^2 (M_G^2 - M_X^2) \geq M_G^2 M_R^2 (M_G^2 - M_X^2)
\end{align*}
\]

as \( M_G \geq M_X \) and \( M_R \geq M_0 \), the second factor can be dropped and so

\[ M_G M_R \leq M_Q M_X \] \hspace{1cm} (256)

For \( M_{llqq}^{\text{max},4} \geq M_{llqq}^{\text{max},2} \) we need

\[
\begin{align*}
&M_G^2 M_X^2 - M_R^2 M_{R}^2 \frac{M_X^2 - M_R^2 M_{R}^2 - M_X^2 M_0^2}{M_X^2} \geq (M_G^2 - M_R^2) \frac{M_{R}^2 - M_0^2}{M_X^2} \\
&(M_X^2 - M_R^2)(M_R^2 M_X^2 - M_X^2 M_0^2) \geq (M_G^2 - M_R^2)(M_X^2 M_R^2 - M_X^2 M_0^2) \\
&-M_X^2 M_0^2 - M_R^2 M_R^2 \geq -M_G^2 M_X^2 M_0^2 - M_G^2 M_{R}^2 M_R^2 \\
&M_X^2 M_0^2 (M_G^2 - M_X^2) \geq M_R^2 (M_G^2 - M_X^2)
\end{align*}
\]

and so

\[ M_X M_0 \geq M_R^2 \] \hspace{1cm} (257)

Finally, for \( M_{llqq}^{\text{max},4} \geq M_{llqq}^{\text{max},3} \) we need

\[
\begin{align*}
&M_G^2 M_X^2 - M_R^2 M_{R}^2 \frac{M_X^2 - M_R^2 M_{R}^2 - M_X^2 M_0^2}{M_X^2} \geq (M_G^2 - M_R^2) \frac{M_{X}^2 - M_{X}^2 M_0^2}{M_X^2} \\
&(M_G^2 M_X^2 - M_R^2 M_{R}^2)(M_G^2 M_{R}^2 - M_X^2 M_0^2) \geq (M_G^2 - M_R^2)(M_X^2 M_R^2 - M_X^2 M_0^2) \\
&-M_G^2 M_X^2 M_0^2 - M_G^2 M_{R}^2 M_R^2 \geq -M_G^2 M_X^2 M_0^2 - M_G^2 M_{R}^2 M_R^2 \\
&M_G^2 M_{R}^2 (M_X^2 - M_G^2) \geq M_X^2 M_0^2 (M_X^2 - M_G^2 M_R^2)
\end{align*}
\]

as \( M_G \geq M_X \) and \( M_R \geq M_0 \), the first factor can be dropped and so

\[ M_X^2 \geq M_Q M_R \] \hspace{1cm} (258)

5.11.6 Fifth collinear end point in \( M(llqq) \)

A fifth collinear configuration can be obtained after flipping the direction of \( q2 \) in the squark decay, the one shown as configuration (c1). The two leptons being parallel and the two quarks as well, the easiest way to compute the mass is from

\[ (M_{llqq}^{\text{max},5})^2 = 4 E_{ll} E_{ll} \] \hspace{1cm} (259)

Then

\[ E_{ll} = \frac{M_G M_Q M_X^2 - M_X^2 M_0^2}{M_Q M_X^2 - 2 M_X} = \frac{M_G M_X^2 - M_X^2}{M_X^2 - 2 M_X} \] \hspace{1cm} (260)
and \(E_{qq}\) is the same as in Section 5.11.5 so that

\[
(M_{llqq}^{\text{max},5})^2 = 4 \frac{M_G^2 - M_X^2}{2M_G} \frac{M_G^2 M_X^2 - M_0^2}{2M_X^2}
\]

or

\[
M_{llqq}^{\text{max},5} = M_G \left(1 - \frac{M_X^2}{M_G^2}\right)^{1/2} \left(1 - \frac{M_0^2}{M_X^2}\right)^{1/2} \tag{261}
\]

This expression is the largest when \(M_X^2 = M_G M_0\) and reaches the absolute maximum (227). As this value lies inside the interval (230), this configuration does not lead to a new true value.

### 5.11.7 Sixth collinear end point in \(M(llqq)\)

A sixth collinear end point is obtained by flipping instead the \(l1\) in the \(\bar{\chi}_2^0\) decay, as shown in configuration (b2). In this case the parallel particles are \(l1, q1\) and \(l2, q2\). The effective mass can best be computed using these massless systems

\[
(M_{llqq}^{\text{max},6})^2 = 4E_{l1q1}E_{l2q2} \tag{262}
\]

The \((l2, q2)\) recoils against a \((l1\bar{\chi}_0^0)\) system in the \(Q = \bar{q}\) frame, from which

\[
E_{l2q2} = \frac{M_G M_Q^2 - M_{10}^2}{2M_Q} \tag{263}
\]

where \(M_{10}\) is the mass of \((l1\bar{\chi}_0^0)\), given in (60). The \((l1, q1)\) recoils against a \((l2\bar{l})\) system in the \(G = \bar{g}\) frame

\[
E_{l1q1} = \frac{M_G^2 - M_{2R}^2}{2M_G} \tag{264}
\]

where the mass \(M_{2R}\) of the \((l2\bar{l})\) can be computed in the \(Q = \bar{q}\) frame, where it recoils against the \(l1\).

\[
p_{l2R}' = p_{l1}' = E_{l1}' = \frac{M_Q M_X^2 - M_{2R}^2}{2M_X} \tag{265}
\]

and

\[
E_{2R}' = M_Q - E_{l1}'
\]

then

\[
M_{2R}^2 = (E_{2R}' - p_{l2R}')^2 = M_Q^2 - M_Q E_{l1}' = M_Q^2 (1 - \frac{M_X^2 - M_{2R}^2}{M_X^2})
\]

or

\[
M_{2R} = \frac{M_Q M_R}{M_X} \tag{265}
\]

Finally, the \(llqq\) mass becomes

\[
(M_{llqq}^{\text{max},6})^2 = 4 \frac{M_G^2 - M_{2R}^2}{2M_G} \frac{M_G^2 M_{2R}^2 - M_{10}^2}{2M_Q^2} = (M_G^2 - M_{2R}^2) \frac{M_Q^2 - M_{10}^2}{M_Q^2} \tag{266}
\]

or

\[
M_{llqq}^{\text{max},6} = M_G \sqrt{\left(1 - \frac{M_{2R}^2}{M_G^2}\right) \left(1 - \frac{M_{10}^2}{M_Q^2}\right)} \tag{266}
\]

This mass takes its largest value for \(\frac{M_G^2}{M_X^2} = M_G M_0\) where it reaches the absolute maximum (227). This value also lies inside the interval (230), hence this configuration does not lead to a new true value.
5.11.8 Conclusion on the $M(llqq)$ end points

As each of the first four end points computed above can reach the absolute maximum and do so for different mass relations, all of them may potentially lead to the true end point.

The conditions associated to the ordering of the end points are summarized in Table 7. Note that for $M_{llqq}^{\max,1}$ only the validity region of $M_{llqq}^{\max,1}$ is mentioned, as in this region it is automatically larger than any of the other solutions.

Table 7: Conditions associated to a given ($M_{llqq}^{\max}$) to give the true end point.

<table>
<thead>
<tr>
<th>$M_{llqq}^{\max}$</th>
<th>Valid region</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \geq 2, 3, 4$</td>
<td>$M_{G} &lt; M_{R} &lt; M_{X} &lt; M_{Q}$</td>
<td>$-\infty &lt; M_{R} &lt; M_{Q} &lt; M_{X}$</td>
</tr>
<tr>
<td>$2 \geq 3 \geq 4$</td>
<td>$M_{G} &lt; M_{R} &lt; M_{X} &lt; M_{Q}$</td>
<td>$M_{G} &lt; M_{R} &lt; M_{X}$</td>
</tr>
<tr>
<td>$2 \geq 4 \geq 3$</td>
<td>$M_{G} &lt; M_{R} &lt; M_{X} &lt; M_{Q}$</td>
<td>$M_{G} &lt; M_{R} &lt; M_{X}$</td>
</tr>
<tr>
<td>$3 \geq 2 \geq 4$</td>
<td>$M_{X} &lt; M_{R} &lt; M_{Q}$</td>
<td>$M_{X} &lt; M_{R}$</td>
</tr>
<tr>
<td>$3 \geq 4 \geq 2$</td>
<td>$M_{Q} &lt; M_{R}$</td>
<td>$M_{Q} &lt; M_{R}$</td>
</tr>
<tr>
<td>$4 \geq 2 \geq 3$</td>
<td>$M_{X} &lt; M_{R}$</td>
<td>$M_{X} &lt; M_{R}$</td>
</tr>
<tr>
<td>$4 \geq 3 \geq 2$</td>
<td>$M_{X} &lt; M_{R}$</td>
<td>$M_{X} &lt; M_{R}$</td>
</tr>
</tbody>
</table>

Note also that in the validity region of $M_{llqq}^{\max,1}$ we have

$$M_{G}^{2} - M_{Q}^{2} \geq \frac{M_{Q}^{2}}{M_{G}^{2}} - M_{Q}^{2} = \frac{M_{Q}^{2}}{M_{G}^{2}}(M_{Q}^{2} - M_{G}^{2}) \geq M_{Q}^{2} - M_{X}^{2}$$

implying that the true ($lq$) end points may be $M_{llqq}^{\max,1}$ or $M_{llqq}^{\max,2}$, both occurring in a different configuration.

An example of the validity regions for the end points of $M(llqq)$ is shown in figure 24. The crossings

Figure 24: Example of the mass regions leading to maxima for $M(llqq)$. 

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of the lines occur at the following values of \( M_G \): It is seen that 1 crosses the others at a point \( M_G \geq M_Q \), whereas 2, 3 and 4 cross each other for \( M_G \leq M_Q \).

The identification of the true end points is as follows:

- \( \max(M_Q, \frac{M^2}{M_G}, \frac{M_G^2}{M_Q}, \frac{M_Q^2}{M_G}, \frac{M_G^2}{M_Q}) \leq M_G \leq \frac{M^2}{M_0} \): in this region, the end point corresponds to a non-collinear configuration and is given by formula (227). It is characterized by clustering around intermediate values of \( M(ll) \) and/or \( M(qq) \).

- if \( M_G \geq \frac{M^2}{M_0} \) the true maximum is given by \( M_{llqq}^{\max,1} \), formula (235) applies. It corresponds to the configuration (a1) and is characterized by low \( ll \) and high \( qq \) masses. The mass conditions are described in Table 7. They imply that the true \( (ll) \) end points may be \( M_{1lq}^{\max} \) or \( M_{2lq}^{\max} \), both occurring in a different configuration and hence not correlated with the \( M_{llqq}^{\max,1} \) end point.

- if the true maximum is given by \( M_{llqq}^{\max,2} \), formula (241) applies. It corresponds to the configuration (d2) and is characterized by high \( ll \) but low \( qq \) masses (but see also next case). The mass conditions are described in Table 7. They imply that \( M_{12lq}^{\max} \) or \( M_{12dq}^{\max} \) is the true end points and appears in the same configuration.

- if the true maximum is given by \( M_{llqq}^{\max,3} \), formula (245) applies. It corresponds to the configuration (e2) and is also characterized by high \( ll \) but low \( qq \) masses. The mass conditions are described in Table 7. They imply that the true \( (ll) \) end points may be \( M_{12lq}^{\max} \) or \( M_{12dq}^{\max} \), both occurring in a different configuration and hence not correlated with the \( M_{llqq}^{\max,3} \) end point.

- if the true maximum is given by \( M_{llqq}^{\max,4} \), formula (254) applies. It corresponds to the configuration (d1) and is characterized by low \( ll \) but high \( qq \) masses. The mass conditions are described in Table 7. They imply that \( M_{1lq}^{\max} \) or \( M_{1dq}^{\max} \) is the true end points and appears in the same configuration.

The main characteristics of the true \( M(ll) \) end points for the collinear configurations are summarized in Table 8.

<table>
<thead>
<tr>
<th>Config</th>
<th>( M(ll) )</th>
<th>( M(qq) )</th>
<th>( M(lq) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_{llqq}^{\max,1} )</td>
<td>( a1 )</td>
<td>min</td>
<td>max</td>
</tr>
<tr>
<td>( M_{llqq}^{\max,2} )</td>
<td>( d2 )</td>
<td>max</td>
<td>min</td>
</tr>
<tr>
<td>( M_{llqq}^{\max,3} )</td>
<td>( a2 )</td>
<td>min</td>
<td>max</td>
</tr>
<tr>
<td>( M_{llqq}^{\max,4} )</td>
<td>( d1 )</td>
<td>max</td>
<td>min</td>
</tr>
</tbody>
</table>

Table 8: Main characteristics when a given \( M_{llqq}^{\max} \) gives the true end point.

### 5.12 Summary and suggestion of a strategy

Rather than choosing the jet originating from the squark or from the gluino by making kinematical cuts (which introduces model dependence), it may be more appropriate to keep both jets and identify the configurations yielding the observed end points (and hence the formula to be applied) by making correlations between the mass distributions. The following is a suggestion for a possible strategy, based on the configurations summarized in Section 5.1.

- The end points of \( (ll) \) and \( (qq) \) should be the most easy to find and are unambiguous. Their distributions are triangular.
• The next step can be the identification of the \((llq)\) true end point (and possibly some others) along the lines presented in Section 5.5.

• The \((llqq)\) true end point can be identified according to the strategy outlined in Section 5.11.8, which relies on a correlation with the \((ll), (qq)\) and \((llq)\) end points.

• Finally, the \((llq)\) and \((llq)\) true end points can be identified according to the strategy outlined in Sections 5.10 and 5.7.9, respectively.

In addition, some consistency checks are available. Out of the 576 possible combinations of the ambiguous collinear solutions (1024 when non-collinear ones are included), many are rejected by means of the mass conditions they have to satisfy. To simplify the notation, we write \(M_{Q}^{\text{max},i} = (x)^i\). The correlations between the true collinear end points giving rise to acceptable solutions two by two are shown in Table 9. Solutions may appear in the same or in different configurations. Note also that additional solutions may exist when non-collinear configurations are included. In most cases the compatibility is easily verified from the tables in the preceding sections and we comment here only on the more complicated cases.

\[
\begin{array}{|c|c|}
\hline
(l1q1) & (l1q1) \text{ or } (l1q1) \\
(l2q1) & (l1q1) \text{ or } (l1q1) \\
(l1q2) & (l1q2) \text{ or } (l1q2) \\
(l2q2) & (l1q2) \text{ or } (l1q2) \\
(l1q1) & (l1q1) \text{ or } (l1q1) \text{ or } (l1q1) \\
(l2q1) & (l1q1) \text{ or } (l2q1) \text{ or } (l1q1) \\
(l1q2) & (l1q2) \text{ or } (l1q2) \text{ or } (l1q2) \\
(l2q2) & (l1q2) \text{ or } (l2q2) \text{ or } (l1q2) \\
(l1q1) & (l2q1) \text{ or } (l1q1) \text{ or } (l2q1) \\
(l2q1) & (l1q1) \text{ or } (l2q1) \text{ or } (l2q2) \\
(l1q2) & (l1q1) \text{ or } (l1q1) \text{ or } (l1q1) \\
(l2q2) & (l1q1) \text{ or } (l2q1) \text{ or } (l2q1) \\
(l1q1) & (l2q1) \text{ or } (l2q1) \text{ or } (l2q1) \\
(l2q1) & (l1q1) \text{ or } (l2q1) \text{ or } (l2q1) \\
(l1q2) & (l1q1) \text{ or } (l1q1) \text{ or } (l1q1) \\
(l2q2) & (l1q1) \text{ or } (l1q1) \text{ or } (l1q1) \\
(l1q1) & (l2q1) \text{ or } (l2q1) \text{ or } (l2q1) \\
(l2q1) & (l1q1) \text{ or } (l2q1) \text{ or } (l2q1) \\
(l1q2) & (l1q1) \text{ or } (l1q1) \text{ or } (l1q1) \\
(l2q2) & (l1q1) \text{ or } (l1q1) \text{ or } (l1q1) \\
\hline
\end{array}
\]

Table 9: Solutions which satisfy two by two the constraints on the masses.

To simplify the formulae, we define the ratios \(r = \frac{M_Q}{M_X}\) and \(s = \frac{M_Q}{M_X}\). For example

\[
M_Q^2 - M_Q^2 \geq M_Q^2 - M_X^2
\]

can be rewritten as

\[
\frac{1}{s^2} - 1 \geq 1 - r^2
\]
or

\[ s \leq \frac{1}{\sqrt{2} - r^2} \]  

(267)

The reverse relation

\[ M_Q^2 - M_M^2 \leq M_M^2 - M_X^2 \]

gives

\[ s \geq \frac{1}{\sqrt{2} - r^2} \quad (\Rightarrow s \geq r) \]  

(268)

The latter excludes that \((l1q2)\) or \((l2q2)\) or \((llq2)\) are compatible with \((l1qq)\) or \((l2qq)\) or \((llqq)\).

The most difficult is the compatibility of \((llq1)\) and \((llq1)\). The second relation for \((llq1)\) is

\[ M_Q^2 - M_M^2 \geq \frac{M_X^2}{M_0^2}(M_Q^2 - M_M^2) \]

\[ \frac{1}{r^2} \left( \frac{1}{s^2} - 1 \right) \geq \frac{M_X^2}{M_0^2} - 1 \]

\[ \frac{1}{r^2s^2}(1 - s^2 + r^2s^2) \geq \frac{M_X^2}{M_0^2} \]

giving a lower bound on \(M_0/M_X\)

\[ \frac{M_0}{M_X} \geq \frac{rs}{\sqrt{1 - (1 - r^2)s^2}} \quad (\geq rs) \]  

(269)

When \((llq1)\) appears among the solutions its second relation is

\[ M_Q^2 - M_M^2 \leq \frac{1}{M_0^2}(M_R^4 - M_Q^2 - M_M^2) \]

\[ \frac{1}{s^2} - 1 \leq r^2 \left( \frac{M_R^2}{M_0^2} \frac{M_M^2}{M_X^2} - 1 \right) \]

\[ \frac{1 - s^2 + r^2s^2}{r^2s^2} \leq \frac{M_R^2}{M_0^2} \frac{M_M^2}{M_X^2} \]

from which

\[ M_R^2 \geq \frac{\sqrt{1 - (1 - r^2)s^2}}{rs} M_0 M_X \]  

(270)

Moreover to prevent \(M_R\) from becoming larger than \(M_X\) leads to the relation

\[ \frac{M_0}{M_X} \leq \frac{rs}{\sqrt{1 - (1 - r^2)s^2}} \]  

(271)

Likewise when \((llq1)\) appears among the solutions its second relation is

\[ M_Q^2 - M_M^2 \leq \frac{M_X^2}{M_R^2}(M_Q^2 - M_M^2) \]

\[ \frac{1}{s^2} - 1 \leq r^2 \left( \frac{M_R^2}{M_0^2} \frac{M_M^2}{M_X^2} - 1 \right) \]

\[ \frac{1 - s^2 + r^2s^2}{r^2s^2} \leq \frac{M_R^2}{M_0^2} \frac{M_M^2}{M_X^2} \]

from which

\[ M_R^2 \leq \frac{rs}{\sqrt{1 - (1 - r^2)s^2}} M_0 M_X \]  

(272)

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To prevent \( M_R \) from becoming smaller than \( M_0 \), we have again relation (271).

On the other hand, the constraints from \((llqq)\) are, for \((llqq)^1\)

\[
\frac{M_0}{M_X} \geq \frac{s}{r} \quad (\Rightarrow s \leq r)
\]  

(273)

For \((llqq)^2\)

\[
M_R^2 \geq M_0 M_G = \frac{1}{rs} M_0 M_X
\]  

(274)

However, to prevent \( M_R \) from becoming larger than \( M_X \), an upper bound on \( M_0 \) is introduced

\[
M_0 \leq \frac{M_X^2}{M_G} = rs M_X
\]

or

\[
\frac{M_0}{M_X} \leq rs
\]  

(275)

For \((llqq)^3\)

\[
\frac{M_0}{M_X} \geq \frac{r}{s}
\]  

(276)

leading to a lower bound

\[
s \geq r
\]  

(277)

Finally for \((llqq)^4\)

\[
M_R^2 \leq rs M_0 M_X
\]  

(278)

and to prevent \( M_R \) from becoming smaller than \( M_0 \) leads to the same condition (275).

These relations show that \((llq)^1\) can be compatible with \((llqq)^1\) and \((llqq)^3\), but not with \((llqq)^2\) or \((llqq)^4\). Moreover, \((llq)^2\) and \((llq)^3\) can both be compatible with \((llqq)^2\) or \((llqq)^4\). They are incompatible with \((llqq)^1\), as it would require

\[
\frac{s^2}{r^2} \leq \frac{r^2 s^2}{1 - (1 - r^2)s^2}
\]

\[
r^4 \geq 1 - (1 - r^2)s^2
\]

\[
(1 - r^2)s^2 \geq 1 - r^4 = (1 - r^2)(1 + r^2)
\]

\[
s^2 \geq 1 + r^2 \geq 1
\]

which is not allowed. Hence we always have

\[
\frac{s}{r} \geq \frac{rs}{\sqrt{1 - (1 - r^2)s^2}}
\]  

(279)

They might be compatible with \((llqq)^3\) provided (271) and (276) are satisfied, i.e.

\[
\frac{r^2}{s^2} \leq \frac{r^2 s^2}{1 - (1 - r^2)s^2}
\]

\[
s^4 \geq 1 - (1 - r^2)s^2
\]

\[
s^4 + (1 - r^2)s^2 - 1 \geq 0
\]

The solution for \( s \) yields the requirement

\[
s^2 \geq \frac{1}{2} \left[ \sqrt{(1 - r^2)^2 + 4 - (1 - r^2)} \right]
\]

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But \((llq1)^2\) and \((llq1)^3\) also require the condition (267) to be fulfilled. Hence

\[
\frac{1}{2} \left[ \sqrt{(1-r^2)^2 + 4 - (1-r^2)} \right] \leq \frac{1}{2} \frac{2 + (1-r^2)(2-r^2)}{2 - r^2} \\
(1-r^2)^2(2-r^2)^2 + 4(2-r^2)^2 \leq 4 + 4(1-r^2)(2-r^2) + (1-r^2)^2(2-r^2)^2 \\
0 \leq 4 + 4(2-r^2)(1-r^2 - 2 + r^2) = 4(1-2+r^2) = -4(1-r^2)
\]

which is impossible. Hence

\[
\frac{r}{s} \geq \frac{rs}{\sqrt{1-(1-r^2)s^2}} \quad \text{if} \quad s \leq \frac{1}{\sqrt{2-r^2}} \tag{280}
\]

Hence, the combination of \((llq1)^2\) or \((llq1)^3\) with \((llqq)^3\) is to be rejected.

There are also some combinations of \((llq)\) and \((llqq)\) which are incompatible. For example, \((llqq)^2\) requires \(M_R/M_X \leq rs\) and \((llqq)^2\) requires \(M_0/M_R \leq rs\). As \(M_R/M_X \geq M_0/M_X\) and \(M_0/M_R \geq M_0/M_X\), both are in contradiction with (269) for \((llqq)^1\). Moreover, \((llqq)^3\) and \((llqq)^4\) require respectively \(M_0/M_R \geq s/r\) and \(M_0/M_R \geq r/s\). Given (279) and (280) they are in contradiction with (270) for \((llq1)^2\). Likewise, \((llqq)^1\) and \((llqq)^3\) are in contradiction with (272) for \((llq1)^3\).

Finally, the combinations of collinear end points which may exist in all mass combinations simultaneously are listed in table 10. This is based on the combinations of the two by two compatible solutions of table 10.

We will now demonstrate that there indeed exists a range of masses for which the solutions of table 10 can coexist. First, all ranges encountered so far constrain only mass ratios. Hence the overall scale is free and we can choose \(M_G\) to set the scale of all masses. It is also observed that the constraints for the \((llqq)\) are always weaker than the corresponding ones for \((llqq)\). Similarly for the \((llq2)\) compared to \((llqq)\). Hence, for these cases we only need to consider the ranges given for \((llqq)\).

To start, we can choose freely the value of \(0 \leq r \leq 1\). The value of \(s\) can then be chosen within an interval determined by \((ll)\) and \((llq)\) (the first constraint listed in tables 1 and 3), for example (267) or (268). In some cases, a stronger constraint may be obtained, for example the relation for \((llqq)^1\), (273) is stronger than (267). Likewise, the relation for \((llqq)^3\) leads to a lower bound (277). The values of \(M_G\), \(M_Q\) and \(M_X\) can then be chosen, provided the suitable conditions above are satisfied. They are shown as the first condition (column) in table 11.

Next, we can choose a value for \(M_0\) (i.e. the ratio \(M_0/M_X\)). The second relation for \((llq1)^1\), (269), can appear together with the one of either \((llqq)^1\) or \((llqq)^3\). It can be shown (a bit tedious) that it can never be stronger than the ones from \((llqq)\) inside the range of \(s\) defined above. Hence, it is the condition for \((llqq)\) which determines the allowed range of \(M_0\). This is (273) for \((llqq)^1\) and (276) for \((llqq)^3\). When the \((llq1)^2\) or \((llq1)^3\) conditions appear together with \((llqq)^2\) or \((llqq)^4\) respectively, the latter requires (275) which is stronger than the corresponding relations for \((llq1)\) and should be used instead. The same holds when \((llqq)^2\) or \((llqq)^4\) appear together with \((llq2)^2\) or \((llq2)^4\).

Finally, we can choose a value for \(M_R\) in the region \(M_R \geq M_0\). In most cases either an upper or a lower bound is given by the second relation for \((llq)\). Exceptions are when \((llqq)^2\) or \((llqq)^4\) appear among the solutions their second relation (274) or (278) respectively are stronger than the bound from \((llq1)\).
and from \((llq)\). All allowed ranges are summarized in Table 11. The position of these solutions in mass space is illustrated in Figure 25. It is seen that each of the 10 solutions occupies a given region of the mass space and that all of them are mutually exclusive. Moreover, there are regions where no collinear solution can exist.

The solutions which satisfy two by two the constraints on the masses and include at least one non-collinear end point (written as \((x)\)) are listed in Table 12.

Most cases are easily verified from the tables of allowed ranges given earlier for collinear configurations, together with the ranges for non-collinear configurations summarized in Table 13. It is easily verified that the non-collinear \((llq)\) or \((lqq)\) configurations are never compatible with a collinear \((llqq)\) one. The non-collinear \((llq1)\) configuration is compatible with all \((lqq)\) configurations. But the non-collinear \((llqq)\) configuration is not compatible with \((llqq)\) or \((lqq1)\), as they require different ranges of \(s\).

The sets of solutions which can go together and include at least one non-collinear end point are listed in Table 14. The range of \(s\) as a function of \(r\) is again mostly obtained from the conditions of \((llq)\). A more restrictive range is sometimes obtained from the \((llqq)\). The regions allowed for \(M_0\) and \(M_R\) give rise to a considerably more complicated patchwork than in the collinear case. The regions of the \((M_L/M_X, M_0/M_R)\) plane where sets of simultaneous solutions exist are illustrated in Figure 26 for the three ranges of \(s\) versus \(r\). It is seen that with the set of solutions obtained there is a complete coverage of the mass space and that each solution is associated to a distinct region. This confirms that the above

\[
\begin{array}{cccccc}
1 & (llq1) & (llq1) & (llq1) & (llq1) & (llq1) \\
2 & (llq1) & (llq1) & (llq1) & (llq1) & (llq1) \\
3 & (llq1) & (llq1) & (llq1) & (llq1) & (llq1) \\
4 & (llq1) & (llq1) & (llq1) & (llq1) & (llq1) \\
5 & (llq1) & (llq1) & (llq1) & (llq1) & (llq1) \\
6 & (llq1) & (llq1) & (llq1) & (llq1) & (llq1) \\
7 & (llq1) & (llq1) & (llq1) & (llq1) & (llq1) \\
8 & (llq1) & (llq1) & (llq1) & (llq1) & (llq1) \\
9 & (llq1) & (llq1) & (llq1) & (llq1) & (llq1) \\
10 & (llq1) & (llq1) & (llq1) & (llq1) & (llq1) \\
\end{array}
\]

Table 11: Solutions which satisfy simultaneously the constraints on the masses.

<table>
<thead>
<tr>
<th>(M_0/M_X)</th>
<th>1, 4</th>
<th>2, 5</th>
<th>7, 9</th>
</tr>
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<tbody>
<tr>
<td>s/r</td>
<td>r/s</td>
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<tr>
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<td>rs</td>
<td>rs</td>
<td>3, 6</td>
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<tr>
<td>3, 6</td>
<td>8, 10</td>
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<td>0</td>
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<table>
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<tr>
<th>(M^2/M_X M_G)</th>
<th>6</th>
<th>6</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/r</td>
<td>4</td>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td>r</td>
<td>1</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>rs</td>
<td>3</td>
<td>3</td>
<td>8</td>
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<tr>
<td>0</td>
<td>0</td>
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<td>0</td>
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Figure 25: Example of the mass regions where solutions can coexist. The notation is \(r = M_X/M_Q\) and \(r = M_Q/M_G\).
Table 12: Solutions which satisfy two by two the constraints on the masses including at least one non-collinear end point.

<table>
<thead>
<tr>
<th>(llq1)$^n$</th>
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Table 13: Summary of mass regions allowed for non-collinear end points. When we write $\frac{r}{s}$, $\frac{s}{r}$, the first is valid for $r \leq s$, the second for $s \leq r$.

set of solutions is really complete.
$$f_{rs} = \frac{r s}{\sqrt{1 - (1-r^2)s^2}}$$

When we write \( r \), \( s \), the first is valid for \( r \leq s \), the second for \( s \leq r \).

5.13 Model dependent expectations

It is interesting to find which solutions could be expected if some model dependence is introduced. Ignoring QCD corrections, the gluino mass is approximately

$$M_G \simeq 2.7 m_{1/2}$$  \hspace{1cm} (281)

and the heaviest squark mass

$$M_{\tilde{Q}}^2 \simeq m_0^2 + 5m_{1/2}^2 + ...$$  \hspace{1cm} (282)

where the D-term has been ignored. In both cases, \( m_{1/2} \) is basically \( M_3 \) as the evolution is dominated by strong interactions. The smallest squark mass is obtained for \( m_0 = 0 \), which leads to the upper bound
on the gluino mass of

\[ M_G \lessapprox 1.2 M_Q \]  

(283)

Under the stronger assumptions of gaugino mass universality and that the $\tilde{\chi}_1^0$ and $\tilde{\chi}_2^0$ are pure gauginos, we have the relation

\[ M_X \simeq 2 M_0 \simeq 0.8 m_{1/2} \]  

(284)

from which we derive the bounds

\[ s = \frac{M_Q}{M_G} \geq 0.83 \text{ , } r = \frac{M_X}{M_Q} \leq 0.36 \]

This implies the region where

\[ s \geq \frac{1}{\sqrt{2 - r^2}} \]  

(285)

Assuming gaugino mass universality, the mass of the $\tilde{l}_L$ is approximately

\[ M_R^2 \simeq m_0^2 + 0.5 m_{1/2}^2 + ... \]  

(286)

leading to a lower bound $M_R \geq 0.7 m_{1/2}$ and hence

\[ \frac{M_R}{M_X} \geq 0.88 \text{ , } \frac{M_0}{M_R} \leq 0.57 \]

which singles out the region

\[ \frac{M_R}{M_X} \geq \frac{M_0}{M_R} \]  

(287)

With these more restrictive assumptions, the preferred set of solutions involves all cases where the true maximum is given by $(l2q2)$ (9 possibilities).

### 6 Cascades from a $\tilde{q}$ ending in 3-body decays of $\tilde{\chi}_2^0$

If the $\tilde{\chi}_2^0$ decays via a direct 3-body decay, the cascade is:

\[ \tilde{q} \rightarrow q \tilde{\chi}_2^0 \text{ , } \tilde{\chi}_2^0 \rightarrow \tilde{l} + l + \tilde{\chi}_1^0 \]  

(288)

The effective mass distributions available are for $ll$, $lq$ and $llq$. 

---

Figure 26: Example of the mass regions where solutions can coexist. The notation is $r = M_X/M_Q$, $r = M_Q/M_G$ and $f(r,s) = rs/\sqrt{1 - (1 - r^2)s^2}$. 

---

81
\[ M_{ll}^{\text{max}} = M_X - M_0 \] (289)

The distinction between the 3-body decay and the sequential 2-body decays could be based on the shape of the mass distribution. For sequential 2-body decays, the di-lepton effective mass distribution increases linearly with the mass. For a direct 3-body decay, the mass distribution of pure Lorentz invariant phase space is in general given by the formula

\[
\frac{dR}{dM_{12}} = \frac{\pi^2}{2E^2 M_{12}} \{ [M_{12}^2 - (m_1 + m_2)^2][M_{12}^2 - (m_1 - m_2)^2] [E^2 - (m_3 + M_{12})^2][E^2 - (m_3 - M_{12})^2] \}^{1/2} \] (290)

where \( M_{12} \) is the effective mass of particles 1 and 2 and \( E \) is the total rest frame energy. In our case \( m_1 = m_2 = 0, m_3 = M_0 \) and \( E = M_X = M(\tilde{\chi}_0^0) \), so that

\[
\frac{dR_3}{dM_{ll}} = \frac{\pi^2}{2M_X^2 M_{ll}} \{ M_{ll}^2 M_{ll}^2 [M_X^2 - (M_0 + M_{ll})^2][M_X^2 - (M_0 - M_{ll})^2] \}^{1/2}
\]

\[ = \frac{\pi^2}{2 M_X^2} \sqrt{[M_X^2 - (M_0 + M_{ll})^2][M_X^2 - (M_0 - M_{ll})^2]} \] (291)

giving a “hat-shaped” distribution with the tangent to the distribution vertical at the minimum mass \( M_{ll} = 0 \) and at its maximum \( M_{ll} = M_X - M_0 \). The distribution may, however, be distorted if the decay is mediated by o-shell particles. For example, if the \( Z^0 \) mass is slightly above the end point, its propagator will pull the distribution towards high masses. In such case, the distribution of the momentum asymmetry of the leptons, \((p_{\text{high}} - p_{\text{low}})/(p_{\text{high}} + p_{\text{low}})\), may still allow the two cases to be distinguished.

6.2 Upper end point in \( M(ll) \)

The upper end point for the \((ll)\) mass distribution differs from the case of sequential 2-body decays. In a direct 3-body decay, the two leptons play an equivalent role and cannot be distinguished.

To maximize the \( M(ll) \), say \( M(l1l2) \), we have to maximize the energy \( E_{l1} \). The configuration is shown in Figure 27. To find the configuration with maximal energy for \( l1 \) in the \( X = \tilde{\chi}_0^0 \) rest frame (starred quantities), let us write the momentum-energy constraints (leptons treated as massless)

\[
E_{l1}^* = p_0^* + E_{l2}^*
\]

\[
E_{l1} = M_X - E_0^* - E_{l2}^*
\] (292)

from which it follows that

\[
2E_{l1}^* = M_X - (E_0^* - p_0^*) \] (293)

The maximum \( E_{l1}^* \) is obtained for the smallest possible value of \( (E_0^* - p_0^*) \), i.e. the largest \( p_0^* \). This is reached when \( p_{l2}^* = 0 \), giving \( p_0^* = E_{l1}^* \). In these conditions, the kinematics is the same as for a system with mass \( M_X \) and consisting of the lepton \( l1 \) and the \( \chi_0^0 \). Therefore (without repeating the calculations done previously)

\[
M_{ll}^2 = 4E_q E_{l1} = \frac{M_{ll}^2 - M_X^2}{M_{ll} M_X} M_\chi^2 \left( M_X^2 - M_X^2 \right)
\]

Figure 27: Configuration leading to the maximum of the mass for \((l1q)\).
or

\[ M_{lq}^{\text{max}} = M_Q \sqrt{1 - \frac{M_X^2}{M_Q^2}} \left(1 - \frac{M_0^2}{M_X^2}\right) \]  

(294)

6.3 Upper end point in \( M(llq) \)

The upper end point of the \((llq)\) mass distribution was discussed in Section 4, where it was already noted that the formula for the end point \( M_{llq}^{\text{max}} \) remains valid for a direct 3-body decay. It is also seen that when \( M_R \rightarrow M_X \) the lower boundary of the interval (73) becomes equal to the upper one, while \( M_{llq}^{\text{max},2} \rightarrow M_{llq}^{\text{max},1} \). Moreover, the validity region for \( M_{llq}^{\text{max},3} \) becomes \( M_Q \leq M_0 \), which is impossible. Hence, the end point is given by

\[ M_{llq}^{\text{max}} = M_Q \sqrt{1 - \frac{M_X^2}{M_Q^2}} \left(1 - \frac{M_0^2}{M_X^2}\right) \]  

(295)

valid in the region where \( M_Q \geq M_X^2/M_0 \). It results in the same relation as for the \((lq)\) end point and provides no additional constraint, which leaves the masses undetermined. It can nevertheless serve as a test that the decay is indeed of 3-body type (provided such events are selected, as \( l2 \) may be soft).

Moreover, additional constraints may be obtained from the relation between \( M_{llq} \) and \( M_{llq} \) discussed in section 4.6.

In the region where \( M_Q \leq M_X^2/M_0 \), the end point is given by a non-collinear configuration with

\[ M_{llq}^{\text{max}} = M_Q - M_0 \]  

(296)

7 Cascades from a \( \tilde{g} \) ending in 3-body decays of \( \tilde{\chi}_2^0 \)

If the \( \tilde{\chi}_2^0 \) decays via a direct 3-body decay, the cascade is:

\[ \tilde{g} \rightarrow \tilde{q} \bar{q} \rightarrow \tilde{q} \tilde{\chi}_2^0, \quad \tilde{\chi}_2^0 \rightarrow l + \tilde{\chi}_1^0 \]  

(297)

The effective mass distributions available are for \( ll, lq, lq1, lq2, llq1, llq2, lqq \) and \( llqq \). The end points for \( ll, lq2 \) and \( llq2 \) have been derived in section 6.

7.1 Upper end point in \( M(q1q2) \)

The end point for \( M(qq) \) is the same as derived in section 5.2

\[ M_{q1q2}^{\text{max}} = M_G \sqrt{1 - \frac{M_Q^2}{M_G^2}} \left(1 - \frac{M_0^2}{M_X^2}\right) \]  

(298)

which is largest when \( M_Q^2 = M_G M_X \) and amounts to \( M_{q1q2}^{\text{max}} = M_G - M_X \). The effective mass distribution is of triangular shape.

7.2 Upper end point in \( M(lq1) \)

It was found in section 6.2 that, to maximize the energy of \( l1 \), one has to put the \( l2 \) at rest in the \( X = \tilde{\chi}_2^0 \) frame, so that

\[ M_{lq1}^2 = 4E_{q1} E_{l1} = \frac{4}{2M_G} \frac{M_G^2 - M_G^2 M_G M_Q M_X^2}{M_Q M_X} = M_0^2 \]  

or

\[ M_{lq1}^{\text{max}} = M_G \sqrt{1 - \frac{M_Q^2}{M_G^2}} \left(1 - \frac{M_0^2}{M_X^2}\right) \]  

(299)

which is largest when \( M_Q M_X = M_G M_0 \) where it reaches \( M_{lq1}^{\text{max}} = M_G - \frac{M_G M_0 M_0}{M_X} \), the second term being the mass of the \((q2 \tilde{\chi}_1^0)\) system with both \( q2 \) and \( \tilde{\chi}_1^0 \) parallel.

To have \( M_{lq1}^{\text{max}} \geq M_{lq2}^{\text{max}} \) requires

\[ M_G^2 - M_Q^2 \geq M_G^2 - M_X^2 \]  

(300)
7.3 Upper end point in $M(llq_1)$

The collinear upper end point is given by the configuration labelled (c1) in section 5.1 and the mass is given by

$$M_{llq_1}^{\text{max}} = M_G \sqrt{(1 - \frac{M_Q^2}{M_G^2})(1 - \frac{M_0^2}{M_X^2})}$$  \hspace{1cm} (301)

It is valid in the region $M_G \geq M_Q M_X / M_0$. This relation is the same as for $M_{llq_1}^{\text{max}}$.

It is also seen that in the limit $M_R \to M_X$, the lower edge of the interval (132) coalesces with the upper one and $M_{llq_1}^{\text{max},2} = M_{llq_1}^{\text{max},1}$. Moreover, the validity region for $M_{llq_1}^{\text{max},3}$ becomes $M_G \leq \sqrt{M_Q^2 - M_X^2 + M_0^2}$ which is $\leq M_Q$ and hence impossible. In the region where $M_G$ is smaller than the solution of equation (146) the end point is obtained in a non-collinear configuration and the mass is given by (145), which is independent of $M_R$.

Also to have $M_{llq_1}^{\text{max},1} \geq M_{llq_2}^{\text{max}}$ requires

$$M_G^2 - M_Q^2 \geq M_Q^2 - M_X^2$$  \hspace{1cm} (302)

7.4 Upper end point in $M(llqq)$

The end points of the $(llqq)$ mass distribution can be obtained by comparing to the $(llq_1)$ of section 5.8 or the $(l2qq)$ of section 5.9. For the case of $(llq_1)$, following the argument given in section 6.2, the maximum is reached when $l_2$ is at rest in the $X = \frac{X}{2}$ frame, i.e. by replacing $M_R$ by $M_0$. For the $(l2qq)$ case, the maximum is reached when $l_1$ is at rest in the $X = \frac{X}{2}$ frame, and $M_R$ should be replaced by $M_X$ in the formulae. It is easily verified that both yield the same result.

As the end points depend on the relative orientation of $q_2$ and $l$ with respect to $q_1$, there will be three possible collinear configurations.

The true end point given by the non-collinear configuration is

$$M_{llqq}^{\text{max}} = M_G - M_0$$  \hspace{1cm} (303)

and applies to the region where

$$\frac{M_X^2}{M_0} \leq M_G \leq \frac{M_Q^2}{M_0} \text{ and } M_G \geq \frac{M_Q^2 M_0}{M_X^2}$$  \hspace{1cm} (304)

A first collinear end point is

$$M_{llq_1}^{\text{max},1} = M_G \sqrt{(1 - \frac{M_Q^2}{M_G^2})(1 - \frac{M_0^2}{M_Q^2})}$$  \hspace{1cm} (305)

valid in the region where $M_G \geq M_Q^2 / M_0$.

A second collinear end point is

$$M_{llq_2}^{\text{max},2} = M_G \sqrt{(1 - \frac{M_X^2}{M_G^2})(1 - \frac{M_0^2}{M_X^2})}$$  \hspace{1cm} (306)

valid in the region where $M_G \leq M_X^2 / M_0$. This end point is not independent of the other, as

$$(M_{llq_2}^{\text{max},2})^2 = (M_{llq_1}^{\text{max},1})^2 + (M_{llq_2}^{\text{max}})^2$$  \hspace{1cm} (307)

The third collinear end point is

$$M_{llq_3}^{\text{max},3} = M_G \sqrt{(1 - \frac{(M_Q M_0)^2}{M_G M_X^2})(1 - \frac{M_X^2}{M_Q^2})}$$  \hspace{1cm} (308)

valid in the region where $M_G \leq M_Q^2 M_0 / M_X^2$. This end point is also not independent of the other, as

$$(M_{llq_3}^{\text{max},3})^2 = (M_{llqq}^{\text{max}})^2 + (M_{llq_2}^{\text{max}})^2$$
To have $M_{llq}^{\text{max},2} \geq M_{llq}^{\text{max}}$ requires
\begin{equation}
M_G \leq \frac{M_Q M_X}{M_0} \tag{310}
\end{equation}
To have $M_{llq}^{\text{max},3} \geq M_{llq}^{\text{max}}$ requires
\begin{equation}
M_G \leq \frac{M_Q^2}{M_X} \tag{311}
\end{equation}
To have $M_{llq}^{\text{max},3} \geq M_{llq}^{\text{max},2}$ requires
\begin{equation}
M_X^2 \leq M_Q M_0 \tag{312}
\end{equation}

7.5 Upper end point in $M(llqq)$

The end points of the $(llqq)$ mass distribution can be derived from section 5.11.

As the end points depend on the relative orientation of $q_2$ and $l$ with respect to $q_1$, there will be three possible collinear configurations.

The true end point given by the non-collinear configuration is
\begin{equation}
M_{llqq}^{\text{max}} = M_G - M_0 \tag{313}
\end{equation}
and applies to the region where
\begin{equation}
\frac{M_0^2}{M_0^2} \leq M_G \leq \frac{M_Q^2}{M_0^2} \quad \text{and} \quad M_G \geq \frac{M_Q^2 M_0}{M_X^2} \tag{314}
\end{equation}
A first collinear end point is independent of $M_R$ and thus remains applicable to the 3-body decay
\begin{equation}
M_{llqq}^{\text{max},1} = M_G \sqrt{(1 - \frac{M_Q^2}{M_G^2})(1 - \frac{M_0^2}{M_Q^2})} \tag{315}
\end{equation}
valid in the region where $M_G \geq M_Q^2/M_0$. It is seen that this is the same as $M_{llq}^{\text{max},1}$.

A second collinear end point is obtained when $l_1$ is at rest in the $X = \tilde{\chi}_2^0$ frame, hence for $M_R \rightarrow M_X$
\begin{equation}
M_{llqq}^{\text{max},2} = M_G \sqrt{(1 - \frac{M_X^2}{M_G^2})(1 - \frac{M_0^2}{M_X^2})} \tag{316}
\end{equation}
valid in the region where $M_G \leq M_X^2/M_0$. It is the same as $M_{llq}^{\text{max},2}$.

The third collinear end point is independent of $M_R$ and thus
\begin{equation}
M_{llqq}^{\text{max},3} = M_G \sqrt{(1 - \frac{(M_Q M_0)^2}{(M_Q M_X)^2})(1 - \frac{M_X^2}{M_Q^2})} \tag{317}
\end{equation}
valid in the region where $M_G \leq M_Q^2 M_0/M_X^2$. It is the same as $M_{llq}^{\text{max},3}$.

The fourth collinear end point obtained when $l_2$ is at rest in the $X = \tilde{\chi}_2^0$ frame, hence by replacing $M_R \rightarrow M_0$
\begin{equation}
M_{llqq}^{\text{max},4} = M_G \sqrt{(1 - \frac{M_X^2}{M_G^2})(1 - \frac{M_0^2}{M_X^2})}
\end{equation}
and is identical to $M_{llqq}^{\text{max},2}$. Hence, this is not an additional end point for the 3-body decay of $\tilde{\chi}_2$.  

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7.6 Summary

When the $\tilde{\chi}_2^0$ decays directly into ($ll\tilde{\chi}_1^0$), many of the relations for the end points provide the same information on the sparticle masses. Only four among the collinear end points give independent relations. They are $M(ll)$, $M(qq)$, $M(lq_1)$, $M(lq_2)$. In a limited range of sparticle masses the $M(lqq)$ also brings independent information, namely in the (unlikely) case where $\frac{m_G}{m_0} \geq \frac{m_0}{m_G}$. These four end points are nevertheless sufficient to determine the four masses involved.

Note that, as already mentioned earlier, the end points in $M(lq_1)$ and $M(lq_2)$ can be distinguished by their correlation with $M(qq)$: the $(lq_2)$ end point is reached for any value of $M(qq)$, whereas the $(lq_1)$ end point is obtained for small values of $M(qq)$. Moreover, the other end points may bring valuable additional constraints to improve the knowledge of these 4 basic end points.

8 Cascades with a 3-body decay of $\tilde{g}$ and 3-body decay of $\tilde{\chi}_2^0$

There is a small portion of the parameter space where the gluino is lighter than any of the squarks and decays to the $\tilde{\chi}_2^0$ which undergoes itself a direct three-body decay to leptons. The cascade is $\tilde{g} \rightarrow \bar{q}q\tilde{\chi}_2^0 \rightarrow \bar{q}ql + l + \tilde{\chi}_1^0$ (318)

The effective mass distributions available for constraining the three sparticle masses involved are $ll$, $qq$, $lq$, $llq$, $lqq$ and $llqq$. In this case, there is no kinematical distinction between the two quarks nor the two leptons (all assumed to be massless).

8.1 Upper end point in $M(ll)$

The $(ll)$ effective mass distribution and end point was discussed in section 6.1, where it was found that

$$M_{ll}^{\text{max}} = M_X - M_0$$ (319)

8.2 Upper end point in $M(qq)$

As there is no intermediate squark, the $(qq)$ decay is similar to the one for $(ll)$ and the end point is given by

$$M_{qq}^{\text{max}} = M_G - M_X$$ (320)

8.3 Upper end point in $M(lq)$

The effective mass $M(lq)$ is maximized in a configuration where the lepton and quark are back-to-back and have their maximal energy

$$M_{lq}^2 = 4E_lE_q$$ (321)

For the quark, this happens when the other quark is at rest in the $G = \tilde{g}$ frame, which implies

$$E_q = \frac{M_G^2 - M_X^2}{2M_G}$$ (322)

For the lepton, the situation is the same as in section 6.2 which implies that its energy in the $G = \tilde{g}$ frame is

$$E_l = \frac{M_G^2 - M_X^2 - M_0^2}{2M_X}$$ (323)

The maximum effective mass for the $(lq)$ system is then

$$(M_{lq}^{\text{max}})^2 = \frac{M_G^2 - M_X^2}{M_G} \frac{M_X^2 - M_0^2}{M_X}$$

or

$$M_{lq}^{\text{max}} = M_G \sqrt{\left(1 - \frac{M_X^2}{M_G^2}\right)\left(1 - \frac{M_0^2}{M_X^2}\right)}$$ (324)

which is largest when $M_X^2 = M_GM_0$ and reaches the value $M_{lq}^{\text{max}} = M_G - M_0$.

As this maximum arises for a configuration where both a quark and a lepton are very soft, it may be difficult to observe experimentally.
8.4 Upper end point in $M(l\bar{l}q)$

The mass of the ($l\bar{l}q$) system is given by the expression

\[ M_{l\bar{l}q}^2 = 2E_q(E_{l1} + p_{11,l}) + 2E_q(E_{l2} + p_{12,l}) \]  

(325)

It is maximized by maximizing $E_q$, i.e. by putting the other quark at rest in the $G = \tilde{g}$ frame, and by maximizing the sum of the lepton longitudinal momenta, which is obtained by taking them parallel and opposite to the quark. In this case one has

\[ M_{l\bar{l}q}^2 = 4E_q E_{l\bar{l}} \]  

(326)

where $E_q$ is given by equation (322) and $E_{l\bar{l}}$ has the same expression as (323). Therefore, the mass is

the same value as for $M_{l\bar{q}}$ and no new constraint is obtained. This end point is largest when $M_{X}^2 = M_G M_0$

where it reaches $M_{l\bar{l}q}^{\text{max}} = M_G - M_0$.

However, this collinear configuration can only lead to the true end point if the boost from $X = \tilde{\chi}_1^0$ is large enough compared to the $\tilde{\chi}_1^0$ velocity. Else the $\tilde{\chi}_1^0$ can be put at rest in the $G = \tilde{g}$ frame in a non-collinear configuration. In the $G = \tilde{g}$ rest frame the velocity of $X = \tilde{\chi}_1^0$ is

\[ \beta_X = \frac{M_G^2 - M_X^2}{M_G^2 + M_X^2} \]

and the velocity of $\tilde{\chi}_1^0$ in the $X = \tilde{\chi}_1^0$ frame is largest when the two leptons are parallel

\[ \beta_0' = \frac{M_X^2 - M_0^2}{M_X^2 + M_0^2} \]

The $\tilde{\chi}_1^0$ cannot be put at rest in the $G = \tilde{g}$ frame provided $\beta_0' \leq \beta_X$, hence

\[ (M_G^2 + M_X^2)(M_X^2 - M_0^2) \leq (M_G^2 - M_X^2)(M_X^2 + M_0^2) \]

\[ -M_G^2 M_0^2 + M_X^4 \leq M_G^2 M_0^2 + M_0^4 \]

or

\[ M_X^2 \leq M_G M_0 \]  

(328)

The region of validity for the mass formula is $M_G \geq M_X^2 / M_0$. If this condition is not fulfilled, the $\tilde{\chi}_1^0$ can be put at rest. For smaller values of $M_G$, the maximum is obtained in a non-collinear configuration and the end point is given by

\[ M_{l\bar{l}q}^{\text{max}} = M_G - M_0 \]  

(329)

8.5 Upper end point in $M(lq\bar{q})$

Following a reasoning analogous to the one of section 8.4, the maximum of $M(lq\bar{q})$ is reached when the two quarks are parallel, the lepton included in the mass is back-to-back with the quarks and the other lepton is at rest in the $X = \tilde{\chi}_2^0$ frame. It leads once more to the same expression for the mass

\[ M_{lq\bar{q}}^{\text{max}} = M_G \sqrt{(1 - \frac{M_X^2}{M_G^2})(1 - \frac{M_0^2}{M_X^2})} \]  

(330)

and provides no additional constraint on sparticle masses. The region of validity of this formula and the end point for non-collinear configurations are the same as for $M_{l\bar{l}q}$. 

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8.6 Upper end point in \( M(llqq) \)

The absolute maximum value of \( M(llqq) \) is reached when the \( \tilde{\chi}_1^0 \) is at rest in the \( G = \tilde{g} \) frame and hence

\[
M_{llqq}^{max} = M_G - M_0 \tag{331}
\]

Unlike the cases considered earlier where several intermediate sparticles were present, in case there is only a \( \tilde{\chi}_2^0 \) the \( \tilde{\chi}_1^0 \) can always be put at rest in a non-collinear configuration where it is emitted opposite to the boost from the \( \chi_2^0 \). This is because the angle between the leptons allows the velocity \( \beta_0^X \) of the \( \chi_1^0 \) in the \( \chi_2^0 \) frame to be adjusted between 0 and a maximum value. The velocity \( \beta_0^X \) of the \( \chi_2^0 \) in the \( G = \tilde{g} \) frame can also be adjusted between 0 and a maximum value by varying the angle between the quarks. Hence there always exist configurations where \( \beta_0^X = \beta_0^X \). The maximum mass will therefore always be given by equation (331).

8.7 Summary

In summary, the end points of the (ll), (qq) and (llqq) mass distributions allow the determination of the mass differences between \( M_G, M_X \) and \( M_0 \). The end point of (llq) provides a non-linear relation between the masses \( M_G, M_X \) and \( M_0 \) which, together with (ll) and (qq), allows all sparticle masses to be determined. Although, if \( M_G > M_X^2/M_0 \), the end points for \( M_{llq}, M_{llqq} \) and \( M_{llqq} \) lead to the same formula, it should, however, be noted that the configurations corresponding to their maxima are not the same (but they involve at least one soft particle) and may all carry useful complementary information.

It may also serve as a confirmation that the decays are indeed 3-body.

Alternatively, if \( M_G < M_X^2/M_0 \), the end points of \( M_{llq}, M_{llqq} \) and \( M_{llqq} \) all measure the mass difference between \( M_G \) and \( M_0 \) and only the \( M_{llq} \) end points allow the sparticle masses to be determined. Only in the unlikely case that \( M_G = M_X^2/M_0 \) is no absolute measurement of the sparticle masses possible. Fortunately, the common hypothesis of gaugino mass universality with the assumption that the lighter neutralinos and chargino are gaugino-like (like in MSUGRA) implies the relations \( M_G \approx 3.4M_X \) and \( M_X \approx 2M_0 \). This means that it is not unlikely that several end points will be given by collinear configurations.

9 End points for massive quarks starting from a \( \tilde{q} \)

The sequential decays may involve a fermion which mass cannot be neglected, e.g. the top mass in the decay chain

\[
i \to t\tilde{\chi}_2^0, \quad \tilde{\chi}_2^0 \to f_1 + \tilde{f}, \quad \tilde{f} \to f_2 + \tilde{\chi}_1^0 \tag{332}
\]

Taking, for instance, the configurations in section 4.1, a massive quark will affect the kinematics of \( q \) and \( X = \tilde{\chi}_2^0 \)

\[
E_q = \frac{M_Q^2 + m_q^2 - M_X^2}{2M_Q}, \quad p_q = \sqrt{E_q^2 - m_q^2} \tag{333}
\]

\[
E_X = \frac{M_Q^2 + M_X^2 - m_q^2}{2M_Q}, \quad p_X = \sqrt{E_X^2 - M_X^2} \tag{334}
\]

in the squark rest frame. The Lorentz transformation of a massless particle \( U \) from the rest frame of \( X \) to the rest frame of \( Q \) is modified correspondingly (for \( U \) parallel to the boost)

\[
E_{U} = \gamma_X E_{U}^{\prime}(1 + \beta_X \cos \theta^*) = \gamma_X (1 \pm \beta_X) E_{U}^{\prime} = \frac{1}{M_X} (E_X \pm p_X) E_{U}^{\prime} \tag{335}
\]

Note that the behaviour for small \( m_q \) can be verified as follows:

\[
p_X^2 = E_X^2 - M_X^2 = \frac{(M_Q^2 + M_X^2 - m_q^2)^2 - 4M_Q^2M_X^2}{4M_Q^2}
\]

\[
= \frac{(M_Q^2 - M_X^2)^2 - 2m_q^2(M_Q^2 - M_X^2) + m_q^4 - 4m_q^2M_X^2}{4M_Q^2} = \frac{(M_Q^2 - M_X^2)^2 - 4m_q^2M_X^2}{4M_Q^2} \tag{336}
\]

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For the lepton and the configurations are the same as encountered in the massless case. All lower end points are obtained and using equation (340) with

\[ E_X \pm p_X = \frac{1}{2M_Q} \left[ (M_{Q}^2 + M_X^2 - m_{q}^2) \pm \sqrt{(M_{Q}^2 - M_X^2 - m_{q}^2)^2 - 4m_{q}^2M_X^2} \right] \]  \hspace{1cm} (337)

which, for \( U \) along the boost direction (the + sign), becomes \( M_Q \) in the limit \( m_q \to 0 \) and hence equation (335) reproduces the result (14), as expected.

The effective mass becomes, in analogy with (44),

\[ M_{lq}^2 = (E_q + E_U)^2 - (-p_q + p_{LU})^2 - (0 + p_{TU})^2 = m_q^2 + 2E_qE_U + 2p_qp_{LU} \]  \hspace{1cm} (338)

where \( p_{LU} \) is \( \pm E_U \), depending on whether the lepton is along or opposite the boost. Hence

\[ M_{lq}^2 = m_q^2 + 2E_U(E_q \pm p_q) = m_q^2 + 2E_qE_U \left( 1 \pm \sqrt{1 - \left( \frac{m_q}{E_q} \right)^2} \right) \]  \hspace{1cm} (339)

When the quark is massive, the \((lq)\) and the \((llq)\) effective mass distributions have both an upper and a lower end point. The upper end point is given by

\[ (M_{lq}^{\text{max}})^2 = m_q^2 + 2E_U E_U \frac{E_X + p_X}{M_X} \left( 1 \pm \sqrt{1 - \left( \frac{m_q}{E_q} \right)^2} \right) \]  \hspace{1cm} (340)

and the configurations are the same as encountered in the massless case. All lower end points are obtained from a configuration displayed in Figure 28. Compared to the upper end point, both leptons are now opposite the boost. Hence, we have to use the minus sign in formulae (335) and (339), so that

\[ (M_{lq}^{\text{min}})^2 = m_q^2 + 2E_U E_U \frac{E_X - p_X}{M_X} \left( 1 \mp \sqrt{1 - \left( \frac{m_q}{E_q} \right)^2} \right) \]  \hspace{1cm} (341)

### 9.1 End points \( M(llq) \) for massive quarks

For the lepton \( l1 \), described in Figure 9, the lepton energy in the rest frame of \( X = \chi_2^0 \) is given by

\[ E_U = \frac{M_X^2 - M_{\chi_2}^2}{2M_X} \]  \hspace{1cm} (342)

and using equation (340) with \( U = l1 \) along the direction of the boost

\[ (M_{l1q}^{\text{max}})^2 = m_q^2 + 2M_Q^2 + m_q^2 - M_{\chi_2}^2 \frac{X}{M_X} p_X \frac{E_X + p_X}{M_X} \left( 1 \pm \sqrt{1 - \left( \frac{m_q}{E_q} \right)^2} \right) \]  \hspace{1cm} (343)

and, put in a form closer to the massless case:

\[ (M_{l1q}^{\text{max}})^2 = m_q^2 + M_Q^2 (1 - \frac{M_{\chi_2}^2}{M_X}) \left( 1 - \frac{M_{\chi_2}^2}{M_{\chi_2}^2} \right) \frac{E_X + p_X}{M_X} \frac{1}{2} \left( 1 \pm \sqrt{1 - \left( \frac{m_q}{E_q} \right)^2} \right) \]  \hspace{1cm} (344)

For the lower end point the direction of \( l1 \) is flipped and, from (341)

\[ (M_{l1q}^{\text{min}})^2 = m_q^2 + M_Q^2 (1 - \frac{M_{\chi_2}^2}{M_X}) \left( 1 - \frac{M_{\chi_2}^2}{M_{\chi_2}^2} \right) \frac{E_X - p_X}{M_X} \frac{1}{2} \left( 1 \mp \sqrt{1 - \left( \frac{m_q}{E_q} \right)^2} \right) \]  \hspace{1cm} (345)
9.2 End points $M(l2q)$ for massive quarks

For the lepton $l2$, described by the configuration labelled $(q3)$, the Lorentz transformation to the rest frame of $R = l$ remains unchanged, but now in the $X = \chi^0_l$ rest frame we get

$$E_{l2}^* = \frac{M_X M_R^2 - M_Q^2}{2M_R}$$

(346)

so that for this case

$$(M_{l2q}^{\text{max}})^2 = m_q^2 + 2 \frac{M_Q^2 + m_q^2 - M_R^2}{2M_Q} \frac{M_X M_R^2 - M_Q^2}{2M_R} \frac{M_R^2}{M_X} \left(1 + \sqrt{1 - \left(\frac{m_q}{E_q}\right)^2}\right)$$

(347)

and, put in a form closer to the massless case:

$$(M_{l2q}^{\text{max}})^2 = m_q^2 + M_Q^2 \left(1 - \frac{M_X^2}{M_Q^2} + \frac{m_q^2}{M_Q^2}\right) \left(1 - \frac{M_R^2}{M_Q^2}\right) \frac{E_X + p_X}{M_Q} \frac{1}{2} \left(1 + \sqrt{1 - \left(\frac{m_q}{E_q}\right)^2}\right)$$

(348)

For the lower end point, the direction of $l2$ is flipped and its energy in the $X = \chi^0_l$ frame is then

$$E_{l2}^* = \frac{M_R M_R^2 - M_Q^2}{2M_R}$$

(349)

Hence

$$(M_{l2q}^{\text{min}})^2 = m_q^2 + 2 \frac{M_Q^2 + m_q^2 - M_R^2}{2M_Q} \frac{M_X M_R^2 - M_Q^2}{2M_R} \frac{M_R^2}{M_X} \left(1 - \sqrt{1 - \left(\frac{m_q}{E_q}\right)^2}\right)$$

(350)

and, put in a form closer to the massless case:

$$(M_{l2q}^{\text{min}})^2 = m_q^2 + M_Q^2 \left(1 - \frac{M_X^2}{M_Q^2} + \frac{m_q^2}{M_Q^2}\right) \left(1 - \frac{M_R^2}{M_Q^2}\right) \frac{E_X - p_X}{M_Q} \frac{1}{2} \left(1 - \sqrt{1 - \left(\frac{m_q}{E_q}\right)^2}\right)$$

(351)

9.3 End points $M(l\bar{l}q)$ for massive quarks

Like in the case of massive quarks, there exist three collinear configurations that can lead to the end points. Here we consider only the first one. In this case, the effective mass becomes (see the configuration labelled $(q1)$)

$$(M_{l\bar{l}q}^{\text{max}})^2 = (E_{l\bar{l}q})^2 - (p_{l\bar{l}q})^2 = m_q^2 + 2(E_q E_{l1} + E_q E_{l2} + E_{l1} E_{l2}) - 2(p_q p_{l1} + p_q p_{l2} + p_{l1} p_{l2})$$

$$= m_q^2 + 2(E_q E_{l1} + E_q E_{l2} + E_{l1} E_{l2}) - 2(-p_q E_{l1} - p_q E_{l2} + E_{l1} E_{l2})$$

$$= m_q^2 + 2(E_q + p_q)(E_{l1} + E_{l2})$$

(352)

where

$$E_q + p_q = E_q \left(1 + \sqrt{1 - \left(\frac{m_q}{E_q}\right)^2}\right)$$

(353)

In the rest frame of $X = \chi^0_l$, the lepton energies are, for $l1$

$$E_{l1}^* = \frac{M_X^2 - M_R^2}{2M_X}$$

(354)

and as $l2$ is opposite the boost of $R$

$$E_{l2}^* = \frac{M_R M_R^2 - M_Q^2}{2M_R} = \frac{M_R^2 - M_Q^2}{2M_X}$$

(355)

In the rest frame of $Q = \bar{q}$ these become, after boosting both forward (see the configuration labelled $(q1)$)

$$E_{l1} + E_{l2} = \frac{1}{M_X} (E_X + p_X) \left[\frac{M_X^2 - M_R^2}{2M_X} + \frac{M_R^2 - M_Q^2}{2M_X}\right] = \frac{1}{M_X} (E_X + p_X) \frac{M_X^2 - M_Q^2}{2M_X}$$

(356)

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and thus

\[(M_{llq}^{\text{max}})^2 = m_q^2 + 2 \frac{M_Q^2 + m_q^2 - M_X^2}{2M_Q} \left( \frac{M_X - M_0}{2M_X} \right) + p_X \left( 1 + \sqrt{1 - \left( \frac{m_q}{E_q} \right)^2} \right) \]

\[= m_q^2 + M_Q^2 \left( 1 - \frac{M_X^2}{M_Q^2} \right) + \left( \frac{m_q}{M_Q} \right)^2 \left( 1 + \sqrt{1 - \left( \frac{m_q}{E_q} \right)^2} \right) \]

(357)

For the lower end point, the direction of both leptons is opposite to their boost. Therefore, \(E_q + p_q\) becomes \(E_q - p_q\), whereas in \(E_1 + E_2\) the sign of \(p_X\) is flipped. Altogether

\[(M_{llq}^{\text{min}})^2 = m_q^2 + M_Q^2 \left( 1 - \frac{M_X^2}{M_Q^2} \right) + \left( \frac{m_q}{M_Q} \right)^2 \left( 1 + \sqrt{1 - \left( \frac{m_q}{E_q} \right)^2} \right) \]

(358)

Depending on the relations among the sparticle masses, other configurations can yield the true end point, but they are not listed here. They can be derived from the general formulae in section 17.

9.4 Conclusion
Used together with the \(M_{ll}\) end point, the above end points in \(M_{lq}\) and \(M_{llq}\) provide sufficient information to reconstruct the sparticle masses.

10 Cascades with a 3-body decay of \(\tilde{g}\) and 3-body decay of \(\tilde{\chi}_2^0\), massive quarks

There is a small portion of the parameter space where the gluino is lighter than any of the squarks and decays to the \(\tilde{\chi}_2^0\) which undergoes itself a direct three-body decay to leptons. The case where all quarks and leptons are massless has been treated in section 8, on which we will heavily rely. However, here we want to generalize the derivation to the case where quarks are massive, some may e.g. be top quarks. The cascade is

\[\tilde{g} \to q_1 q_2 \tilde{\chi}_2^0 \to q_1 q_2 \ell + l + \tilde{\chi}_1^0 \]

(359)

The formulae will equally apply to the decay via a chargino, in which case the two quarks are distinct (hence they are noted as \(q_1\) and \(q_2\)) and one of the leptons is a \(\nu\). The effective mass distributions available for constraining the three sparticle masses involved are \(ll, qq, lq_1, lq_2, llq_1, llq_2, llq\) and \(llqq\). In this case, there is no kinematical distinction between the two leptons (assumed to be massless).

10.1 Upper end point in \(M(ll)\)
The \((ll)\) effective mass distribution and end point are unchanged compared to section 8, hence

\[M_{ll}^{\text{max}} = M_X - M_0 \]

(360)

10.2 Upper end point in \(M(qq)\)
Also the \((qq)\) decay is the same as in section 8, hence

\[M_{qq}^{\text{max}} = M_G - M_X \]

(361)

10.3 Upper end point in \(M(lq)\)
The effective mass \(M(lq_1)\) is maximized in a configuration where the lepton and quark are back-to-back and have their maximal energy

\[M_{lq_1}^2 = m_{q_1}^2 + 2(E_{q_1} + p_{q_1,L})E_l \]

(362)
For the lepton, the configuration is the same as in section 8.3 which implies that its energy in the $G = \tilde{g}$ frame is equivalent to the one for $\text{10.4 Upper end point in } M(\text{llq})$

The mass of the (llq) system is maximized by maximizing $E_l(q1)$, i.e. by putting $q2$ at rest in the $G = \tilde{g}$ frame, and by maximizing the sum of the lepton longitudinal momenta, which is obtained by taking them parallel and opposite to the quark. As the leptons are considered massless, this configuration is equivalent to the one for $M(llq1)$. Therefore, the mass is given by the same formula as equation (365)

$$(M_{llq1}^{max})^2 = m_{q1}^2 + (E_{q1} + p_{q1})(E_X + p_X)(1 - \frac{M_0^2}{M_X^2})$$

with $E_X$ and $p_X$ defined in (363). No new constraint is obtained.

However, this collinear configuration can only lead to the true end point if the boost from $X = \tilde{\chi}_1^0$ is large enough compared to the $\tilde{\chi}_1^0$ velocity. Else the $\tilde{\chi}_1^0$ can be put at rest in the $G = \tilde{g}$ frame in a non-collinear configuration. In the $G = \tilde{g}$ rest frame the velocity of $X = \tilde{\chi}_1^0$ is

$$\beta_X = \frac{p_X}{E_X}$$

and the velocity of $\tilde{\chi}_1^0$ in the $X = \tilde{\chi}_1^0$ frame is largest when the two leptons are parallel

$$\beta_0 = \frac{M_X^2 - M_q^2}{M_X^2 + M_q^2}$$

The $\tilde{\chi}_1^0$ cannot be put at rest in the $G = \tilde{g}$ frame provided $\beta_0^0 \leq \beta_X$, in which case the end point is given by formula (366). Else, if $\beta_0^0 \geq \beta_X$, the $\tilde{\chi}_1^0$ velocity can be made equal to $\beta_X$ by introducing an opening angle between the two leptons and keep the $\tilde{\chi}_1^0$ at rest in the $G = \tilde{g}$ frame, which leads to the end point

$$M_{llq1}^{max} = M_G - M_0 - m_{q2}$$

10.5 Upper end point in $M(llq)$

Following a reasoning analogous to the one of section 8.4, the maximum of $M(llq)$ is reached when the two quarks are parallel, the lepton included in the mass is back-to-back with the quarks and the other lepton is at rest in the $X = \tilde{\chi}_1^0$ frame. As for massive quarks the di-quark mass is not fixed (is non-zero), it can be chosen to maximize $E_X$ or equivalently to minimize $M(qq)$. Now

$$M_{qq}^2 = m_{q1}^2 + m_{q2}^2 + 2E_{q1}E_{q2} - 2p_{q1}p_{q2}$$
This can be minimized e.g. with respect to \( p_{q_1} \)

\[
\frac{\partial M_{qq}^2}{\partial p_{q_1}} = 2 \frac{p_{q_1}}{E_{q_1}} E_{q_2} - 2 p_{q_2} = 0
\]

which yields

\[
\frac{p_{q_1}}{E_{q_1}} = \frac{p_{q_2}}{E_{q_2}} \text{ and also } \frac{E_{q_1}}{m_{q_1}} = \frac{E_{q_2}}{m_{q_2}}
\]

The effective mass then gives the expected result

\[
M_{qq}^2 = m_{q_1}^2 + m_{q_2}^2 + 2 \frac{m_{q_2}^2 E_{q_1}^2}{m_{q_1}^2} m_{q_1} - 2 \frac{m_{q_2}^2}{m_{q_1}^2} p_{q_1}^2 = m_{q_1}^2 + m_{q_2}^2 + 2 \frac{m_{q_2}^2 m_{q_1}^2}{m_{q_1}^2} = (m_{q_1} + m_{q_2})^2
\]

For this fixed di-quark mass, we have

\[
E_{qq} = \frac{M_G^2 + (m_{q_1} + m_{q_2})^2 - M_X^2}{2M_G}, \quad E_X = \frac{M_G^2 - (m_{q_1} + m_{q_2})^2 + M_X^2}{2M_G}
\]

\[
p_{qq} = \sqrt{(M_G^2 - (m_{q_1} + m_{q_2})^2 - M_X^2)^2 - 4(m_{q_1} + m_{q_2})^2 M_X^2} = p_X
\]

It leads to the expression for the mass

\[
(M_{qq}^{\text{max}})^2 = M_G^2 + 2(E_{qq} + p_{qq}) E_d
\]

or

\[
(M_{qq}^{\text{max}})^2 = (m_{q_1} + m_{q_2})^2 + (E_{qq} + p_{qq})(E_X + p_X)(1 - \frac{M_0^2}{M_X^2})^2
\]

(368)

and provides, in principle, an additional constraint on sparticle masses. The region of validity of this formula can be derived by requiring \( \beta_X \geq \beta_0 \)

\[
\frac{p_X}{E_X} \geq \frac{M_X^2 - M_0^2}{M_X^2 + M_0^2}
\]

This inequality can be solved to put a condition on \( M_G \)

\[
(M_X^2 + M_0^2)^2(E_X^2 - M_X^2) \geq (M_X^2 - M_0^2)^2 E_X^2
\]

\[
4M_X^2 M_0^2 E_X^2 \geq M_X^2 (M_X^2 + M_0^2)^2
\]

\[
4M_0^2 E_X^2 \geq (M_X^2 + M_0^2)^2
\]

Introducing the value of \( E_X \) gives

\[
M_G^2 \left[ M_G^2 - (m_{q_1} + m_{q_2})^2 + M_X^2 \right] \geq (M_X^2 + M_0^2)^3 M_G^2
\]

\[
M_G^2 M_0^4 - [2M_0^2 (m_{q_1} + m_{q_2})^2 - 2M_0^2 M_X^2 + (M_X^2 + M_0^2)^2] M_G^2 + [(m_{q_1} + m_{q_2})^2 - M_X^2]^2 M_0^2 \geq 0
\]

\[
M_G^2 M_0^4 - [2M_0^2 (m_{q_1} + m_{q_2})^2 + M_X^2 + M_0^2] M_G^2 + [(m_{q_1} + m_{q_2})^2 - M_X^2]^2 M_0^2 \geq 0
\]

The solution of the equality is

\[
M_G^2 = \frac{1}{2M_0^2} \left[ 2M_0^2 (m_{q_1} + m_{q_2})^2 + M_X^2 + M_0^2 \pm \sqrt{\Delta} \right]
\]

where the discriminant is

\[
\Delta = \left[ 2M_0^2 (m_{q_1} + m_{q_2})^2 + M_X^2 + M_0^2 \right]^2 - 4M_0^4 \left[ (m_{q_1} + m_{q_2})^2 - M_X^2 \right]^2
\]

\[
= 4M_0^4 (M_X^2 + M_0^2)^2 (m_{q_1} + m_{q_2})^2 + (M_X^2 + M_0^2)^2 - 4M_0^4 M_X^2
\]

\[
= 4M_0^4 (M_X^2 + M_0^2)^2 (m_{q_1} + m_{q_2})^2 + (M_X^2 + M_0^2)^2 - 4M_0^4 M_X^2
\]

To illustrate this result, we can take the limit \( m_{q_1} = m_{q_2} = 0 \). The two solution are then \( M_G = M_X^2 / M_0 \), the same as obtained in section 8.5, and \( M_G = M_0 \), which is meaningless. The validity range for the collinear configuration is thus for \( M_G \) larger than the "++" solution of the above equation.

If this inequality is not fulfilled, the maximum mass is given by

\[
M_{qq}^{\text{max}} = M_G - M_0
\]

(369)

and the configuration is non-collinear.
10.6 Upper end point in $M(llqq)$

The absolute maximum value of $M(llqq)$ is reached when the $\tilde{\chi}_1^0$ is at rest in the $G = \tilde{g}$ frame and hence

$$M_{llqq}^{\text{max}} = M_G - M_0$$  \hspace{1cm} (370)

Unlike the cases considered earlier where several intermediate sparticles were present, in case there is only a $\tilde{\chi}_0^0$, the $\tilde{\chi}_1^0$ can always be put at rest in a non-collinear configuration where it is emitted opposite to the boost from the $\tilde{\chi}_0^0$ in the $\tilde{\chi}_2^0$ frame to be adjusted between 0 and a maximum value. The velocity $\beta_X$ of the $\tilde{\chi}_2^0$ in the $G = \tilde{g}$ frame can also be adjusted between 0 and a maximum value by varying the angle between the quarks. Hence there always exist configurations where $\beta_0^0 = \beta_X$. The maximum mass will therefore always be given by equation (331).

10.7 Summary

In summary, the end points of the $(ll)$, $(qq)$ and $(llqq)$ mass distributions allow the determination of the mass differences between $M_G$, $M_X$ and $M_0$. If the velocity of $X = \tilde{\chi}_2^0$ is large enough, the end points of $(lq)$, $(llq)$ and $(lqq)$ provide a non-linear relation between the masses $M_G$, $M_X$ and $M_0$ which, together with $(ll)$ and $(qq)$, allow all sparticle masses to be determined. Although the end points for $M_{lq}$ and $M_{llq}$ lead to the same formula, it should, however, be noted that the configurations corresponding to their maxima are not the same (but they involve at least one soft particle) and may all carry useful complementary information. It may also serve as a confirmation that the decays are indeed 3-body.

Alternatively, if $M_G$ is small compared to $M_X$, i.e. the velocity of $X = \tilde{\chi}_2^0$ is small, the end points of $M_{lq}$, $M_{llq}$, $M_{lqq}$ and $M_{llqq}$ all measure the mass difference between $M_G$ and $M_0$ and no absolute measurement of the sparticle masses is possible. Fortunately, the common hypothesis of gaugino mass universality with the assumption that the lighter neutralinos and chargino are gaugino-like (like in MSUGRA) implies the relations $M_G \approx 3.4 M_X$ and $M_X \approx 2 M_0$. This means that it is not unlikely that several end points will be given by collinear configurations.

11 Sequential 2-body decays from a $\tilde{q}$ to final states with $h^0$

In a large part of the parameter space, the $\tilde{\chi}_2^0$ decays dominantly to the $h^0$. The decay chain is then

$$\tilde{q} \rightarrow q\tilde{\chi}_2^0 \rightarrow qh^0\tilde{\chi}_1^0$$  \hspace{1cm} (371)

The Higgs is supposed to be observed through its decay into $bb$ and its mass reconstructed. Note that the formulae derived for the $h^0$ are equally applicable to the $\tilde{\chi}_2^0$ decaying into $\tilde{\chi}_1^0Z^0$.

11.1 End point in $M(h^0q)$

The configuration leading to a maximum in $M(h^0q)$ is shown in figure 29. The only available kinematical variable is then $M(hq)$. In the squark rest frame this is

$$M_{hq}^2 = M_h^2 + 2 E_q (E_h + p_{hL})$$  \hspace{1cm} (372)

where the longitudinal momentum is signed and the quark energy is given in (54).
In the rest frame of the $X = \tilde{\chi}^0_2$, the energy and momentum of the $h^0$ are

$$E_h^* = \frac{M_X^2 - M_0^2 + M_h^2}{2M_X}$$

$$p_h^* = \frac{1}{2M_X} \sqrt{(M_X^2 - M_0^2 - M_h^2)^2 - 4M_0^2 M_h^2}$$  \hspace{1cm} (373)

The Lorentz transformation to the $\tilde{q}$ rest frame gives

$$E_h = \gamma_X (E_h^* + \beta_X p_h^* \cos \theta^*)$$

$$p_{hL} = \gamma_X (\beta_X E_h^* + p_h^* \cos \theta^*)$$  \hspace{1cm} (374)

From this,

$$E_h + p_{hL} = \gamma_X (1 + \beta_X)(E_h^* + p_h^* \cos \theta^*)$$

$$= \frac{M_Q^2 + M_X^2}{2M_Q M_X} \left( 1 + \frac{M_Q^2 - M_X^2}{M_Q^2 + M_X^2} \right) (E_h^* + p_h^* \cos \theta^*)$$

$$= \frac{M_Q}{M_X} (E_h^* + p_h^* \cos \theta^*)$$  \hspace{1cm} (375)

and the effective mass becomes

$$M_{hq}^2 = M_h^2 + \frac{2 M_Q^2 - M_X^2}{M_Q} \frac{M_Q}{M_X} (E_h^* + p_h^* \cos \theta^*)$$

$$= M_h^2 + \frac{M_Q^2 - M_X^2}{M_X} (E_h^* + p_h^* \cos \theta^*)$$  \hspace{1cm} (376)

where $E_h^*$ and $p_h^*$ are given by the expression (373). Due to the non-zero mass of the Higgs, the $(hq)$ mass distribution has both a maximum and a minimum value

$$(M_{hq}^{\text{max}})^2 = M_h^2 + \frac{2 M_Q^2 - M_X^2}{2M_X^2} \left( M_X^2 - M_0^2 + M_h^2 + \sqrt{(M_X^2 - M_0^2 - M_h^2)^2 - 4M_0^2 M_h^2} \right)$$  \hspace{1cm} (377)

$$(M_{hq}^{\text{min}})^2 = M_h^2 + \frac{2 M_Q^2 - M_X^2}{2M_X^2} \left( M_X^2 - M_0^2 + M_h^2 - \sqrt{(M_X^2 - M_0^2 - M_h^2)^2 - 4M_0^2 M_h^2} \right)$$  \hspace{1cm} (378)

A similar formula is obtained when considering the decay to a $Z^0$. If it is assumed that the decay follows approximately phase space, the mass distribution corresponds to 2 particles out of 3 and should drop steeply near the upper end. The maximum mass should, therefore, be well visible.

Even if we assume that both end points for $h^0$ are measurable, they lead to only two constraints on the three unknown masses. They do not, on their own, allow to determine all masses involved. They may, however, bring valuable information if used in conjunction with another decay mode, e.g. the $Z^0$ or di-leptons, provided the latter has a sufficient branching ratio. Else, if the squark itself originates from the decay of a gluino more constraints become available, allowing to determine unambiguously the masses. An interesting, but not so infrequent, situation may even arise, where the decay modes of the squark $\tilde{q} \rightarrow q\tilde{\chi}_0^0 \rightarrow q\ell\bar{\ell}$ and $\tilde{q} \rightarrow q\tilde{\chi}_0^0 \rightarrow qh^0\tilde{\chi}_0^0$ compete, allowing the $h^0$ decay channel, together with the di-leptons, to determine the mass of the $\tilde{\chi}_0^0$.

Although the usage of such extra information would be preferable, in case it does not exist, there would still be a possibility to infer all masses from a detailed shape analysis of the $M(lq)$ distribution. But this requires to take correctly into account the spin effect in the decay, as well as to control very accurately the backgrounds, distortions due to acceptance and efficiency, etc.

12 Sequential 2-body decays from a $\tilde{g}$ to final states with $h^0$

The decay chain considered is now

$$\tilde{g} \rightarrow q\bar{q}, \tilde{g} \rightarrow q\tilde{\chi}_0^0, \tilde{\chi}_0^0 \rightarrow h^0\tilde{\chi}_0^0$$  \hspace{1cm} (379)

where the additional kinematical quantities available are $M(q\bar{q}0)$, $M(h^0q1)$ and $M(h^0q1q2)$.  

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12.1 Overview of the collinear configurations

The following collinear configurations will need to be considered for the calculation of end points.

\[ (ha) \quad (hb) \]

\[ \begin{align*}
M_{\text{h}0}^\text{min} \\
M_{\text{h}\text{q}2}^\text{max} \\
M_{\text{h}0\text{q}2}^\text{max} \\
M_{\text{h}0\text{q}2}^\text{max,2} \\
\end{align*} \]

\[ \begin{align*}
M_{\text{h}0}^\text{min} \\
M_{\text{h}\text{q}2}^\text{max} \\
M_{\text{h}\text{q}2}^\text{max,2} \\
M_{\text{h}0\text{q}2}^\text{max} \\
\end{align*} \]

12.2 End point in \( M(q_1q_2) \)

The configuration for this end point is the same as in section 5.2 and the end point formula is

\[ M_{\text{q}1\text{q}2}^\text{max} = M_G \sqrt{(1 - M_Q^2/M_G^2)(1 - M_X^2/M_Q^2)} \]  (380)

As this decay proceeds via an intermediate scalar, the mass distribution is expected to be triangular.

12.3 End points in \( M(h^0q_1) \)

As seen from the discussion in section 2.5, the effective mass is

\[ M_{\text{h}q_1}^2 = M_h^2 + 2E_{q_1}(E_h + p_{hL}) \]  (381)

defined in the rest frame of \( G = \bar{g} \) and with the quark energy given by (121). Then

\[ E_h + p_{hL} = \gamma_Q(1 + \beta_Q)(E_h' \pm p_{hL}') \]  (382)

where the \( \pm \) give respectively the maximum and minimum and the primed quantities are defined in the rest frame of \( Q = \bar{q} \).

12.3.1 Upper end point in \( M(h^0q_1) \)

The configuration leading to a maximum in \( M(h^0q_1) \) is called \( (ha) \). As the decay configurations of the \( \bar{q} \) and \( \chi_2^0 \) are the same as encountered in section 11, the \( E_h' + p_{hL}' \) is given by formula (375) with \( p_h^* \) defined in (373). Hence, for the configuration leading to the maximum

\[ E_h + p_{hL} = \frac{M_G M_Q}{M_X} (E_h^* + p_h^* \cos \theta^*) \]  (383)
and the effective mass is
\[
M_{hq1}^2 = M_h^2 + 2 \frac{M_G^2 - M_Q^2 M_G}{2M_G^2} M_X (E_h^* + p_h^0 \cos \theta^*)
\]
\[
= M_h^2 + \frac{M_G^2 - M_Q^2}{M_Q^2} M_X (E_h^* + p_h^0 \cos \theta^*)
\]  
(384)

The maximum value of the mass distribution is then
\[
(M_{hq1}^{\text{max}})^2 = M_h^2 + \frac{M_G^2 - M_Q^2}{2M_Q^2} \left( M_X^2 - M_0^2 + M_Q^2 + \sqrt{(M_X^2 - M_0^2 - M_Q^2)^2 - 4M_Q^2 M_h^2} \right)
\]
(385)

The comparison of the two (hq) end points shows that \( M_{hq1} \geq M_{hq2} \) provided
\[
M_G^2 - M_Q^2 \geq M_Q^2 - M_h^2
\]
(386)

which is valid for both maximum and minimum end points.

A minimum can be obtained by flipping the direction of the \( h^0 \) in the \( X = \tilde{\chi}_0^0 \) rest frame \((\cos \theta^* = -1)\), as in configuration \((hb)\), but this will not give the true minimum (see below).

12.3.2 Distinguishing upper end points of \( M(h^0 q1) \) and \( M(h^0 q2) \)

It is seen from the configurations that the upper end point of \( M(h^0 q2) \) is reached independent of the orientation with respect to \( q1 \), hence for any mass \( M(q1q2) \). On the other hand, the upper end point of \( M(h^0 q1) \) requires the \( M(q1q2) \) to be minimal. This allows the two end points to be distinguished.

12.3.3 Lower end point in \( M(h^0 q1) \)

The configuration leading to the minimum \( M(h^0 q1) \) is labelled \((hd)\) (see section 2.5)
\[
E_h + p_{hL} = \frac{M_G}{M_Q} M_X (E_h^* - p_h^0 \cos \theta^*)
\]
(387)

the effective mass is
\[
M_{hq1}^2 = M_h^2 + 2 \frac{M_G^2 - M_Q^2 M_G M_X}{2M_G M_Q} (E_h^* - p_h^0 \cos \theta^*)
\]
\[
= M_h^2 + \frac{M_G^2 - M_Q^2}{M_Q^2} M_X (E_h^* - p_h^0 \cos \theta^*)
\]  
(388)

and the minimum value of the mass distribution is given by
\[
(M_{hq1}^{\text{min}})^2 = M_h^2 + \frac{M_G^2 - M_Q^2}{2M_Q^2} \left( M_X^2 - M_0^2 + M_Q^2 - \sqrt{(M_X^2 - M_0^2 - M_Q^2)^2 - 4M_Q^2 M_h^2} \right)
\]  
(389)

A secondary maximum can be obtained by flipping the direction of the \( h^0 \) in the \( X = \tilde{\chi}_0^0 \) rest frame \((\cos \theta^* = -1)\), as in configuration \((hb)\), but this is always below the maximum (385).

12.4 End point for the sum \( M(h^0 q1) + M(h^0 q2) \)

It is seen from the figure that for configuration \((ha)\) both \( M(h^0 q1) \) and \( M(h^0 q2) \) take their maximum value, hence
\[
[M_{hq1} + M_{hq2}]^{\text{max}} = M_{hq1}^{\text{max}} + M_{hq2}^{\text{max}}
\]  
(390)

which may allow the determination of one of the end points, once the other is found as the true \( M(h^0 q) \) end point.
12.5 End points in $M(h^0q1q2)$

12.5.1 Upper end points in $M(h^0q1q2)$

The configuration leading to a maximum in $M(h^0q1q2)$ may correspond to any of the configurations labelled $(ha)$, $(hc)$ or $(hd)$. The one corresponding to the minimum is labelled $(hb)$.

Following the conventions outlined in section 2.5, The effective mass for the configurations $(ha)$ and $(hb)$ is

\[(M_{hq})^2 = M_h^2 + 2E_qE_h + 2E_q2E_h + 2E_qph_{h,L} + 2E_q2p_{h,L} \]
\[= M_h^2 + 2(E_q + E_q)(E_h + p_{h,L}) \quad (391)\]

and for the configurations $(hc)$ and $(hd)$

\[(M_{hq})^2 = M_h^2 + 2E_qE_h + 2E_qE_h + 2E_q2E_h + 2E_q2E_q2 + 2E_q2p_{h,L} - 2E_q2p_{h,L} \]
\[= M_h^2 + 4E_qE_h + 2E_q(E_h + p_{h,L}) + 2E_q2(E_h - p_{h,L}) \quad (392)\]

It can already be noticed that this expression contains $M_{hq}^2$.

For the configurations $(ha)$ and $(hb)$

\[\begin{align*}
E_{q2} &= \frac{M_Q M_Q^2 - M_h^2}{2M_Q} = \frac{M_h^2 - M_h}{2M_Q} \\
E_{q1} + E_{q2} &= \frac{M_h^2 - M_h}{2M_Q} + \frac{M_h^2 - M_h}{2M_Q} = \frac{M_h^2 - M_h}{2M_Q}
\end{align*} \]

and the expression for $(E_h + p_{h,L})$ is given by (383). The mass is then

\[(M_{hq})^2 = M_h^2 + \frac{M_h^2 - M_h}{M_Q}(E_h + p_h^* cos \theta^*) \quad (393)\]

which is independent of $M_Q$.

For the configurations $(hc)$ and $(hd)$

\[E_{q2} = \frac{M_h^2 - M_h}{2M_Q} \]

the expression for $(E_h + p_{h,L})$ is given by (387) and, following section 2.5

\[E_h - p_{h,L} = \gamma_Q (1 - \beta Q)(E_h + p_{h,L}) = \frac{M_Q}{M_G} (E_{h}^* + p_h^* cos \theta^*) = \frac{M_Q}{M_G} (E_{h}^* + p_h^* cos \theta^*) \]

From this

\[(M_{hq})^2 = M_h^2 + 4 \frac{M_h^2 - M_h}{2M_Q} \frac{M_h^2 - M_h}{2M_Q} \]
\[+ 2 \frac{M_h^2 - M_h}{2M_Q} \frac{M_h^2 - M_h}{2M_Q} (E_h^* - p_h^* cos \theta^*) + 2 \frac{M_h^2 - M_h}{2M_Q} \frac{M_h^2 - M_h}{2M_Q} \frac{M_h^2 - M_h}{2M_Q} (E_h^* + p_h^* cos \theta^*) \]

or

\[(M_{hq})^2 = M_h^2 + \frac{(M_h^2 - M_h)(M_h^2 - M_h)}{M_Q} \]
\[+ \frac{M_h^2 - M_h}{M_Q} M_Q (E_h^* - p_h^* cos \theta^*) + \frac{M_h^2 - M_h}{M_Q} M_Q (E_h^* + p_h^* cos \theta^*) \quad (394)\]

in which one recognizes $M_{hq}^2$ and $M_{hq2}^2$, so that it can also be written as

\[(M_{hq})^2 = M_{hq}^2 + M_{hq2}^2 + \frac{M_h^2 - M_h}{M_Q} M_Q (E_h^* - p_h^* cos \theta^*) \quad (395)\]
12.5.2 Absolute maximum of $M(hq1q2)$

Depending on the values of the masses, the $\tilde{X}_0$ can be put at rest or sent backward in the $G = \tilde{g}$ rest frame. In the latter case, the above formulae for the maxima are no longer applicable and the upper end point corresponds to a non-collinear configuration with mass

$$M_{hq2}^{\text{max}} = M_G - M_0$$

(396)

This happens in the situation where $\beta_Q \leq \beta_0^-$, where $\beta_0^-$ is the neutralino velocity in the $Q = \tilde{q}$ rest frame. For any $M(hq2)$, maximum or minimum, the condition is

$$\frac{M_G^2 - M_Q^2}{M_{hq2}^2} \leq \sqrt{(M_G^2 + M_Q^2 - M_{hq2}^2)^2 - 4M_G^2M_{hq2}^2}$$

$$-4M_G^2M_Q^2(M_G^2 + M_Q^2 - M_{hq2}^2)^2 \leq -(M_G^2 + M_Q^2)^24M_G^2M_Q^2$$

$$M_G(M_G^2 + M_Q^2 - M_{hq2}^2) \geq (M_G^2 + M_Q^2)M_0$$

$$M_Q^2(M_G - M_0) + M_GM_0(M_0 - M_G) \geq M_GM_{hq2}^2$$

or

$$(M_{hq2}^2)^2 \leq \frac{(M_G - M_0)(M_Q^2 - M_GM_0)}{M_G}$$

(397)

The case where $M_{hq2} = M_{hq2}^{\text{min}}$ corresponds to configuration $(hc)$, the one where $M_{hq2} = M_{hq2}^{\text{max}}$ to configurations $(ha)$ or $(hd)$. It is interesting to note that, when $M_h \to 0$ we have $M_{hq2}^{\text{min}} = 0$ and the condition becomes $M_Q^2 - M_GM_0 \geq 0$, i.e. the same as obtained for $M_{11qq}^{\text{max}}$ equation (190) (after replacing $M_R$ by $M_0$). For the same limiting case, we have

$$(M_{hq2}^{\text{max}})^2 = \frac{M_Q^2 - M_X^2}{M_X^2}(M_X^2 - M_0^2)$$

for which the condition yields

$$(M_Q^2 - M_GM_0)(M_G - M_0)M_X^2 \geq M_G(M_X^2 - M_0^2)(M_X^2 - M_0^2)
- M_GM_X^2M_0 - M_G^2M_X^2M_0 \geq -M_GM_X^2 - M_GM_Q^2M_0
M_GM_X^2(M_X^2 - M_GM_0) - M_G^2M_0(M_X^2 - M_GM_0) \geq 0
(M_X^2 - M_GM_0)(M_GM_X^2 - M_0^2) \geq 0$$

where the first factor reproduces the condition for $M_{11qq}^{\text{max}^2}$, equation (190) in a configuration similar to $(ha)$ and the second one the condition for $M_{11qq}^{\text{max}^3}$, equation (191) in a configuration similar to $(hd)$.

This clarifies the relation between the massless case and the case where $M_h \neq 0$. In analogy with the massless case, let us now compute the values of $M_G$ at which the collinear configurations do not give the true maximum any more. This occurs when

$$(M_{hq2}^2)^2 = \frac{(M_Q^2 - M_X^2)(M_X^2 - M_0^2)}{M_G}
M_0M_Q^2 - (M_Q^2 + M_G^2 - M_{hq2}^2)M_G + M_{hq2}^2M_0 = 0$$

which has the solutions

$$M_G = \frac{1}{2M_0}\left[M_Q^2 + M_0^2 - M_{hq2}^2 \pm \sqrt{(M_Q^2 + M_0^2 - M_{hq2}^2)^2 - 4M_Q^2M_0^2}\right]$$

(398)

To understand which solution is associated to each collinear configuration, let us take again the limit $M_h \to 0$. Then, for $M_{hq2}^{\text{max}} = (M_{hq2}^{\text{max}})^2 = 0$, the "+" solution gives $M_G = M_Q^2/M_0$ which corresponds to configuration $(hc)$. For the configuration with $M_{hq2}^{\text{max}}$

$$(M_{hq2}^2)^2 = \frac{M_Q^2 - M_X^2}{M_X^2}(M_X^2 - M_0^2)$$
so that
\[ M_Q^2 + M_0^2 - (M_{hq2})^2 = M_Q^2 + M_0^2 - \frac{M_Q^2 - M_X^2}{M_X^2}(M_X^2 - M_0^2) = \frac{1}{M_X^2}(M_Q^2M_0^2 + M_X^2) \]
and
\[
M_G = \frac{1}{2M_0} \left[ \frac{M_Q^2M_0^2 + M_X^2}{M_X^2} \pm \frac{(M_Q^2M_0^2 + M_X^2)^2}{M_X^2} - 4M_Q^2M_0^2 \right]^{-1/2} \\
= \frac{1}{2M_X^2M_0}[M_Q^2M_0^2 + M_X^2 \pm (M_Q^2M_0^2 - M_X^2)]
\]
The "+" solution gives \( M_G = M_X^2/M_0 \) and corresponds to configuration \((h\bar{d})\); the "-" solution gives \( M_G = M_X^2/M_0 \) and corresponds to configuration \((h\bar{a})\).

12.5.3 First collinear end point in \(M(h^0q1q2)\)
The first collinear end point is obtained from configuration \((hc)\) with the mass given by equation (394) with \( \cos \theta^* = -1 \)
\[
(M_{hq1q2}^{max})^2 = M_h^2 + \frac{(M_G^2 - M_Q^2)(M_G^2 - M_X^2)}{M_Q^2} + \frac{M_G^2 - M_Q^2}{M_X^2}M_X(E_h^* + p_h^*) + \frac{M_G^2 - M_X^2}{M_X^2}M_X(E_h^* - p_h^*)
\]
(399)
which can also be written as
\[
(M_{hq1q2}^{max})^2 = M_q^2 + (M_{hq2}^{max})^2 + \frac{M_G^2 - M_Q^2}{M_Q^2}M_X(E_h^* + p_h^*)
\]
(400)

Depending on the values of the masses, the \(\chi^3_q\) can be put at rest or sent backward in the \(G = \bar{g}\) rest frame. The condition of validity of this equation is given by the relation
\[
M_G \geq \frac{1}{2M_0} \left[ M_Q^2 + M_0^2 - (M_{hq2}^{max})^2 + \sqrt{(M_Q^2 + M_0^2 - (M_{hq2}^{max})^2)^2 - 4M_Q^2M_0^2} \right]
\]
(401)
where \(M_{hq2}^{min}\) is given by equation (378).

Alternative way of computing the end point for configuration \((hc)\)
To compute \(M(hq)\), note that in this configuration it is equivalent to compute the maximum mass of \(q1\) opposite to the \(hq2\) system in its minimal mass, equation (378). Then
\[
(M_{hq}^{max})^2 = (E_{q1} + E_h)q^2 - (p_{q1} + p_h)^2 = M_{hq}^2 = 2E_{q1}E_h - 2p_{q1}p_h
\]
(402)
Using primed variables in the rest frame of \(Q = \bar{g}\)
\[
E_{hq}^\prime = E_{q2}^\prime + E_h^\prime, p_{hq}^\prime = E_{q2}^\prime - p_h^\prime
\]
(403)
so that in the rest frame of \(G = \bar{g}\)
\[
E_{hq} = \gamma_Q(E_{hq} + \beta_Q p_{hq}^\prime \cos \theta^*), p_{hq,L} = \gamma_Q(\beta_Q E_{hq}^\prime + p_{hq}^\prime \cos \theta^*)
\]
(404)
where the angle \(\theta^*\) of the \(hq\) system in the \(Q = \bar{g}\) rest frame is measured with respect to the direction of \(Q\). To obtain \(M_{hq}^{max}\) we have \(\cos \theta^* = +1\) and
\[
E_{hq} + p_{hq,L} = \gamma_Q(1 + \beta_Q)(E_{hq}^\prime + p_h^\prime) = \frac{M_G^2 + M_Q^2}{2M_G M_Q} \left[ 1 + \frac{M_G^2 - M_X^2}{M_G^2 + M_X^2} \right] (E_{hq}^\prime + p_h^\prime) = \frac{M_G}{M_Q}(E_{hq} + p_h)
\]
(405)
as \( \cos \theta^* = -1 \) for this configuration and the quantities \( E^*_h \) and \( p^*_h \) are defined in (373). Hence

\[
(M_{hqq}^{\text{max}})^2 = (M_{hqq}^{\text{min}})^2 + \frac{2M_Q^2 - M_G^2}{2M_G} \left[ M_G (M_Q^2 - M_X^2) + \frac{M_G M_X}{M_Q^2} (E^*_h + p^*_h) \right]
\]

\( = (M_{hqq}^{\text{min}})^2 + \frac{(M_G^2 - M_Q^2)}{M_Q^2} \left[ (M_Q^2 - M_X^2) + M_X (E^*_h + p^*_h) \right] \quad (406) \]

which exhibits the fact that when \( M_G = M_Q \) the \( q_1 \) is at rest in the \( G \) rest frame and the mass for this configuration is \( M_{hqq}^{\text{max}} = M_{hqq}^{\text{min}} \). It is seen that the same formula is derived as above.

### 12.5.4 Second collinear end point in \( M(h^0q1q2) \)

The second collinear end point is obtained from configuration \( (ha) \) with the mass given by equation (393) with \( \cos \theta^* = +1 \)

\[
(M_{hqlq2}^{\text{max},2})^2 = M_h^2 + \frac{M_Q^2 - M_G^2}{M_X^2} (E^*_h + p^*_h) \quad (407)
\]

The region of validity of this equation is bounded by

\[
M_G \leq \frac{1}{2M_0} \left[ M_Q^2 + M_G^2 - (M_{hqlq2}^{\text{max},2})^2 - \sqrt{(M_Q^2 + M_G^2 - (M_{hqlq2}^{\text{max},2})^2)^2 - 4M_Q^2 M_G^2} \right] \quad (408)
\]

where \( M_{hqlq2}^{\text{max}} \) is given by equation (377).

### 12.5.5 Third collinear end point in \( M(h^0q1q2) \)

The third collinear end point is obtained from configuration \( (hd) \) with the mass given by equation (394) with \( \cos \theta^* = +1 \)

\[
(M_{hqlq2}^{\text{max},3})^2 = M_h^2 + \frac{(M_G^2 - M_Q^2)(M_G^2 - M_X^2)}{M_Q^2} + \frac{M_Q^2 - M_G^2}{M_Q^2} M_X (E^*_h - p^*_h) \quad (409)
\]

which can also be written as

\[
(M_{hqlq2}^{\text{max},3})^2 = M_{qq}^2 + (M_{hqlq2}^{\text{max},2})^2 + \frac{M_Q^2 - M_G^2}{M_Q^2} M_X (E^*_h - p^*_h) \quad (410)
\]

The region of validity of this equation is bounded by

\[
M_G \leq \frac{1}{2M_0} \left[ M_Q^2 + M_G^2 - (M_{hqlq2}^{\text{max},3})^2 - \sqrt{(M_Q^2 + M_G^2 - (M_{hqlq2}^{\text{max},3})^2)^2 - 4M_Q^2 M_G^2} \right] \quad (411)
\]

where \( M_{hqlq2}^{\text{max}} \) is given by equation (377).

As there is an overlap between the regions of validity of \( M_{hqq}^{\text{max},2} \) and \( M_{hqq}^{\text{max},3} \), we still have to verify where one or the other gives the true end point. To have \( M_{hqq}^{\text{max},3} \geq M_{hqq}^{\text{max},2} \), we need

\[
M_{QQ}^2 + \frac{M_G^2 - M_Q^2}{M_X} (E^*_h + p^*_h) + \frac{M_Q^2 - M_G^2}{M_Q^2} M_X (E^*_h - p^*_h) \geq \frac{M_G^2 - M_Q^2}{M_X} (E^*_h + p^*_h)
\]

\[
M_{QQ}^2 = \frac{M_G^2 - M_Q^2}{M_X} (E^*_h + p^*_h) + \frac{M_Q^2 - M_G^2}{M_Q^2} M_X (E^*_h - p^*_h) \geq 0
\]

\[
\frac{M_Q^2 - M_G^2}{M_Q^2 M_X} \left[ (M_Q^2 - M_X^2) M_X - M_Q^2 (E^*_h + p^*_h) + M_X^2 (E^*_h - p^*_h) \right] \geq 0
\]

or

\[
(M_Q^2 - M_X^2) M_X - M_Q^2 (E^*_h + p^*_h) + M_X^2 (E^*_h - p^*_h) \geq 0
\]
It is easily verified that this corresponds to the condition for $M_{1qq}^{max,3} \geq M_{1qq}^{max,2}$ in the limiting case $M_h \to 0$. It can be rewritten in more compact form as

$$(M_Q^2 - M_X^2)(M_X - E_h^*) \geq (M_Q^2 + M_X^2)p_h^*$$

or

$$\frac{M_Q^2 - M_X^2}{M_Q^2 + M_X^2} \geq \frac{p_h^*}{M_X - E_h^*} = \frac{p_h^*}{E_0^*}$$

which allows for a simple physical interpretation as the left side is $\beta_X^*$ in the $Q = \tilde{q}$ rest frame and the right side is $\beta_0^*$ of the neutralino in the $X = \chi_0^2$ rest frame. Hence, the requirement is

$$\beta_X^* \geq \beta_0^*$$

### 12.5.6 Lower end point in $M(h^0q_1q_2)$

The minimum mass of $(hqq)$ is obtained for configuration $(hb)$ with $\cos\theta' = -1$

$$(M_{hqq}^{min})^2 = M_h^2 + \frac{M_Q^2 - M_X^2}{M_X}(E_h^* - p_h^*)$$

(412)

and is independent of $M_Q$.

### 13 Direct 3-body decays of a $\tilde{g}$ to final states with $h^0$

This decay chain, which dominates in a large part of region 3 in figure 1 is

$$\tilde{g} \to q_1q_2\chi_2^0, \chi_2^0 \to h^0\chi_1^0$$

(413)

The available kinematical quantities are $M(q_1q_2)$, $M(hq)$ and $M(hqq)$. In $M(hq)$ both quarks play the same role. The same formulae are valid if the Higgs is replaced by a $Z^0$.

#### 13.1 End point in $M(q_1q_2)$

As the $\tilde{g}$ undergoes a direct 3-body decay, the configuration leading to a maximum in $M(qq)$ is the one shown in figure 30. with the $\chi_2^0$ at rest in the $G = \tilde{g}$ reference frame. The end point is then given by

$$M_{qq}^{max} = M_G - M_X$$

(414)

If the decay is assumed to behave approximately according to phase space, the distribution of $M_{qq}$ corresponds to 2 bodies out of 3 and falls steeply near the upper edge. The upper end point should, therefore, be well visible.

#### 13.2 End points in $M(hq)$

The configuration leading to a maximum in $M(hq)$ is the one shown in figure 31. In this case, one of the quarks is at rest in the $G = \tilde{g}$ rest frame. If we approximate the quarks by zero-mass particles, this configuration becomes equivalent with the one of section 11 with the squark mass replaced by the gluino mass.
Figure 31: Configuration leading to a maximum of the mass for \((hq)\) in 3-body decays of the gluino.

The end points are then given by

\[
M_{hq}^2 = M_h^2 + \frac{M_G^2 - M_X^2}{M_X^2} M_X (E_h^* + p_h^* \cos \theta^*)
\]

or

\[
(M_{hq}^{max})^2 = M_h^2 + \frac{M_G^2 - M_X^2}{2M_X^2} \left( M_X^2 + M_h^2 - M_0^2 + \sqrt{(M_X^2 - M_0^2 - M_h^2)^2 - 4M_0^2 M_h^2} \right)
\]

\[
(M_{hq}^{min})^2 = M_h^2 + \frac{M_G^2 - M_X^2}{2M_X^2} \left( M_X^2 + M_h^2 - M_0^2 - \sqrt{(M_X^2 - M_0^2 - M_h^2)^2 - 4M_0^2 M_h^2} \right)
\]

13.3 End points in \(M(hqq)\)

The configuration leading to a maximum in \(M(hqq)\) is the one shown in figure 32, with the \(\tilde{\chi}_1^0\) at rest in the \(G = \tilde{g}\) reference frame. The end point is then given by

\[
M_{hqq}^{max} = M_G - M_0
\]

If the decay is assumed to behave approximately according to phase space, the distribution of \(M_{hqq}\) corresponds to 3 bodies out of 4 and peaks near the upper edge. The upper end point should, therefore, be well visible.

14 Sequential 2-body decays from \(\tilde{q}\) to \(\tilde{g}\) to \(\tilde{q}\) to final states with \(h^0\)

There exists a region of the parameter space, where some squarks are heavier and some other lighter (typically \(\tilde{b}_1\) or \(\tilde{t}_1\)) than the gluino, the lighter squark decays to a \(\tilde{\chi}_2^0\) which itself decays dominantly to the \(h^0\) or \(Z^0\). This very long decay chain is then

\[
\tilde{q} \to q1\tilde{g} \to q1q2\tilde{q} \to q1q2q3\tilde{\chi}_2^0 \to q1q2q3h^0\tilde{\chi}_1^0
\]

It follows that the flavour of \(q1\) may or may not differ from the flavour of \(q2\) and \(q3\), which have to be a particle-antiparticle state. The Higgs is supposed to be observed through its decay into \(bb\) and its mass reconstructed. Note that the formulae derived for the \(h^0\) are equally applicable to the \(\tilde{\chi}_2^0\) decaying into \(\tilde{\chi}_1^0 Z^0\).

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### 14.1 Overview of the collinear configurations

There are 8 collinear configurations leading to maxima in the effective mass distributions: all the ones shown in section 12.1 with two possible orientations, along or opposite to $q_1$. They are displayed below. There are also 11 possible mass distributions: $(q_1 q_2)$, $(q_1 q_3)$, $(q_2 q_3)$, $(hq_3)$, $(hq_2)$, $(hq_1)$, $(q_1 q_2 q_3)$, $(hq_1 q_2)$, $(hq_2 q_3)$, $(hq_1 q_2 q_3)$. The mass combinations containing only $q_2$, $q_3$ and $h^0$ have been derived in sections 11 and 12 (with the obvious replacement $q_1 \rightarrow q_2$ and $q_2 \rightarrow q_3$) and will not be repeated here.

![Diagram of collinear configurations](image_url)
As the calculation of the effective masses will require the knowledge of \( E_{h} + p_{h,L} \) and \( E_{h} - p_{h,L} \), let us first compute them for all relevant configurations.

For configurations \((ha1)\) and \((hb1)\):

\[
E_{h} + p_{h,L} = \frac{M_{P}}{M_{G}}(E^\ast_{h} + p_{h,L}^\ast) = \frac{M_{P}}{M_{G}} \frac{M_{Q}}{M_{G}}(E^\ast_{h} - p_{h,L}^\ast) = \frac{M_{P}}{M_{G}} \frac{M_{Q}}{M_{G}} M_{X}(E^\ast_{h} - p_{h}^\ast \cos \theta^\ast) \\
= \frac{M_{P}}{M_{G}^2} M_{X}(E^\ast_{h} - p_{h}^\ast \cos \theta^\ast)
\]

\[
E_{h} - p_{h,L} = \frac{M_{G}}{M_{P}}(E^\ast_{h} + p_{h,L}^\ast) = \frac{M_{G}}{M_{P}} \frac{M_{Q}}{M_{G}}(E^\ast_{h} + p_{h,L}^\ast) = \frac{M_{G}}{M_{P}} \frac{M_{Q}}{M_{G}} M_{X}(E^\ast_{h} + p_{h}^\ast \cos \theta^\ast) \\
= \frac{M_{G}^2}{M_{P} M_{X}^2} M_{X}(E^\ast_{h} + p_{h}^\ast \cos \theta^\ast)
\]
For configurations \((hc1)\) and \((hd1)\):

\[
E_h + p_{h,L} = \frac{M_P}{M_G} (E'_h + p'_{h,L}) = \frac{M_P M_Q}{M_G M_Q} (E'_h + p'_{h,L}) = \frac{M_P M_Q}{M_G M_Q} (E'_h + p'_h \cos \theta^*) \\
= \frac{M_P M_Q}{M_G M_X} (E'_h + p'_h \cos \theta^*)
\]

\[
E_h - p_{h,L} = \frac{M_C}{M_P} (E'_h - p'_{h,L}) = \frac{M_C M_Q}{M_P M_Q} (E'_h - p'_{h,L}) = \frac{M_C M_Q}{M_P M_Q} (E'_h - p'_h \cos \theta^*) \\
= \frac{M_C}{M_P} (E'_h - p'_h \cos \theta^*)
\]

For configurations \((ha2)\) and \((bb2)\):

\[
E_h + p_{h,L} = \frac{M_P}{M_G} (E'_h + p'_{h,L}) = \frac{M_P M_Q}{M_G M_Q} (E'_h + p'_{h,L}) = \frac{M_P M_Q}{M_G M_Q} (E'_h + p'_h \cos \theta^*) \\
= \frac{M_P}{M_X} (E'_h + p'_h \cos \theta^*)
\]

\[
E_h - p_{h,L} = \frac{M_C}{M_P} (E'_h - p'_{h,L}) = \frac{M_C M_Q}{M_P M_Q} (E'_h - p'_{h,L}) = \frac{M_C M_Q}{M_P M_Q} (E'_h - p'_h \cos \theta^*) \\
= \frac{M_C}{M_P} (E'_h - p'_h \cos \theta^*)
\]

For configurations \((hc2)\) and \((hd2)\):

\[
E_h + p_{h,L} = \frac{M_P}{M_G} (E'_h + p'_{h,L}) = \frac{M_P M_Q}{M_G M_Q} (E'_h + p'_{h,L}) = \frac{M_P M_Q}{M_G M_Q} (E'_h + p'_h \cos \theta^*) \\
= \frac{M_P M_Q}{M_G M_X} (E'_h + p'_h \cos \theta^*)
\]

\[
E_h - p_{h,L} = \frac{M_C}{M_P} (E'_h - p'_{h,L}) = \frac{M_C M_Q}{M_P M_Q} (E'_h - p'_{h,L}) = \frac{M_C M_Q}{M_P M_Q} (E'_h - p'_h \cos \theta^*) \\
= \frac{M_C}{M_P} (E'_h - p'_h \cos \theta^*)
\]

14.2 End points in \(M(qq)\)

14.2.1 End point in \(M(q2q3)\)

The maximum of \(M(q2q3)\) is reached in the configurations labelled \((hc1)\), \((hd1)\), \((hc2)\) and \((hd2)\). The end point was given in section 12.2, formula (380).

14.2.2 End point in \(M(q1q2)\)

The maximum of \(M(q1q2)\) is reached in the configurations labelled \((ha1)\), \((hb1)\), \((hc1)\) and \((hd1)\). The end point formula is

\[
M_{q1q2}^{max} = M_P \sqrt{(1 - \frac{M_G}{M_P}) (1 - \frac{M_Q}{M_P})} 
\]

By comparing this formula with (380), it is seen that to have \(M_{q1q2}^{max} \geq M_{q2q3}^{max}\) requires

\[
\frac{M_P^2 - M_G^2}{M_P^2} \geq \frac{M_Q^2 - M_X^2}{M_Q^2} 
\]

and thus

\[
M_P^2 - M_G^2 \geq \frac{M_Q^2}{M_Q} (M_Q^2 - M_X^2) \]
14.2.3 End point in \( M(q_1q_3) \)

The maximum of \( M(q_1q_3) \) is reached in the configurations labelled \((hc_2)\) and \((hd_2)\). The end point formula is similar to the one for \( M(l_2q) \) in section 4.3.1 and reads

\[
M_{q_1q_3}^{\text{max}} = M_P \sqrt{(1 - \frac{M_G^2}{M_P^2})(1 - \frac{M_X^2}{M_Q^2})}
\]  

(426)

To have \( M_{q_1q_3}^{\text{max}} \geq M_{q_4q_2}^{\text{max}} \) requires

\[
M_Q^2 \geq M_G M_X
\]

(427)

and for \( M_{q_1q_3}^{\text{max}} \geq M_{q_2q_2}^{\text{max}} \)

\[
M_P^2 - M_G^2 \geq M_Q^2 - M_Q^2
\]

(428)

14.2.4 Distinguishing upper end points of \( M(qq) \)

14.3 End points in \( M(h_0q) \)

14.3.1 Upper end point in \( M(h_0q_3) \)

The maximum of \( M(h_0q_3) \) is reached in the configurations labelled \((ha_1)\) and \((ha_2)\). The end point was given in section 11, formula (377).

14.3.2 Upper end point in \( M(h_0q_2) \)

The maximum of \( M(h_0q_2) \) is reached in the configurations labelled \((ha_1)\), \((hd_1)\), \((ha_2)\) and \((hd_2)\). The end point was given in section 12.3, formula (385).

14.3.3 Upper end point in \( M(h_0q_1) \)

The maximum of the \((h_0q_1)\) mass distribution is reached in the configuration labelled \((ha_2)\). As seen from the discussion in section 2.5, the effective mass is

\[
M_{h_0q_1}^2 = M_h^2 + 2E_{q_1}(E_h + p_{h,L})
\]

(429)

defined in the rest frame of \( P = \vec{q}' \) and with the quark energy given by

\[
E_{q_1} = \frac{M_P^2 - M_G^2}{2M_P}
\]

(430)

Hence, for the configuration leading to the maximum mass, using (422) for \( E_h + p_{h,L} \), the maximum effective mass is

\[
(M_{h_0q_1}^{\text{max}})^2 = M_h^2 + 2 \frac{M_P^2 - M_G^2}{2M_P} M_P (E_h^* + p_h^* \cos \theta^*)
\]

\[
= M_h^2 + \frac{M_P^2 - M_G^2}{M_X^2} M_X (E_h^* + p_h^*)
\]

(431)

with \( E_h^* \) and \( p_h^* \) defined in (373), or

\[
(M_{h_0q_1}^{\text{max}})^2 = M_h^2 + \frac{M_P^2 - M_G^2}{2M_X^2} \left( M_h^2 - M_0^2 + M_h^2 + \sqrt{(M_h^2 - M_0^2 - M_h^2)^2 - 4M_0^2 M_h^2} \right)
\]

(432)

The comparison with the other \((hq)\) end points shows that \( M_{h_0q_1} \geq M_{h_0q_2} \) provided

\[
M_P^2 - M_G^2 \geq M_Q^2 - M_Q^2
\]

(433)

and that \( M_{h_0q_1} \geq M_{h_0q_3} \) provided

\[
M_P^2 - M_G^2 \geq M_Q^2 - M_X^2
\]

(434)
14.3.4 Distinguishing upper end points of $M(h^0 q)$

It is seen from the configurations that the upper end point of $M(h^0 q3)$ is reached independent of the orientation with respect to $q1$ and to $q2$, hence for any mass $M(q1q2)$, $M(q1q3)$ and $M(q2q3)$. Also the end point in $M(h^0 q2)$ is reached independent of the orientation with respect to $q1$, hence for any mass $M(q1q2)$ and $M(q1q3)$, but for minimal $M(q2q3)$. On the other hand, the upper end point of $M(h^0 q1)$ requires the $M(q1q2)$, $M(q1q3)$ and $M(q2q3)$ to be minimal. This allows this end point to be distinguished from the ones in $hq1$ and $hq2$, as all three associated $M(qq)$ values have to be minimal. If only two of the associated $M(qq)$ values are near the minimum and the end point is independent of the third one, it is $M(h^0 q2)$. If the end point does not depend on any of the $M(qq)$ values, it is $M(h^0 q3)$. In practice, as the $M(qq)$ mass distributions tend not to be very populated near their lower edge, the statistics may severely limit this way of identifying the end points. The identification of the flavour of the quark jets, in particular the distinction of $q1$ from the other two, might help by restricting the number of possibilities.

14.3.5 End point for the sum of $M(hq)^2$

An additional information can be obtained from the sum of squares of the three $(hq)$ masses. It is seen by inspection of the configurations of section 14.1 that in one of them, namely $(ha2)$, all three mass combinations reach their maximum simultaneously. Hence

$$\left(M_{hq1}^2 + M_{hq2}^2 + M_{hq3}^2\right)_{\text{max}} = \left(M_{hq1}^{\text{max}}\right)^2 + \left(M_{hq2}^{\text{max}}\right)^2 + \left(M_{hq3}^{\text{max}}\right)^2$$

$$= 3M_h^2 + \left[\frac{M_h^2 - M_G^2}{M_X^2} + \frac{M_h^2 - M_Q^2}{M_X^2} + \frac{M_h^2 - M_P^2}{M_X^2}\right]M_X(E_h^* + p_h^*)$$

or

$$\left(M_{hq1}^2 + M_{hq2}^2 + M_{hq3}^2\right)_{\text{max}} = 3M_h^2 + \frac{M_P^2 - M_G^2}{M_X^2}M_X(E_h^* + p_h^*)$$

Therefore, even without any identification of the quark flavours, at least two $(hq)$ end points can be used to reconstruct the sparticle masses.

14.3.6 Lower end point in $M(h^0 q1)$

A configuration leading to the minimum $M(h^0 q1)$ is labelled $(ha1)$ (see section 2.5). Using (420) for $E_h + p_{hL}$ the minimum value of the mass distribution is given by

$$\left(M_{hq1}^{\text{min}}\right)^2 = M_h^2 + \frac{2M_P^2 - M_G^2}{2M_P^2}M_PM_X(E_h^* - p_h^*)$$

$$= M_h^2 + \frac{M_P^2 - M_G^2}{M_X^2}M_X(E_h^* - p_h^*)$$

or

$$\left(M_{hq1}^{\text{min}}\right)^2 = M_h^2 + \frac{M_P^2 - M_G^2}{2M_G^2}\left(M_X^2 - M_0^2 + M_h^2 - \sqrt{(M_X^2 - M_0^2 - M_h^2)^2 - 4M_0^2M_h^2}\right)$$

14.4 End points in $M(q1q2q3)$

The formulae for the maximum of $M(qqq)$ can be directly translated from the ones for $(llq)$ in section 4.4.

14.4.1 Absolute maximum of $M(q1q2q3)$

The absolute maximum for the effective mass is

$$M_{qqq}^{\text{max}} = M_P - M_X$$

and is obtained for non-collinear configurations, provided the sparticle masses satisfy the following conditions

$$\frac{M_Q^2}{M_X} < M_P < \frac{M_G^2}{M_X} \quad \text{and} \quad M_P \geq \frac{M_G^2M_X}{M_Q^2}$$
14.4.2 First collinear end point in $M(q_1q_2q_3)$

The first maximum of $M(qqq)$ is found in the configurations labelled $(ha1)$ and $(hb1)$. It is given by

$$M_{qqq}^{\text{max}} = M_P \sqrt{\left(1 - \frac{M_G^2}{M_P^2}\right)\left(1 - \frac{M_X^2}{M_G^2}\right)}$$  \hspace{1cm} (440)

and applies to the region where $M_P \geq M_G^2/M_X$.

14.4.3 Second collinear end point in $M(q_1q_2q_3)$

The second maximum of $M(qqq)$ is found in the configurations labelled $(hc2)$ and $(hd2)$. It is given by

$$M_{qqq}^{\text{max},2} = M_P \sqrt{\left(1 - \frac{M_Q^2}{M_P^2}\right)\left(1 - \frac{M_X^2}{M_Q^2}\right)}$$  \hspace{1cm} (441)

and applies to the region where $M_P \leq M_Q^2/M_X$.

This relation is not independent of the other ones, as

$$(M_{qqq}^{\text{max},2})^2 = (M_{q_1q_3}^{\text{max}})^2 + (M_{q_2q_3}^{\text{max}})^2$$  \hspace{1cm} (442)

To have $M_{qqq}^{\text{max},2} \geq M_{qqq}^{\text{max}}$ requires

$$M_P \leq \frac{M_G M_Q}{M_X}$$  \hspace{1cm} (443)

14.4.4 Third collinear end point in $M(q_1q_2q_3)$

The third maximum of $M(qqq)$ is found in the configurations labelled $(hc1)$ and $(hd1)$. It is given by

$$M_{qqq}^{\text{max},3} = M_P \sqrt{\left(1 - \frac{M_G^2}{M_P^2}\right)\left(1 - \frac{(M_G M_X)^2}{(M_P M_Q)^2}\right)}$$  \hspace{1cm} (444)

and applies to the region where $M_P \leq \frac{M_G M_X}{M_Q}$.

This relation is also not independent of the other ones, as

$$(M_{qqq}^{\text{max},3})^2 = (M_{q_1q_2}^{\text{max}})^2 + (M_{q_2q_3}^{\text{max}})^2$$  \hspace{1cm} (445)

To have $M_{qqq}^{\text{max},3} \geq M_{qqq}^{\text{max}}$ requires

$$M_P \leq \frac{M_Q^2}{M_G}$$  \hspace{1cm} (446)

an for $M_{qqq}^{\text{max},3} \geq M_{qqq}^{\text{max},2}$, we need to have

$$M_Q^2 \leq M_G M_X$$  \hspace{1cm} (447)

i.e. the condition which implies $M_{q_1q_3}^{\text{max}} \leq M_{q_1q_2}^{\text{max}}$.

14.5 End points in $M(h^0q_2q_3)$

The end points in the $M(h^0q_2q_3)$ mass distribution have been computed in section 12.5, where it was shown that three configurations can lead to the end points.

The first end point is obtained in the configurations labelled $(hc1)$ and $(hc2)$ and the formula for the maximum is given in (399).

The second end point is obtained in the configurations labelled $(ha1)$ and $(ha2)$ and the formula for the maximum is given in (407).

The third end point is obtained in the configurations labelled $(hd1)$ and $(hd2)$ and the formula for the maximum is given in (409).
14.6 End points in \( M(h^0q1q2) \)

14.6.1 Upper end points in \( M(h^0q1q2) \)

The configuration leading to a true maximum in \( M(h^0q1q2) \) can be derived by comparison with the configurations for the \((l2qq)\) end points in section 5.9. They may correspond to any of the configurations labelled \((hd1)\), \((ha2)\) or \((ha1)\).

Following the conventions outlined in section 2.5, the effective mass for the configurations \((hd1)\) and \((ha1)\) is

\[
(M_{hq1q2})^2 = M_0^2 + 2E_{q1}E_q + 2E_{q1}E_h + 2E_{q1}E_{qh} + 2E_{q1}p_{h,L} + 2E_{q2}p_{h,L}
\]

\[
(M_{hq1q2})^2 = M_0^2 + 2E_{q1}E_h + 2E_{q1}E_{qh} + 2E_{q1}p_{h,L} + 2E_{q2}p_{h,L}
\]

It can be noticed that this expression contains \( M_q^2 \). For these configurations

\[
E_{q2} = \frac{M_P M_0^2 - M_q^2}{2M_G}
\]

For the configuration \((ha2)\)

\[
(M_{ha1q2})^2 = M_0^2 + 2E_{q1}E_h + 2E_{q1}E_{qh} + 2E_{q1}p_{h,L} + 2E_{q2}p_{h,L}
\]

\[
(M_{ha1q2})^2 = M_0^2 + 2(E_{q1} + E_{q2})(E_h + p_{h,L})
\]

with

\[
E_{q2} = \frac{M_G M_0^2 - M_q^2}{2M_G} = \frac{M_G^2 - M_q^2}{2M_P}
\]

14.6.2 Absolute maximum of \( M(h^0q1q2) \)

In all three configurations leading to end points the two particles not included in the mass, \( \bar{\chi}_1^0 \) and \( q3 \), are parallel and thus at their minimal mass \( M_{q30} \). Depending on the values of the masses, the \((q3\bar{\chi}_1^0)\) system can be put at rest or sent backward in the \( P = \hat{q} \) rest frame. In the latter case, the above formulæe for the maxima are no longer applicable and the upper end point corresponds to a non-collinear configuration with mass

\[
M_{q30}^\text{max} = M_P - M_{q30}
\]

where the mass \( M_{q30} \) can be directly computed in the rest frame of \( Q = \hat{q} \).

\[
E_0^0 = \frac{M_0^2 + M_0^2 - M_q^2}{2M_X}
\]

\[
p_0^0 = p_0^h = \frac{1}{2M_X} \sqrt{(M_0^2 - M_0^2 - M_q^2)^2 - 4M_X^2 M_q^2}
\]

and

\[
M_{q30}^2 = M_0^2 + 2E_{q3}(E_0^0 + p_{0,L}) = M_0^2 + 2 \frac{M_0^2 - M_0^2}{M_0^2} \frac{M_M}{M_X} (E_0^0 + p_0^0 \cos \theta^*)
\]

or

\[
M_{q30}^2 = M_0^2 + \frac{M_0^2 - M_q^2}{M_X} (E_0^0 - p_0^0)
\]

It remains to be checked under which conditions the non-collinear configurations give the true end point. This can be computed from the three collinear configurations in the case where the \((q3\bar{\chi}_1^0)\) is at rest in the \( P = \hat{q} \) frame, i.e. where \( p_{q30} = 0 \).

For configuration \((hd1)\), we have

\[
E_{q3} = \frac{M_G M_G M_0^2 - M_q^2}{2M_0}, \quad p_{q3,L} = -E_{q3}
\]

\[
E_0 + p_{0,L} = \frac{M_P M_0^2}{M_G M_X} (E_0^0 + p_0^0 \cos \theta^*) = \frac{M_P M_0^2}{M_G M_X} (E_0^0 - p_0^0)
\]

\[
E_0 - p_{0,L} = \frac{M_0^2 M_X}{M_P M_0^2} (E_0^0 - p_0^0 \cos \theta^*) = \frac{M_0^2 M_X}{M_P M_0^2} (E_0^0 + p_0^0)
\]
where the expressions for $E_0 + p_0$ and $E_0 - p_0$ are obtained from (421) with $\cos\theta^* = -1$. Then, to have the true end point in a non-collinear configuration, the requirement is $p_{q30} \leq 0$

$$2p_{q30} = 2p_{q3,L} + (E_0 + p_0, L) - (E_0 - p_0, L)$$

$$= -\frac{M^2_P}{M_P M^2_Q} (M^2_Q - M^2_X) + \frac{M_P M^2_Q}{M^2_M} (E_0^* - p_0^*) - \frac{M^2_P M_X}{M_P M^2_Q} (E_0^* + p_0^*) \leq 0$$

$$- M^2_Q M_X (M^2_Q - M^2_X) + M^2_P M^2_Q (E_0^* - p_0^*) - M^2_P M^2_X (E_0^* + p_0^*) \leq 0$$

or

$$M^2_P \leq \frac{M^2_Q M_X [M^2_Q - M^2_X + M_X (E_0^* + p_0^*)]}{M^2_Q (E_0^* - p_0^*)} \quad \text{(453)}$$

We may verify what this leads to in the case $M_h = 0$. Then, as seen from equation (22),

$$E_0^* + p_0^* = M_X \quad E_0^* - p_0^* = \frac{M^2}{M_X}$$

which gives

$$M_P \leq \frac{M^2_Q M_X}{M^2_Q M_0}$$

which agrees with condition (202) for the (l2qq) case.

For configuration (ha2), we have

$$E_{q3} = M_G M_Q M^2_Q - M^2_X}{2M_Q} = \frac{M^2_Q - M^2_X}{2M_P}, \quad p_{q3,L} = -E_{q3}$$

$$E_0 + p_0, L = \frac{M_P}{M_X} (E_0^* + p_0^* \cos\theta^*) = \frac{M_P}{M_X} (E_0^* + p_0^*)$$

$$E_0 - p_0, L = \frac{M_X}{M_P} (E_0^* - p_0^* \cos\theta^*) = \frac{M_X}{M_P} (E_0^* - p_0^*)$$

where the expressions for $E_0 + p_0$ and $E_0 - p_0$ are obtained from (422) with $\cos\theta^* = -1$. Then, to have the true end point in a non-collinear configuration, the requirement is $p_{q30} \geq 0$

$$2p_{q30} = 2p_{q3,L} + (E_0 + p_0, L) - (E_0 - p_0, L)$$

$$= -\frac{M^2_Q - M^2_X}{M_P} + \frac{M_P}{M_X} (E_0^* - p_0^*) - \frac{M_X}{M_P} (E_0^* + p_0^*) \geq 0$$

$$- M_X (M^2_Q - M^2_X) + M^2_P (E_0^* - p_0^*) - M^2_P (E_0^* + p_0^*) \geq 0$$

or

$$M^2_P \geq \frac{M_X (M^2_Q - M^2_X) + M^2_X (E_0^* + p_0^*)}{E_0^* - p_0^*} \quad \text{(454)}$$

We may verify what this gives in the case $M_h = 0$

$$M_P \geq \frac{M_Q M_0}{M_X}$$

which agrees with condition (202) for the (l2qq) case.

Finally, for configuration (ha1), we have

$$E_{q3} = \frac{M_P}{M_G} \frac{M^2_Q - M^2_X}{2M_Q} = \frac{M^2_Q}{2M^2_Q} (M^2_Q - M^2_X), \quad p_{q3,L} = +E_{q3}$$

$$E_0 + p_0, L = \frac{M_P}{M_X} (E_0^* - p_0^* \cos\theta^*) = \frac{M_P}{M_X} (E_0^* + p_0^*)$$

$$E_0 - p_0, L = \frac{M^2}{M_P M_X} (E_0^* + p_0^* \cos\theta^*) = \frac{M^2}{M_P M_X} (E_0^* - p_0^*)$$
where the expressions for $E_0 + p_0$ and $E_0 - p_0$ are obtained from (420) with $\cos \theta^* = -1$. Then, to have
the true end point in a non-collinear configuration, the requirement is $p_{q30} \leq 0$

\[
2p_{q30} = 2p_{q3, L} + (E_0 + p_{0, L}) - (E_0 - p_{0, L})
\]

\[
= \frac{M_P}{M_G} (M_Q^2 - M_G^2) + \frac{M_P M_X}{M_G} (E_0^* + p_0^*) - \frac{M_G^2}{M_P M_X} (E_0^* - p_0^*) \geq 0
\]

\[
M_P^2 M_X (M_Q^2 - M_G^2) + M_P^2 M_X^2 (E_0^* + p_0^*) - M_G^2 (E_0^* - p_0^*) \geq 0
\]

or

\[
M_P^2 \geq \frac{M_G^2 M_0}{M_Q M_X}
\] (455)

We may verify what this leads to in the case $M_h = 0$

\[
M_P \geq \frac{M_G^2 M_0}{M_Q M_X}
\]

which agrees with condition (203) for the $(l2qq)$ case.

### 14.6.3 First collinear end point in $M(h^0q_1q_2)$

The first collinear end point is obtained from configuration $(hd1)$ with the mass given by equation (448). Now, for configuration $(hd1)$ using (421) for $E_h + p_{0, L}$ and $E_h - p_{0, L}$

\[
(M_{hq_1q_2}^{\text{max}})^2 = M_h^2 + \frac{4(M_P^2 - M_G^2) M_P (M_Q^2 - M_G^2)}{2M_P M_G - 2M_G M_Q}
\]

\[
+ \frac{2M_P^2 - M_G^2}{2M_P M_G M_X} (E_h^* + p_h^*) + \frac{2M_P}{M_G} \frac{M_G^2 - M_Q^2}{M_Q M_X} M_X (E_h^* + p_h^*)
\]

or

\[
(M_{hq_1q_2}^{\text{max}})^2 = M_h^2 + \frac{(M_P^2 - M_G^2) (M_Q^2 - M_G^2)}{M_G}
\]

\[
+ \frac{(M_P^2 - M_G^2) M_Q^2}{M_G^2 M_X} M_X (E_h^* + p_h^*) + \frac{M_Q^2}{M_G} M_X (E_h^* + p_h^*)
\] (456)

which can also be written as

\[
(M_{hq_1q_2}^{\text{max}})^2 = (M_{q_1q_2}^{\text{max}})^2 + (M_{hq_2}^{\text{max}})^2 + \frac{M_Q^2}{M_G M_X} M_X (E_h^* + p_h^*)
\] (457)

Depending on the values of the masses, the $q_3X_0^0$ system can be put at rest or sent backward in the $G = \bar{g}$
rest frame. As seen above, the condition of validity of this equation is given by the relation

\[
M_P^2 \geq \frac{M_G^2 M_X [M_G^2 - M_X (E_0^* + p_0^*)]}{M_Q^2 (E_0^* - p_0^*)}
\] (458)

### 14.6.4 Second collinear end point in $M(h^0q_1q_2)$

The second collinear end point is obtained from configuration $(ha2)$ with the mass given by equation (449). Now, for configuration $(ha2)$ using (422) for $E_h + p_{h, L}$ with $\cos \theta^* = +1$

\[
(M_{hq_1q_2}^{\text{max}})^2 = M_h^2 + 2 \left[ \frac{M_P^2 - M_G^2}{2M_P} + \frac{M_G^2 - M_Q^2}{2M_P} \right] M_P M_X (E_h^* + p_h^*)
\]

or

\[
(M_{hq_1q_2}^{\text{max}})^2 = M_h^2 + \frac{M_P^2 - M_G^2}{M_X} M_X (E_h^* + p_h^*)
\] (459)

As derived above, the region of validity of this equation is bounded by

\[
M_P^2 \leq \frac{M_X [M_G^2 - M_X (E_0^* + p_0^*)]}{E_0^* - p_0^*}
\] (460)
14.6.5 Third collinear end point in $M(h^0q1q2)$

The third collinear end point is obtained from configuration $(h1a)$ with the mass given by equation (448). Now, for configuration $(ha1)$ using (420) for $E_h + p_{h,L}$ and $E_h - p_{h,L}$

\[
(M_{hqlq2}^{\text{max},3})^2 = M_h^2 + 4 \frac{(M_P^2 - M_G^2)(M_P^2 - M_Q^2)}{2 M_P M_G} M_P (M_P^2 - M_Q^2) + 2 \frac{M_P^2 - M_G^2}{2 M_P} M_P M_X (E_h - p_h^*) + 2 \frac{M_P^2 - M_G^2}{2 M_P} M_P M_X (E_h^* + p_h^*)
\]

or

\[
(M_{hqlq2}^{\text{max},3})^2 = M_h^2 + \frac{(M_P^2 - M_G^2)(M_P^2 - M_Q^2)}{M_Q^2} M_P M_X (E_h - p_h^*) + \frac{M_P^2 - M_G^2}{M_G^2} M_P M_X (E_h^* + p_h^*)
\]

which can also be written as

\[
(M_{hqlq2}^{\text{max},3})^2 = (M_{hqlq2}^{\text{max}})^2 + \frac{M_P^2 - M_G^2}{M_G^2} M_P M_X (E_h^* - p_h^*)
\]

As seen above, the region of validity of this equation is bounded by

\[
M_P^2 \geq \frac{M_G^2 (E_h^* - p_h^*)}{M_X (E_h^* - p_h^*) + M_X (M_G^2 - M_X^2)}
\]

As there is an overlap between the regions of validity of $M_{hqlq2}^{\text{max},2}$ and $M_{hqlq2}^{\text{max},3}$, we still have to verify where one or the other gives the true end point. To have $M_{hqlq2}^{\text{max},3} \geq M_{hqlq2}^{\text{max},2}$, we need

\[
\frac{M_P^2 - M_G^2}{M_G^2} [M_G^2 - M_Q^2 + M_X (E_h^* - p_h^*)] + \frac{M_G^2 - M_Q^2}{M_X} (E_h^* + p_h^*) \geq \frac{M_P^2 - M_G^2}{M_X} (E_h^* + p_h^*)
\]

\[
\frac{M_P^2 - M_G^2}{M_G^2} [M_G^2 - M_Q^2 + M_X (E_h^* - p_h^*)] \geq \frac{M_P^2 - M_G^2}{M_X} (E_h^* + p_h^*)
\]

\[
(M_P^2 - M_G^2) M_X [M_G^2 - M_Q^2 + M_X (E_h^* - p_h^*)] \geq (M_P^2 - M_G^2) M_G^2 (E_h^* + p_h^*)
\]

or

\[
M_X [M_G^2 - M_Q^2 + M_X (E_h^* - p_h^*)] \geq M_G^2 (E_h^* + p_h^*)
\]

It is seen that in the limit $M_h = 0$, this becomes

\[
M_X (M_G^2 - M_Q^2) \geq M_G^2 (M_X - M_0^2)
\]

\[
M_G M_0 \geq M_Q M_X
\]

which agrees with (223) for the case of $(l2qq)$.

14.7 End points in $M(h^0q1q3)$

14.7.1 Upper end points in $M(h^0q1q3)$

The collinear configuration leading to a true maximum in $M(h^0q1q3)$ can be derived by comparison with the configurations for the $(llq1)$ end points in section 5.7. They may correspond to any of the configurations labelled $(hc2)$, $(ha2)$ or $(hd2)$.

Following the conventions outlined in section 2.5, the effective mass for the configurations $(hc2)$ and $(hd2)$ is

\[
(M_{hqlq3})^2 = M_h^2 + 2 E_{q1} E_{q3} + 2 E_{q1} E_h + 2 E_{q3} E_h + 2 E_{q1} E_{q3} + 2 E_{q1} p_{h,L} - 2 E_{q3} p_{h,L}
\]

\[
M_h^2 + 4 E_{q1} E_{q3} + 2 E_{q1} (E_h + p_{h,L}) + 2 E_{q3} (E_h - p_{h,L})
\]

(465)
It can be noticed that this expression contains $M_{q1q3}^2$. For these configurations

$$E_{q3} = \frac{M_P M_Q M_{G}^2 - M_{Q}^2}{2M_Q} = \frac{M_P M_{Q}^2 - M_{G}^2}{2M_Q}$$

Moreover (double primed in $G = \tilde{g}$, primed in $Q = \tilde{q}$), using (423) for $E_h + p_{h,L}$ and $E_h - p_{h,L}$

$$M_{h_{q1q3}}^2 = M_h^2 + 4 \frac{M_{Q}^2 - M_{Q}^2 M_P M_{G}^2 - M_{X}^2}{2M_P} + 2 \frac{M_{Q}^2 - M_{Q}^2 M_P M_{X}^2}{2M_P} (E_h - p_{h,cos\theta^*})$$

or

$$M_{h_{q1q3}}^2 = M_h^2 + (M_P^2 - M_{Q}^2) \frac{M_{Q}^2 - M_{X}^2}{M_P^2} + \frac{M_{P}^2 - M_{Q}^2}{M_{X}^2} M_X (E_h^* + p_{h,cos\theta^*}) + \frac{M_{Q}^2 - M_{X}^2}{M_{X}^2} M_X (E_h^* + p_{h,cos\theta^*})$$

For the configuration $(h02)$

$$(M_{h_{q1q3}})^2 = M_h^2 + 2E_{q1}E_h + 2E_{q3}E_h + 2E_{q1}p_{h,L} + 2E_{q3}p_{h,L}$$

$$M_{h_{q1q3}}^2 = M_h^2 + 2(E_{q1} + E_{q3})(E_h + p_{h,L})$$

with

$$E_{q3} = \frac{M_G M_Q M_{G}^2 - M_{Q}^2}{2M_Q} = \frac{M_{Q}^2 - M_{X}^2}{2M_P}$$

and using (422) for $E_h + p_{h,L}$

$$M_{h_{q1q3}}^2 = M_h^2 + 2 \left[ \frac{M_{P}^2 - M_{Q}^2}{2M_P} + \frac{M_{P}^2 - M_{X}^2}{2M_P} \right] \frac{M_P (E_h^* + p_{h,cos\theta^*})}{M_X}$$

or

$$M_{h_{q1q3}}^2 = M_h^2 + \frac{M_{P}^2 - M_{Q}^2 + M_{P}^2 - M_{X}^2}{M_{X}^2} M_X (E_h^* + p_{h,cos\theta^*})$$

### 14.7.2 Absolute maximum of $M(h_{0q1q3})$

In all three configurations leading to end points the two particles not included in the mass, $\tilde{\chi}^0_1$ and $q2$, form a system $(q2\tilde{\chi}^0_1)$ with mass $M_{q20}$ which can, depending on the values of the masses, be put at rest in the $P = \tilde{q}'$ rest frame for a non-collinear configuration. In this case, the above formulae for the maxima are no longer applicable and the upper end point corresponds to a non-collinear configuration with mass

$$M_{h_{q1q2}}^\max = M_P - M_{q20}$$

However, as was the case for the $(l0q1)$ end points in section 5.7 the mass $M_{q20}$ varies in the interval where the non-collinear configuration applies. We will here again compute $M_{q20}$ as a function of the sparticle masses and the limits of this interval, relying heavily on what was done for the massless case of $(llq1)$. We start again by computing the condition for which the $M_{q20}$ system is put at rest in the $P = \tilde{q}'$ frame for the collinear configurations of interest.

For all three configurations

$$E_{q2} = \frac{M_G M_{G}^2 - M_{Q}^2}{2M_G} = \frac{M_{Q}^2 - M_{G}^2}{2M_P} , \ p_{q2,L} = -E_{q2}$$

(470)
For configurations \((hc2)\) and \((hd2)\), the expressions of \(E_0 + p_{0,L}\) and \(E_0 - p_{0,L}\) for the neutralino can be obtained from (423).

\[
2p_{q20} = 2p_{q2,L} + (E_0 + p_{0,L}) - (E_0 - p_{0,L}) = -\frac{M_G^2 - M_Q^2 - M_P M_X (E_0^* - p_0^* \cos \theta^* - \frac{M_Q^2}{M_X} (E_0^* + p_0^* \cos \theta^*) = 0}{\frac{M_P M_X}{M_G^2}}
\]

\[
- M_G^2 M_X (M_G^2 - M_Q^2) + M_P M_X (E_0^* - p_0^*) - M_Q^2 (E_0^* + p_0^* \cos \theta^*) = 0
\]

For \((hc2)\), \(\cos \theta^* = 1\) and the non-collinear configuration occurs when \(p_{q20} \leq 0\), hence

\[
M_P^2 \leq \frac{M_G^2 [M_X (M_G^2 - M_Q^2) + M_Q^2 (E_0^* + p_0^*)]}{M_X (E_0^* - p_0^*)}
\]

(471)

We may verify what this leads to in the case \(M_h = 0\). Then, as seen from equation (454),

\[
M_P \leq \frac{M_G M_Q}{M_0}
\]

which agrees with the upper bound of the interval (132) for the (llq1) case.

For \((hd2)\), \(\cos \theta^* = -1\) and the non-collinear configuration occurs when \(p_{q20} \geq 0\), hence

\[
M_P^2 \geq \frac{M_G^2 [M_X (M_G^2 - M_Q^2) + M_Q^2 (E_0^* + p_0^*)]}{M_X (E_0^* + p_0^*)}
\]

(472)

In the limit \(M_h = 0\),

\[
M_P^2 \geq \frac{M_G^2 [M_G^2 M_X^2 - M_G^2 M_P^2 + M_P^2 M_Q^2]}{M_X (E_0^* + p_0^*)}
\]

which agrees with condition (133) for the (llq1) case.

For configuration \((ha2)\), the expressions of \(E_0 + p_{0,L}\) and \(E_0 - p_{0,L}\) for the neutralino can be obtained from (422) with \(\cos \theta^* = -1\) and for a non-collinear configuration \(p_{q20} \geq 0\)

\[
2p_{q20} = 2p_{q2,L} + (E_0 + p_{0,L}) - (E_0 - p_{0,L}) = -\frac{M_G^2 - M_Q^2 + M_P M_X (E_0^* - p_0^*) - \frac{M_X}{M_P} (E_0^* + p_0^*) \geq 0}{\frac{M_P}{M_X}}
\]

or

\[
M_P^2 \geq \frac{M_X [M_G^2 - M_Q^2 + M_X (E_0^* + p_0^*)]}{E_0^* - p_0^*}
\]

(473)

In the limit \(M_h = 0\),

\[
M_P^2 \geq \frac{M_G^2 [M_G^2 - M_Q^2 + M_G^2]}{M_Q^2}
\]

which agrees with the lower bound of the interval (132) for the (llq1) case.

However, as was seen in section 5.7.1, these bounds do not correspond to the domain in which the non-collinear configurations give the true end point and we will rederive them now. It is easily verified that the derivation of equation (145) in section 5.7.1 does not rely on \(t_2\) (here \(h^0\) being massless. Therefore, the maximum of \(M_{h,q1q3}\) can be obtained by a straightforward conversion from the (llq1) to the (hq1q3) case. The neutralino momentum is given by

\[
p_0 = \frac{M_0 (M_G^2 - M_Q^2)}{2M_P \sqrt{M_P^2 - M_G^2 + M_Q^2}}
\]

(474)

and the effective mass by

\[
M_{h,q1q3}^2 = \sqrt{M_P^2 - M_G^2 + M_Q^2 - M_0}
\]

(475)
The second collinear end point is obtained from configuration (ha2) with the mass given by equation (466) where \( \cos \theta^* = -1 \).

\[
(M_{hq_4q_3}^{\text{max}})^2 = M_h^2 + (M_P^2 - M_G^2) \frac{M_G^2 - M_X^2}{M_Q^2} + \frac{M_P^2 - M_G^2}{M_Q^2} M_X (E_0^* + p_0^*) + \frac{M_G^2 - M_Q^2}{M_X^2} M_X (E_0^* - p_0^*)
\]

This can also be written as

\[
(M_{hq_4q_3}^{\text{max}})^2 = (M_{hq_4q_3}^{\text{max}})^2 + \frac{M_P^2 - M_G^2}{M_Q^2} M_X (E_0^* + p_0^*) + \frac{M_G^2 - M_Q^2}{M_X^2} M_X (E_0^* - p_0^*)
\]

which is valid, as seen above, for values of \( M_P \) larger than the solution of equation (476).

14.7.4 Second collinear end point in \( M(h^lq_1q_3) \)

The second collinear end point is obtained from configuration (ha2) with the mass given by equation (468) where \( \cos \theta^* = +1 \).

\[
(M_{hq_1q_3}^{\text{max}})^2 = M_h^2 + (M_P^2 - M_G^2) \frac{M_G^2 - M_X^2}{M_Q^2} - \frac{M_P^2 - M_G^2}{M_Q^2} M_X (E_0^* + p_0^*)
\]

This can also be written as

\[
(M_{hq_1q_3}^{\text{max}})^2 = -M_h + (M_{hq_1}^{\text{max}})^2 + (M_{hq_3}^{\text{max}})^2
\]

which is valid for values of \( M_P \) smaller than the solution of equation (478).
14.7.5 Third collinear end point in $M(h^0q1q3)$

The third collinear end point is obtained from configuration (hd2) with the mass given by equation (466) where $\cos \theta^* = +1$.

\[
(M^\text{max,3}_{hq1q3})^2 = M_h^2 + (M_p^2 - M_Q^2) \frac{M_Q^2 - M_X^2}{M_Q^2} + \frac{M_p^2 - M_Q^2}{M_Q^2} M_X(E_h^* - p_h^*) + \frac{M_Q^2 - M_X^2}{M_X^2} M_X(E_h^* + p_h^*)
\]

(483)

This can also be written as

\[
(M^\text{max,3}_{hq1q3})^2 = (M^\text{max}_{hq1q3})^2 + \frac{M_p^2 - M_Q^2}{M_Q^2} M_X(E_h^* + p_h^*)
\]

(484)

and gives the true end point provided $M_p$ is smaller than the solution of equation (477).

As there is an overlap between the regions of validity of $M^\text{max,2}_{hq1q3}$ and $M^\text{max,3}_{hq1q3}$, we still have to find the condition under which e.g. $M^\text{max,3}_{hq1q3} \geq M^\text{max,2}_{hq1q3}$

\[
(M_p^2 - M_Q^2) \frac{M_Q^2 - M_X^2}{M_Q^2} + \frac{M_p^2 - M_Q^2}{M_Q^2} M_X(E_h^* - p_h^*) + \frac{M_Q^2 - M_X^2}{M_X^2} M_X(E_h^* + p_h^*) \geq \frac{M_p^2 - M_Q^2}{M_Q^2} M_X(E_h^* + p_h^*)
\]

(485)

or

\[
M_X(M_Q^2 - M_X^2) + M_X^2(E_h^* - p_h^*) \geq M_Q^2(E_h^* + p_h^*)
\]

In the limit $M_h = 0$, this gives

\[
M_X^2 \leq M_Q M_0
\]

which agrees with the condition (174) for the (llq1) case.

14.8 End points in $M(h^0q1q2q3)$

For this section, we will use the conclusions already obtained in section 5.11 for the (llqq) system. Among the 8 possible configurations, 4 can lead to true end points, namely the ones where one of the four particles included in the mass recoils against all the other three. These are (hb1), (ha2), (hc1) and (hd2).

14.8.1 Absolute maximum of $M(h^0qqq)$

The absolute maximum of $M(h^0qqq)$ is reached when the $\tilde{\chi}_0$ is at rest in the frame of $P = q'$, for which

\[
M^\text{max}_{hq1q3} = M_p - M_0
\]

(486)

The regions of validity for the non-collinear configurations can be most easily obtained by finding the conditions under which the $p_{0,L}$ is to zero in the frame of $P = q'$.

For the configuration (hb1), the $E_0 + p_{0,L}$ and $E_0 - p_{0,L}$ are given by (420) with $\cos \theta^* = +1$ and its validity requires $p_{0,L} \leq 0$

\[
2p_{0,L} = \frac{M_p M_X}{M_G^2} (E_0^* - p_0^*) - \frac{M_G^2}{M_p M_X} (E_0^* + p_0^*) \leq 0
\]

or

\[
M_p^2 \leq \frac{M_G^2}{M_X} E_0^* + p_0^*
\]

(487)
As a check, for $M_h = 0$ this leads to

$$M_P M_0 \leq M_G^2$$

which agrees with the upper bound of (230) in the $(llqq)$ case.

For the configuration $(ha2)$, the $E_0 + p_{0,L}$ and $E_0 - p_{0,L}$ are given by (422) with $\cos \theta^* = -1$ and its validity requires $p_{0,L} \geq 0$

$$2p_{0,L} = \frac{M_P}{M_X}(E_0^* - p_0^*) - \frac{M_X}{M_P}(E_0^* + p_0^*) \geq 0$$

or

$$M_P^2 \geq M_X^2 \frac{E_0^* + p_0^*}{E_0^* - p_0^*}$$

(488)

As a check, for $M_h = 0$ this leads to

$$M_P M_0 \geq M_X^2$$

which agrees with the lower bound of (230) in the $(llqq)$ case.

For the configuration $(hc1)$, the $E_0 + p_{0,L}$ and $E_0 - p_{0,L}$ are given by (421) with $\cos \theta^* = +1$ and its validity requires $p_{0,L} \leq 0$

$$2p_{0,L} = \frac{M_P M_Q^2}{M_X^2}(E_0^* + p_0^*) - \frac{M_Q^2 M_X}{M_P M_Q^2}(E_0^* - p_0^*) \leq 0$$

or

$$M_P^2 \geq \frac{M_Q^4 M_X^2}{M_Q^2} \frac{E_0^* + p_0^*}{E_0^* - p_0^*}$$

(489)

As a check, for $M_h = 0$ this leads to

$$M_P \geq \frac{M_Q^2 M_0}{M_Q^2}$$

which agrees with the bound of (231) in the $(llqq)$ case.

For the configuration $(hd2)$, the $E_0 + p_{0,L}$ and $E_0 - p_{0,L}$ are given by (423) with $\cos \theta^* = -1$ and its validity requires $p_{0,L} \geq 0$

$$2p_{0,L} = \frac{M_P}{M_X^2}(E_0^* + p_0^*) - \frac{M_Q^2}{M_P M_X^2}(E_0^* - p_0^*) \geq 0$$

or

$$M_P^2 \geq \frac{M_Q^4}{M_X^2} \frac{E_0^* + p_0^*}{E_0^* - p_0^*}$$

(490)

As a check, for $M_h = 0$ this leads to

$$M_P \geq \frac{M_Q^2 M_0}{M_X^2}$$

which agrees with the bound of (232) in the $(llqq)$ case.

14.8.2 First collinear end point in $M(h^0qqq)$

The first end point of $M(h^0qqq)$ is obtained in the configuration labelled $(hh1)$. In this configuration, the $(q2q3)$ is massless and the total mass can be written as $(E_{23} = E_2 + E_3)$

$$(M_{qqqq}^{max})^2 = 4E_{q1}E_{23} + 2E_{q1}(E_h + p_{h,L}) + 2E_{23}(E_h - p_{h,L})$$

(491)
With a massless system.

As seen above, its region of validity is forming a massless system.

The second end point of

The third end point of

or

14.8.4 Third collinear end point in \( M(h^0qqq) \)

The third end point of \( M(h^0qqq) \) is obtained in the configuration labelled \((hq2)\), with the three quarks forming a massless system.

\[
M_{hqqq}^{max,2} = M_h^2 + 2E_{qqq}(E_h + p_{h,L})
\] (495)

Now

\[
E_{qqq} = \frac{M_p - M_G^2}{2M_p} + \frac{M_G M_q^2 - M_Q^2}{2M_G} + \frac{M_G M_Q M_p - M_{GQ}^2}{2M_G} = \frac{M_p - M_X}{2M_p}
\] (496)

and the \( E_h + p_{h,L} \) and \( E_h - p_{h,L} \) are given by (422) with \( \cos \theta^* = +1 \).

\[
(M_{hqqq}^{max,2})^2 = M_h^2 + \frac{2M_p^2 - M_X^2}{2M_p} M_x (E_h^* + p_h^*)
\] (497)

Its region of validity is

\[
M_p^2 \leq \frac{M_X^2 E_h^* + p_h^*}{E_0^* - p_0^*}
\] (498)

14.8.4 Third collinear end point in \( M(h^0qqq) \)

The third end point of \( M(h^0qqq) \) is obtained in the configuration labelled \((hq1)\), with the \((q1q3)\) forming a massless system.

\[
M_{hqqq}^{max,3} = M_h^2 + 4E_{13} E_{q2} + 2E_{13}(E_h + p_{h,L}) + 2E_{q2}(E_h - p_{h,L})
\] (499)

With

\[
E_{13} = \frac{M_p^2 - M_G^2}{2M_p} + \frac{M_G M_q^2 - M_Q^2}{2M_G}
\]

\[
E_{q2} = \frac{M_p M_G^2 - M_Q^2}{2M_G}
\] (500)
and the $E_h + p_{h,l}$ and $E_h - p_{h,l}$ given by (421) with $\cos \theta^* = -1$, the effective mass becomes

\[
(M_{hqq}^{\text{max},3})^2 = M_h^2 + 4 \left( \frac{M_p^2 - M_q^2}{2M_p} + \frac{M_q^2 - M_X^2}{2M_q} \right) \frac{M_p M_0^2 - M_0 Q}{2M_G} \\
+ 2 \left( \frac{M_p^2 - M_q^2}{2M_p} + \frac{M_q^2 - M_X^2}{2M_q} \right) \frac{M_0 M_0^2}{2M_G M_X} M_X(E_h^* - p_h^*) \\
+ 2 \frac{M_p M_0^2 - M_0 Q}{2M_G} M_X(E_h^* + p_h^*) = M_h^2 + \left( \frac{M_p^2 - M_q^2}{M_0^2} + \frac{M_q^2 - M_X^2}{M_X} \right) (M_0^2 - M_q^2) \\
+ \left( \frac{M_p^2 - M_q^2}{M_0^2} + \frac{M_q^2 - M_X^2}{M_X} \right) \frac{M_X(E_h^* - p_h^*) + M_0^2 - M_q^2}{M_0^2} M_X(E_h^* + p_h^*) \\
\]

or

\[
(M_{hqq}^{\text{max},3})^2 = M_h^2 + \frac{M_p^2 M_q^2 - M_0^2 M_X^2}{M_0^2 M_X} (M_0^2 - M_q^2) \\
+ \frac{M_q^2 - M_0^2}{M_X^2} M_X(E_h^* - p_h^*) + \frac{M_0^2 - M_q^2}{M_0^2} M_X(E_h^* + p_h^*) \\
(501)
\]

It is valid in the region

\[
M_h^2 \leq \frac{M_0^2 M_X^2}{M_0^2} \frac{E_h^* + p_h^*}{E_h^* - p_h^*} \\
(502)
\]

As the validity range for $M_{hqq}^{\text{max},3}$ and $M_{hqq}^{\text{max},2}$ overlap, it is necessary to find where e.g. $M_{hqq}^{\text{max},3} \geq M_{hqq}^{\text{max},2}$

\[
\frac{M_p^2 M_q^2 - M_0^2 M_X^2}{M_0^2 M_X} (M_0^2 - M_q^2) + \frac{M_q^2 M_0^2 - M_0^2 M_X^2}{M_0^2 M_X} (E_h^* - p_h^*) \\
+ \frac{M_q^2 - M_0^2}{M_X} M_X(E_h^* - p_h^*) + \frac{M_0^2 - M_q^2}{M_0^2} M_X(E_h^* + p_h^*) \geq M_h^2 - M_q^2 \\
+ \frac{M_0^2}{M_X} (M_0^2 - M_q^2) (E_h^* + p_h^*) + \frac{M_q^2}{M_0^2} (M_0^2 - M_q^2) (E_h^* + p_h^*) \\
+ \frac{M_q^2}{M_X} (M_0^2 - M_q^2) (E_h^* + p_h^*) \geq 0
\]

and using the fact that $M_p \geq M_G$ and $M_Q \geq M_X$

\[
M_X(M_0^2 - M_q^2) + M_q^2 (E_h^* - p_h^*) \geq M_q^2 (E_h^* + p_h^*) \\
(503)
\]

14.8.5 Fourth collinear end point in $M(h^0qqq)$

The fourth end point of $M(h^0qqq)$ is obtained in the configuration labelled $(hd2)$, with the $(q1q2)$ forming a massless system.

\[
M_{hqq}^{\text{max},3} = M_h^2 + 4E_{12}E_{q3} + 2E_{12}(E_h + p_{h,l}) + 2E_{q3}(E_h - p_{h,l}) \\
(504)
\]

With

\[
E_{12} = \frac{M_p^2 - M_q^2}{2M_p} + \frac{M_0^2}{M_0} \frac{M_0^2 - M_q^2}{2M_G} = \frac{M_p^2 - M_q^2}{2M_p} \\
E_{q3} = \frac{M_p M_0 M_q^2 - M_0 X^2}{2M_Q} = \frac{M_p M_0^2 - M_X^2}{2M_Q} \\
(505)
\]

120
and the $E_h + p_{h,L}$ and $E_h - p_{h,L}$ given by (423) with $\cos\theta^* = +1$, the effective mass becomes

$$(M'_{hqq}^{max,4})^2 = M_h^2 + \frac{4M_P^2 - M_Q^2}{2M_P} M_P \frac{M_Q^2 - M_X^2}{2M_Q}$$

$$+ 2\frac{M_P^2 - M_Q^2}{2M_P} M_P M_X (E_h^* - p_h^*) + 2\frac{M_P^2 - M_Q^2}{2M_Q} M_P M_X (E_h^* + p_h^*)$$

or

$$(M'_{hqq}^{max,4})^2 = M_h^2 + (M_P^2 - M_Q^2) \frac{M_Q^2 - M_X^2}{M_Q^2}$$

$$+ \frac{M_P^2 - M_Q^2}{M_Q^2} M_X (E_h^* - p_h^*) + \frac{M_P^2 - M_X^2}{M_X^2} M_X (E_h^* + p_h^*)$$

(506)

This end point is valid in the region

$$M_P^2 \leq M_Q^2 E_h^* + p_h^*$$

(507)

As the validity range for $M'_{hqq}^{max,4}$ overlaps with $M'_{hqq}^{max,3}$ and $M'_{hqq}^{max,2}$, it is necessary to find which one is larger. E.g. $M'_{hqq}^{max,4} \geq M'_{hqq}^{max,2}$ if

$$\frac{(M_P^2 - M_Q^2)(M_Q^2 - M_X^2)}{M_Q^2} + \frac{M_P^2 - M_Q^2}{M_Q^2} M_X (E_h^* - p_h^*) + \frac{M_P^2 - M_X^2}{M_X^2} M_X (E_h^* + p_h^*) \geq 0$$

(508)

And $M'_{hqq}^{max,4} \geq M'_{hqq}^{max,3}$ if

$$\frac{(M_P^2 - M_Q^2)(M_Q^2 - M_X^2)}{M_Q^2} + \frac{M_P^2 - M_Q^2}{M_Q^2} M_X (E_h^* - p_h^*) + \frac{M_P^2 - M_X^2}{M_X^2} M_X (E_h^* + p_h^*) \geq 0$$

and using the fact that $M_P \geq M_Q$ and $M_X - E_h^* + p_h^* \geq M_X - E_h^* - p_h^*$, the term in brackets is always positive and the condition is

$$M_Q^2 \geq M_P^2 M_X^2$$

(509)
15 Sequential $\tilde{g}$ decays involving a $\tilde{\chi}_1^\pm$ which decays leptonically

The decay chain involving a $\tilde{\chi}_1^\pm$ leads to two indistinguishable final states.

\[
\tilde{g} \rightarrow q_1\tilde{q} \rightarrow q_1l_1\tilde{\chi}_1^\pm \rightarrow q_1l_1l_2\tilde{\nu} \rightarrow q_1l_1l_2\nu\tilde{\chi}_1^0\quad (510)
\]

The available mass combinations are $(qq)$, $(lqlq)$, $(lqlq)$ and $(lqq)$. Note that the decay chain starting from a squark has only the mass distribution $M(lq)$ available. As this distribution has two end points, one from the decay via $\tilde{\nu}$ and one from $\tilde{l}$, it should allow the determination of $M(\tilde{\chi}_1^\pm)$ and of $M(\tilde{\nu})$, assuming the masses of all other sparticles are known, for instance from the decay chains via $\tilde{\chi}_2^0$. Here we consider only the longer decay chain starting from a gluino for which more constraints are available and hence there is a possibility to determine all masses involved and get an independent measurement.

The configurations are the same as shown in section 5.1, except that one of the leptons, $l2$ for the decay via $\tilde{\nu}$ and $l1$ for the decay via $\tilde{l}$, is now a $\nu$ and is undetected. The formulae for the end points can be straightforwardly derived from the ones for the decays via $\tilde{\chi}_2^0$.

15.1 Upper end point in $M(q1q2)$

The upper end point of $M(q1q2)$ is of the same form as in section 5.2

\[
M_{qq}^{\text{max}} = M_Q \sqrt{(1 - \frac{M_Q^2}{M_Z^2})(1 - \frac{M_2^2}{M_Q^2})} \quad (512)
\]

where $M_C$ is the chargino mass. Its largest value is reached for $M_Q^2 = M_CM_C$ and amounts to $M_{qq}^{\text{max}} = M_Q - M_C$. As $q2$ is produced in the decay of a $\tilde{q}$, the $(qq)$ mass distribution should again be of triangular shape.

15.2 Upper end point in $M(lqlq)$

The $(lqlq)$ mass distribution should reflect the two competing leptonic decays and display two end points.

For the decay through a $\tilde{\nu}$, the end point is similar to the one of $(llq)$ of section 4.2

\[
M_{lqlq,\tilde{\nu}}^{\text{max}} = M_Q \sqrt{(1 - \frac{M_Q^2}{M_Z^2})(1 - \frac{M_N^2}{M_C^2})} \quad (513)
\]

where $M_N$ is the $\tilde{\nu}$ mass. It reaches its largest value when the $\tilde{\nu}$ is at rest in the $Q = \tilde{q}$ frame, i.e. for $M_Q^2 = M_Q M_N$ where it becomes $M_{lqlq,\tilde{\nu}}^{\text{max}} = M_Q - M_N$.

For the decay through a $\tilde{l}$, the end point is similar to the one of $(l2q)$ of section 4.3

\[
M_{lqlq,\tilde{l}}^{\text{max}} = M_Q \sqrt{(1 - \frac{M_Q^2}{M_Z^2})(1 - \frac{M_0^2}{M_R^2})} \quad (514)
\]

which reaches its largest value $M_{lqlq,\tilde{l}}^{\text{max}} = M_Q - \frac{M_Q M_0}{M_R}$ when $M_C = \frac{M_0 M_0}{M_R}$.

The condition for which $M_{lqlq,\tilde{l}}^{\text{max}} > M_{lqlq,\tilde{\nu}}^{\text{max}}$ is

\[
\frac{M_N}{M_C} \leq \frac{M_0}{M_R} \quad (515)
\]

hence either of the two may be larger than the other.

In this case, the two end points cannot be identified by correlations with other mass distributions. It may be possible to distinguish them from the shape of their mass distributions, but this will be more delicate, as already mentioned earlier. For example, if we neglect spin effects, the distribution of $M(lqlq)$ for the decay via $\tilde{\nu}$ should be triangular, whereas the one for the decay via $\tilde{l}$ should have its maximum well below the end point (2 particles out of 4). But spin correlations may well modify these expectations.
15.3 Upper end point in $M(lq1)$

The distribution of $M(lq1)$ should also display two end points.

The end point for the decay via a $\tilde{\nu}$ is similar to the one of $(l1q1)$ in section 5.3

$$M_{lq1,\tilde{\nu}}^{\text{max}} = M_G \sqrt{\left( \frac{1}{1 - \frac{M_N^2}{M_C^2}} \right) \left( \frac{1}{1 - \frac{M_N^2}{M_C^2}} \right)}$$

with its largest value $M_{lq1,\tilde{\nu}}^{\text{max}} = M_G - \frac{M_0 M_N}{M_C}$ reached for $M_Q = \frac{M_0 M_N}{M_C}$.

For the decay through a $\tilde{l}$, the end point is similar to the one of $(l2q1)$ of section 5.4

$$M_{lq1,l}^{\text{max}} = M_G \sqrt{\left( \frac{1}{1 - \frac{M_N^2}{M_C^2}} \right) \left( \frac{1}{1 - \frac{M_N^2}{M_C^2}} \right)}$$

which reaches its largest value $M_{lq1,l}^{\text{max}} = M_G - \frac{M_0 M_Q}{M_R}$ when $M_Q = \frac{M_0 M_N}{M_R}$.

The condition for which $M_{lq1,\tilde{\nu}}^{\text{max}} \geq M_{lq1,l}^{\text{max}}$ is

$$\frac{M_N}{M_C} \leq \frac{M_0}{M_R}$$

hence either of the two may be larger than the other. This condition is the same as the one for $M_{lq2,\tilde{\nu}}^{\text{max}}$.

The distinction between the two $(lq1)$ end points suffers from the same difficulties as discussed in the context of $(lq2)$ above. However, the $(lq1)$ end points can be distinguished from the $(lq2)$ ones by their correlation with $M(qq)$. The $(lq2)$ end points are independent from the $M(qq)$ values, whereas the $(lq1)$ end points occur in the region where $M(qq)$ is small.

To have $M_{lq1,\tilde{\nu}}^{\text{max}} \geq M_{lq2,\tilde{\nu}}^{\text{max}}$ or $M_{lq1,l}^{\text{max}} \geq M_{lq2,l}^{\text{max}}$, the requirement is

$$M_Q^2 - M_N^2 \geq M_Q^2 - M_C^2$$

15.4 End point for the sum of $M(lq)^2$

The true end point of $M(lq)$ is either due to the decay of $\tilde{\nu}$ (i.e. $l1q$) or of $\tilde{l}$ (i.e. $l2q$). Now, as seen in section 5.1, there exists one configuration, namely $(d2)$, which maximizes simultaneously $l2q1$ and $l2q2$. From this

$$[M(l1q1)^2 + M(l1q2)^2]^{\text{max}} = (M_{l1q1}^{\text{max}})^2 + M_{l1q2}^{\text{max}} = (M_G^2 - M_C^2)(1 - \frac{M_0^2}{M_R^2})$$

for the decay via a $\tilde{l}$.

But also in configuration $(d1)$, both $l1q1$ and $l1q2$ are maximal

$$[M(l1q1)^2 + M(l1q2)^2]^{\text{max}} = (M_{l1q1}^{\text{max}})^2 + M_{l1q2}^{\text{max}} = (M_G^2 - M_C^2)(1 - \frac{M_N^2}{M_C^2})$$

for the decay via a $\tilde{\nu}$. This end point will be larger than the one from the decay via a $\tilde{l}$ provided $M_N/M_C \leq M_0/M_R$.

The distribution of the sum $M(l1q1)^2 + M(l1q2)^2$ is easy to construct and is also somewhat simpler than the $M(lq)$ distribution as it contains only two end points instead of four. It may allow to introduce an additional constraint or replace one of the end points if it is too difficult to extract from $M(lq)$. However, it suffers again from the same problem of distinguishing between the decay via $\tilde{\nu}$ and via $\tilde{l}$.

15.5 Upper end points in $M(lqq)$

The analysis of the $(lqq)$ mass distribution is rather involved, as it comprises the three end points similar to the ones of $(l1qq)$ in section 5.8 as well as the three end points similar to the ones of $(l2qq)$ in section 5.9.
15.5.1 Upper end points in $M(lqq)$ for decays via a $\bar{\nu}$

The formulae for the end points are derived from the ones of $((lqq))$ in section 5.8. The absolute maximum, obtained in non-collinear configurations is given by

$$M_{lqq,\bar{\nu}}^{\text{max}} = M_G - M_N$$

and its validity range is

$$\frac{M_G^2}{M_N^2} \leq M_G \leq \frac{M_Q^2}{M_N^2} \text{ and } M_G \geq \frac{M_G^2 M_N}{M_C^2}$$

(523)

A first collinear upper end point is

$$M_{lqq,\bar{\nu}}^{\text{max},1} = M_G \sqrt{(1 - \frac{M_Q^2}{M_G^2})(1 - \frac{M_N^2}{M_Q^2})}$$

(524)

Its validity region is $M_G \geq \frac{M_Q^2}{M_N}$.

A second collinear upper end point is

$$M_{lqq,\bar{\nu}}^{\text{max},2} = M_G \sqrt{(1 - \frac{M_C^2}{M_G^2})(1 - \frac{M_N^2}{M_C^2})}$$

(525)

Its validity region is $M_G \leq \frac{M_Q^2}{M_N}$ and it leads to the true end point provided $M_G^2 > M_Q M_N$.

A third collinear upper end point is

$$M_{lqq,\bar{\nu}}^{\text{max},3} = M_G \sqrt{(1 - \frac{(M_Q M_N)^2}{(M_G M_C)^2})(1 - \frac{M_C^2}{M_Q^2})}$$

(526)

Its validity region is $M_G \leq \frac{M_Q M_N}{M_C}$ and it leads to the true end point provided $M_G^2 < M_Q M_N$.

15.5.2 Upper end points in $M(lqq)$ for decays via a $\bar{l}$

The formulae for the end points are derived from the ones of $((l2qq))$ in section 5.9. The formulae can be derived from the maxima for the decays via a $\bar{\nu}$ after replacing $M_{l10}$ by $M_G M_0 / M_R$, the minimum mass of the $(\nu\chi^0)$ system with the two unobserved particles parallel.

The absolute maximum, obtained in non-collinear configurations is given by

$$M_{lqq,\bar{l}}^{\text{max}} = M_G - \frac{M_G M_0}{M_R}$$

(527)

its validity range is

$$\frac{M_C M_R}{M_0} \leq M_G \leq \frac{M_Q M_R}{M_C M_0} \text{ and } M_G \geq \frac{M_G^2 M_0}{M_C M_R}$$

(528)

A first collinear upper end point is

$$M_{lqq,\bar{l}}^{\text{max},1} = M_G \sqrt{(1 - \frac{M_Q^2}{M_G^2})(1 - \frac{(M_C M_0)^2}{(M_Q M_R)^2})}$$

(529)

Its validity region is $M_G \geq \frac{M_Q M_R}{M_C M_0}$. It is smaller than the end point of $M_{lqq,\bar{\nu}}^{\text{max}}$ provided $M_N / M_C \leq M_0 / M_R$.

A second collinear upper end point is

$$(M_{lqq,\bar{l}}^{\text{max},2})^2 = (M_{lqq,\bar{l}}^{\text{max},2})^2 + (M_{lqq,\bar{l}}^{\text{max},2})^2$$

(530)

or

$$M_{lqq,\bar{l}}^{\text{max},2} = M_G \sqrt{(1 - \frac{M_C^2}{M_G^2})(1 - \frac{M_0^2}{M_R^2})}$$

(531)
Its validity region is $M_G \leq M_C M_R/M_0$ and is smaller than the end point of $M_{iqq,\omega}^{max,2}$ provided $M_N/M_C \leq M_0/M_R$.

A third collinear upper end point is

$$\left(M_{iqq,\omega}^{max,3}\right)^2 = (M_{qq}^{max})^2 + (M_{i42,\omega}^{max})^2$$

or

$$M_{iqq,\omega}^{max,3} = M_G \sqrt{(1 - \frac{(M_Q M_0)^2}{(M_G M_R)^2})(1 - \frac{M_C^2}{M_Q^2})}$$

Its validity region is $M_G \leq \frac{M_Q^2 M_0}{M_C M_R}$ and is smaller than the end point of $M_{iqq,\omega}^{max,2}$ provided $M_N/M_C \leq M_0/M_R$.

15.5.3 Conclusion on the $M(lqq)$ end points

It is seen from previous sections that if $M_N/M_C \leq M_0/M_R$ all the true end points of $M_{iq}$ and of $M_{iqq}$ are from the decay via $a \tilde{\nu}$. Conversely, if $M_N/M_C \geq M_0/M_R$ all the true end points of $M_{iq}$ and of $M_{iqq}$ are from the decay via $a \tilde{t}$. But the distributions will most likely not allow to identify which of the two cases applies. This ambiguity can, however, be resolved if the information from the chargino decay is used in conjunction with the one from the leptonic decay of $\chi_2^0$ by performing the test of the two decay hypotheses, e.g. on the basis of a chi-squared.

16 Sequential $\tilde{g}$ decays involving a $\chi_1^{\pm}$ which decays via $W$

The decay chain involving a $\chi_1^{\pm}$ leads also to the final state

$$\tilde{g} \rightarrow \tilde{q}_1 \tilde{q} \rightarrow \tilde{q}_1 q_2^* \chi_1^\pm \rightarrow \tilde{q}_1 q_2^* W^\pm \chi_1^0 \rightarrow \tilde{q}_1 q_2^* q_3^* q_4^* \chi_1^0$$

$$\rightarrow \tilde{q}_1 q_2^* l^\pm \nu \chi_1^0$$

The leptonic decay of the $W$ produces a final state which is indistinguishable from the ones coming from the chargino leptonic decay of section 15. This case will be discussed separately below.

16.1 End point for hadronic decays of the $W$

We will first assume that the $W$ is reconstructed from its hadronic jets. The available mass combinations are $(qq)$, $(Wq_1)$, $(Wq_2)$ and $(Wqq)$. Note that the decay chain starting from a squark has only the mass distribution $M(Wq)$ available. Here we consider only the longer decay chain starting from a gluino for which more constraints are available and hence there is a possibility to determine all masses involved and get an independent measurement.

The configurations are the same as shown in section 12.1, and the formulae for the end points can be straightforwardly derived from the ones in sections 11 and 12 for the decays via a $\chi_2^0$ into $h^0$ by replacing $M_h$ by $M_W$ and $M_X$ by $M_C$. Therefore, we will not repeat them here.

16.2 End points for leptonic decays of the $W$

The collinear configurations giving rise to the end points resemble the ones for the decay into $\chi_2^0$ followed by the decay via a slepton, except that the $\chi_1^0$ is emitted in the last but one rather than the last decay step. As a result, all formulae have to be rederived. There are 6 useful configurations (with non-zero effective masses) listed below. The available mass combinations are $(qq)$, $(lq_1)$, $(lq_2)$ and $(lqq)$. 

![Diagram](wa1)

![Diagram](wa2)
As the main difference compared to previous cases is the position of the neutralino in the decay chain, it is useful to express the lepton four-vector in the rest frame of the chargino. In the decay frame of the $W$, as both decay products can be considered massless, the lepton energy is given by

$$E_l = \frac{M_W}{2} = p_l^*$$  (536)

The transformation to the $C = \tilde{\chi}_1^+$ rest frame is

$$E_l' = \gamma_W (E_l^* + p_l^* \cos \theta^*) = \gamma_W (1 + \beta_W \cos \theta^*) E_l^*$$
$$p_{l,L}^* = \gamma_W (\beta_W E_l^* + p_l^* \cos \theta^*) = \gamma_W (\beta_W + \cos \theta^*) E_l^*$$  (537)

where $\theta^*$ is the angle of $l$ in the $W$ rest frame and the primed quantities are defined in the $C = \tilde{\chi}_1^+$ rest frame, where

$$E_W' = \frac{M_W^2 - M_0^2 + M_{1q}^2}{2M_C} \cdot p_W' = \frac{1}{2M_C} \sqrt{(M_C^2 - M_0^2 - M_W^2)^2 - 4M_0^2 M_W^2}$$  (538)

For the collinear configuration, all we will need is

$$\gamma_W (1 \pm \beta_W) = \frac{E_W' \pm p_W'}{M_W}$$  (539)
from which

\[
E'_l = \frac{E'_W \pm p'_W}{2} = \frac{1}{4M_C} \left[ M_C^2 - M_0^2 + M_W^2 \pm \sqrt{(M_C^2 - M_0^2 - M_W^2)^2 - 4M_0^2M_W^2} \right]
\]

\[
p'_{lL} = \pm \frac{E'_W \pm p'_W}{2} = \pm \frac{1}{4M_C} \left[ M_C^2 - M_0^2 + M_W^2 \pm \sqrt{(M_C^2 - M_0^2 - M_W^2)^2 - 4M_0^2M_W^2} \right]
\]

(540)

where the \( \pm \) corresponds respectively to \( \cos \theta^* = \pm 1 \) in the decay of the \( W \).

16.3 Upper end point in \( M(q1q2) \)

The upper end point of \( M(q1q2) \) is the same as in section 15.1 and will not be repeated here. It applies to the configurations labelled \( (wa1), (wa2), (wb1) \) and \( (wb2) \).

16.4 Upper end point in \( M(lq2) \)

The \( (lq2) \) mass distribution has an upper end point for the configurations labelled \( (wb1) \) and \( (wd) \). The effective mass can be computed in the rest frame of \( G = \tilde{q} \).

\[
(M_{lq2}^{\text{max}})^2 = 4E_{q2}E_l
\]

(541)

where

\[
E_{q2} = \frac{M_Q^2 - M_C^2}{2M_Q}
\]

\[
E_l = \frac{M_Q}{M_C} E'_l
\]

(542)

The maximum effective mass is obtained for \( \cos \theta^* = +1 \)

\[
(M_{lq2}^{\text{max}})^2 = 4 \frac{M_Q^2 - M_C^2}{2M_Q} \frac{1}{M_C^2} (E'_W + p'_W)
\]

or

\[
(M_{lq2}^{\text{max}})^2 = (M_Q^2 - M_C^2) \frac{E'_W + p'_W}{M_C}
\]

\[
= (M_Q^2 - M_C^2) \frac{M_C^2 - M_0^2 + M_W^2 + \sqrt{(M_C^2 - M_0^2 - M_W^2)^2 - 4M_0^2M_W^2}}{2M_C^2}
\]

(543)

Note that this maximum is reached independently from \( M(qq) \).

16.5 Upper end point in \( M(lq1) \)

The distribution of \( M(lq1) \) displays an end point for the collinear configuration labelled \( (wd) \). The effective mass, computed in the rest frame of \( G = \tilde{g} \) is now

\[
(M_{lq1}^{\text{max}})^2 = 4E_{q1}E_l
\]

(544)

where

\[
E_{q1} = \frac{M_Q^2 - M_C^2}{2M_Q}
\]

\[
E_l = \frac{M_Q}{M_C} E'_l
\]

(545)
The maximum effective mass is again obtained for $\cos \theta^* = +1$

\[
(M_{lq1}^{\text{max}})^2 = \frac{4}{2 M_G} \left( M_G^2 - M_Q^2 \right) \frac{1}{2} \left( E'_W + p'_W \right)
\]

or

\[
(M_{lq1}^{\text{max}})^2 = \left( M_Q^2 - M_C^2 \right) \frac{E''_W + p''_W}{M_C} \frac{M_G^2 - M_Q^2 + M_Q^2 + \sqrt{(M_G^2 - M_Q^2)^2 - 4 M_Q^2 M_C^2}}{2 M_C^2}
\]

We can now compare $M_{lq1}^{\text{max}}$ and $M_{lq2}^{\text{max}}$ to find when one is larger than the other. To have $M_{lq1}^{\text{max}} \geq M_{lq2}^{\text{max}}$ requires

\[
M_Q^2 - M_C^2 \geq M_Q^2 - M_C^2
\]

To determine experimentally to which case the true end point of $M(lq)$ belongs, its correlation with $M(qq)$ can be analyzed. If the end point in $M(lq)$ is reached for any value of $M(qq)$, it is given by $M_{lq2}^{\text{max}}$; if it is correlated with the minimum of $M(qq)$, it is given by $M_{lq1}^{\text{max}}$.

### 16.6 Upper end points in $M(lqq)$

There are potentially three collinear configurations which may give rise to the true end point for $M(lqq)$, namely $(wb2)$, $(wd)$ or $(wb1)$.

#### 16.6.1 Absolute maximum in $M(lqq)$

The absolute maximum of the $M(lqq)$ distribution is reached when the system $(\nu \bar{\chi}^0)$ has its minimal mass (both particles parallel) and is at rest in the $G = \tilde{g}$ frame. For this configuration, the $\nu$ is emitted backwards in the $W$ frame. The $(\nu \bar{\chi}^0)$ energy and momentum in the $C = \tilde{\chi}^\pm_1$ frame are

\[
E'_{\nu 0} = M_G - E'_l, \quad p'_{\nu 0} = p'_l = E'_l
\]

hence the mass is

\[
M_{\nu 0}^2 = E'_{\nu 0}^2 - p'_{\nu 0}^2 = M_C(M_G - 2 E'_l)
\]

As $l$ is emitted forward in the $W$ decay, we get from the above derivation that

\[
E'_l = \frac{E''_W + p''_W}{2}
\]

from which

\[
M_{\nu 0}^2 = M_C(M_G - E''_W - p''_W) = M_C(E'_0 - p'_0)
\]

The absolute maximum of $M(lqq)$ is then

\[
M_{lqq}^{\text{max}} = M_G - M_{\nu 0}
\]

The regions of validity for the non-collinear configurations with the $(\nu \bar{\chi}^0)$ system at rest are easily derived from section 5.8 by replacing $M_R$ by $M_{\nu 0}$ and $M_X$ by $M_C$. They are

\[
\frac{M_C^2}{M_{\nu 0}} \leq M_G \leq \frac{M_Q^2}{M_{\nu 0}}
\]

and

\[
M_G \geq \frac{M_C^2}{M_Q^2} M_{\nu 0}
\]
16.6.2 First collinear end point in \( M(lqq) \)

The first collinear end point in \( M(lqq) \) corresponds to the configuration labelled (wb2), for which \( \cos \theta^* = +1 \). The effective mass is given by

\[
(M_{lqq})^2 = 4E_{q1}E_{q2} + 2E_{q1}(E_l + p_{l,L}) = 4E_{q1}(E_{q2} + E_l)
\]

(555)

where

\[
E_l = \frac{M_G M_Q}{M_Q M_Q} E'_l = \frac{M_G M_C}{M_Q M_Q} \frac{1}{2} (E'_W + p'_W)
\]

\[
E_{q2} = \frac{M_G M_Q^2 - M_Q^2}{2M_Q}
\]

(556)

The mass then becomes

\[
(M_{lqq}^{max})^2 = 4 \frac{M_G^2 - M_Q^2}{2M_G} M_Q \left[ \frac{M_G^2 - M_Q^2}{2M_Q} + \frac{M_C}{M_Q} \frac{1}{2} (E'_W + p'_W) \right]
\]

or

\[
(M_{lqq}^{max})^2 = (M_G^2 - M_Q^2) \frac{M_G^2 - M_Q^2}{2M_Q} + M_C (E'_W + p'_W)
\]

(557)

16.6.3 Second collinear end point in \( M(lqq) \)

The second collinear end point in \( M(lqq) \) corresponds to the configuration labelled (wd), for which \( \cos \theta^* = +1 \). The effective mass is given by

\[
(M_{lqq})^2 = 2E_{q1}(E_l + p_{l,L}) + 2E_{q2}(E_l + p_{l,L}) = 4(E_{q1} + E_{q2})E_l
\]

(558)

where

\[
E_l = \frac{M_G M_Q}{M_Q M_C} E'_l = \frac{M_G}{M_C} \frac{1}{2} (E'_W + p'_W)
\]

\[
E_{q1} + E_{q2} = \frac{M_G^2 - M_Q^2}{2M_G} + \frac{M_Q M_G^2 - M_Q^2}{2M_Q} = \frac{M_G^2 - M_Q^2}{2M_G}
\]

(559)

The mass then becomes

\[
(M_{lqq}^{max,2})^2 = (M_G^2 - M_Q^2) \frac{E'_W + p'_W}{M_C}
\]

(560)

16.6.4 Third collinear end point in \( M(lqq) \)

The third collinear end point in \( M(lqq) \) corresponds to the configuration labelled (wb1), for which \( \cos \theta^* = +1 \). The effective mass is given by

\[
(M_{lqq})^2 = 4E_{q1}E_{q2} + E_{q2}(E_l + p_{l,L}) = 4E_{q2}(E_{q1} + E_l)
\]

(561)

where

\[
E_l = \frac{M_Q}{M_G} \frac{1}{M_Q M_C} (E'_W + p'_W)
\]

\[
E_{q2} = \frac{M_Q M_Q^2 - M_Q^2}{2M_Q}
\]

(562)

leading to the effective mass

\[
(M_{lqq}^{max,3})^2 = 4 \frac{M_G}{M_Q} M_Q^2 - M_Q^2 \left[ \frac{M_G^2 - M_Q^2}{2M_G} + \frac{M_Q^2}{M_G M_C} \frac{1}{2} (E'_W + p'_W) \right]
\]

or

\[
(M_{lqq}^{max,3})^2 = (M_Q^2 - M_Q^2) \frac{M_Q^2 + E'_W + p'_W}{M_C}
\]

(563)
16.6.5 Conclusion on the \( M(lqq) \) end points

17 Sequential decays to 5-particle final states, massive case

There are many examples where the mass of some or all stable final state particles cannot be neglected, as they may involve top quarks, \( h^0 \) or \( Z^0 \) etc. Some special cases where some particles remain massless have been considered above. We will now treat the case where any of them can be massive, generalizing the discussion of section 5 with 5 stable particles, including a \( \tilde{\chi}^0_1 \), in the final state. For the derivation of the formulae, we will heavily rely on what was learned in section 5.

Let us label the states in the decay chain as follows

\[
D_{i+1} \rightarrow S_i + D_j \rightarrow S_i + S_k + D_4 \rightarrow S_i + S_k + S_l + D_5 \rightarrow S_i + S_k + S_l + S_m + D_5
\]

where \( D_i \) stands for decaying states (except \( D1 \) which is stable) with mass \( M_i \) and \( S_i \) for stable SM particles with mass \( m_i \).

As the decays are still supposed to be 2-body, the energy and momentum of particles \( S_i \) and \( D_i \) in the rest frame of \( D_{i+1} \) are given by

\[
E_{S_i} = \frac{M_{i+1}^2 - M_i^2 + m_i^2}{2M_{i+1}}, \quad E_{D_i} = \frac{M_{i+1}^2 + M_i^2 - m_i^2}{2M_{i+1}}
\]

\[
p_{S_i} = \sqrt{(M_{i+1}^2 - M_i^2 - m_i^2)^2 - 4M_i^2 m_i^2}
\]

As discussed in section 2.5, an easy way to operate the Lorentz transformations is to use \( E + p_L \) and \( E - p_L \) which transform as

\[
E_{S_i} + p_{S_i,L} = \frac{E_{D_{i+1}} + p_{D_{i+1}}}{M_{i+1}} (E_{S_i} + p_{S_i} \cos \theta')
\]

\[
E_{S_i} - p_{S_i,L} = \frac{E_{D_{i+1}} - p_{D_{i+1}}}{M_{i+1}} (E_{S_i} - p_{S_i} \cos \theta')
\]

where primed quantities are in the rest frame of \( D_i + 1 \) and unprimed quantities include the boost of \( D_i + 1 \). From this the \( E \) and \( p_L \) can be extracted.

17.1 Two-body effective masses

The effective mass \( M_{ij} \) of particles \( S_i \) and \( S_j \) \((i > j)\) is given by

\[
M_{ij}^2 = m_i^2 + m_j^2 + 2E_i E_j + 2p_i p_j,_{L}
\]

where the orientation is chosen such that \( p_i \) is opposite to the \( z \)-axis and where \( p_{j,L} \) is signed. This mass is maximized in the collinear configuration where particles \( i \) and \( j \) are back to back and the boost between them is maximal.

17.2 Three-body effective masses

The effective mass \( M_{ijk} \) of particles \( S_i, S_j \) and \( S_k \) \((i > j > k)\) is given by

\[
M_{ijk}^2 = m_i^2 + m_j^2 + m_k^2 + 2E_i E_j + 2E_i E_k + 2E_j E_k + 2p_i p_j,_{L} + 2p_i p_k,_{L} + 2p_j p_k,_{L} - 2p_{j,L} p_{k,L}
\]

where the orientation is again chosen such that \( p_i \) is opposite to the \( z \)-axis.

In this case, there are three collinear configurations which can lead to the true end point, depending on the sparticle masses, and in addition there is a region where the maximum is determined by a non-collinear configuration. The collinear configurations are the ones where one of the particles \( i, j \) or \( k \) is sent in the direction opposite to the other two and the boost between them is maximal. The non-collinear configuration corresponds to putting the system of the particle(s) between the two extreme ones in the decay and not included in the mass at rest in the frame of the parent. An exception is the mass \( M_{124} \), for which the system \( S_4 \tilde{\chi}_1^0 \) is not at rest, like was encountered in the case of \( llq1 \). For the first cases, defining
$p_{ijk} = -p_i + p_{j,L} + p_{k,L}$, the region of validity of the collinear configurations is $p_{ijk} \leq 0$ for the first one (with $i$ opposite to $j$ and $k$) and $p_{ijk} \geq 0$ for the other two. Between these bounds the end point is given by the non-collinear configuration. The maximum effective mass is in this case

$$M_{123}^{\text{max}} = M_4 - M_1$$
$$M_{134}^{\text{max}} = M_5 - M(D1, S2)$$
$$M_{234}^{\text{max}} = M_5 - M_1$$

(569)

where $M(D1, S2)$ is the effective mass of $D1$ and $S2$ when both particles are parallel (i.e. their minimal mass).

The case of $M_{124}$ is more complicated and both the value of the maximum mass and the non-collinearity conditions have to be generalized to include massive particles. As the maximum mass is reached in the configuration shown in figure 19, the energy and momentum of $S3$ and $S4$ are fixed. The effective mass can be computed from

$$M_{124}^2 = m_i^2 + m_{12}^2 + 2E_4E_{12} + 2p_4p_{12}$$

(570)

where the index 12 stands for the system of particles $S1$ and $S2$.

The overall energy/momentum conservation equations are

$$M_5 = E_{34} + E_{12} + E_0 , \quad 0 = -p_{34} + p_{12} + p_0$$

(571)

where the index 0 is for particle $D1$ (usually the neutralino). They can be introduced in the on-shell condition

$$M_i^2 = E_i^2 - p_i^2 = M_5^2 + m_{34}^2 + m_{12}^2 - 2M_5E_{34} - 2(M_5 - E_{34})E_{12} + 2p_{34}p_{12}$$

from which

$$m_{12}^2 = M_5^2 - m_{34}^2 + 2M_5E_{34} + 2(M_5 - E_{34})E_{12} - 2p_{34}p_{12}$$

which can be used to simplify $M_{124}$

$$M_{124}^2 = M_1^2 - M_5^2 + m_{34}^2 + 2M_5E_{34} + 2(M_5 - E_{34})E_{12} + 2(p_4 - p_{34})p_{12}$$
$$= M_1^2 + m_4^2 - M_5^2 - E_{34}^2 + p_{34}^2 + 2M_5E_{34} + 2(M_5 - E_3)E_{12} - 2p_{34}p_{12}$$

or

$$M_{124}^2 = M_1^2 + m_4^2 - (M_5 - E_{34})^2 + p_{34}^2 + 2(M_5 - E_3)E_{12} - 2p_{34}p_{12}$$

(572)

From the energy/momentum conservation equations it is seen that

$$\frac{\partial E_{12}}{\partial p_0} = -\frac{\partial E_0}{\partial p_0} = -\frac{p_0}{E_0}$$
$$\frac{\partial p_{12}}{\partial p_0} = -1$$

so that the maximum of $M_{124}$ is defined by ($S3$ and $S4$ are fixed)

$$\frac{\partial M_{124}^2}{\partial p_0} = -2(M_5 - E_3)p_0 + 2p_3 = 0$$
$$p_3E_0 = (M_5 - E_3)p_0$$
$$p_3^2(p_0^2 + M_1^2) = (M_5 - E_3)^2p_0^2$$
$$p_0^2[(M_5 - E_3)^2 - p_3^2] = M_1^2p_3^2$$

or

$$p_0 = \frac{M_1|p_3|}{\sqrt{M_5^2 + M_1^2 - 2M_5E_3}}$$

(573)
which is the generalization of (144). This can now be used to compute the maximum of $M_{124}$.

\[ E_0^2 = p_0^2 + M_1^2 = M_1^2 \left( \frac{p_3^2}{M_5^2 + M_3^2 - 2M_5E_3} + 1 \right) \]
\[ = M_1^2 \frac{M_2^2 + E_3^2 - 2M_5E_3}{M_5^2 + M_3^2 - 2M_5E_3} = M_1^2 \frac{(M_5 - E_3)^2}{M_5^2 + M_3^2 - 2M_5E_3} \]
\[ E_{12} = M_5 - E_{34} - E_0 = M_5 - E_{34} - \frac{M_1(M_5 - E_3)}{\sqrt{M_5^2 + M_3^2 - 2M_5E_3}} \]
\[ p_{12} = p_{34} - p_0 = p_{34} - \frac{M_1p_3}{\sqrt{M_5^2 + M_3^2 - 2M_5E_3}} \]

Then, from (572)

\[ M_{124}^2 = M_1^2 + m_4^2 - (M_5 - E_{34})^2 + p_{34}^2 + 2(M_5 - E_3)(M_5 - E_{34}) - 2p_3p_{34} \]
\[ = 2M_1 \frac{(M_5 - E_3)^2 - p_3^2}{\sqrt{M_5^2 + M_3^2 - 2M_5E_3}} \]
\[ = M_1^2 + m_4^2 + (M_5 - E_3)(M_5 - E_{34}) - 2M_5 + 2E_3 \]
\[ = 2M_1 \frac{M_2^2 + M_3^2 - 2M_5E_3}{\sqrt{M_5^2 + M_3^2 - 2M_5E_3}} \]
\[ = M_1^2 + m_4^2 - (p_4^2 - p_3^2) + (M_5 - E_3 - E_4)(M_5 - E_3 + E_4) - 2M_1 \sqrt{M_5^2 + M_3^2 - 2M_5E_3} \]
\[ = M_1^2 + m_4^2 - (p_4^2 - p_3^2) + (M_5 - E_3)^2 - E_4^2 - 2M_1 \sqrt{M_5^2 + M_3^2 - 2M_5E_3} \]
\[ = M_1^2 - p_3^2 + M_2^2 + E_3^2 - 2M_5E_3 - 2M_1 \sqrt{M_5^2 + M_3^2 - 2M_5E_3} \]
\[ = M_1^2 + M_2^2 + M_3^2 - 2M_5E_3 - 2M_1 \sqrt{M_5^2 + M_3^2 - 2M_5E_3} \]
\[ = \left( M_1 - \sqrt{M_5^2 + M_3^2 - 2M_5E_3} \right)^2 \]

or finally

\[ M_{124} = \sqrt{M_5^2 + M_3^2 - 2M_5E_3} - M_1 \] (574)

which generalizes equation (145).

The region of validity of the three collinear configurations which lead to possible true end points is defined as follows. For the first configuration, defining $p_{124} = -p_3 + p_{1,L} + p_{2,L}$, the region of validity is $-p_{124} - p_{3,L} \geq p_0$, where $p_0$ is given by equation (573). For the other two configurations it is $-p_{124} - p_{3,L} \leq p_0$. Between these bounds the end point is given by the non-collinear configuration, equation (574).

### 17.3 Four-body effective masses

The four-body effective mass is a straightforward extension of the $(llqq)$ case. There are now four collinear configurations leading to the true end point and the non-collinear configuration where the $\chi^0_l$ is at rest in the frame of $D_5$.

### 18 Full reconstruction of events

If a long enough decay chain is available, like the one described in figure 33, it is possible to fully reconstruct individual events and to determine all masses using all events, not only the ones leading to end points. In figure 33 it is assumed that $m(\bar{g}) > m(\bar{g})$. 

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Reconstruction of the $\tilde{\chi}^0_1$ momentum from a squark decay chain

We first reconstruct the momentum of the $\tilde{\chi}^0_1$ under the assumption that the masses of all sparticles are known. Starting with the decay of $X = \tilde{\chi}^0_2$ and assuming zero mass for the quarks and leptons, we get from $(C2)$

$$M^2_R = p_0^2 + p_1^2 + 2p_0p_1 = M^2_0 + 0 + 2p_0p_1$$

$$\Rightarrow 2p_0p_1 = M^2_R - M^2_0 = 2E_{12}(E_0 - |p_0|\cos\theta_{120})$$

and from $(C3)$

$$M^2_\chi = (p_0 + p_1)^2 + p_2^2 + 2p_0p_2 = M^2_R + 0 + 2p_0p_2$$

$$\Rightarrow 2p_0p_2 = M^2_\chi - M^2_R - 2p_0p_1 = 2E_{11}(E_0 - |p_0|\cos\theta_{110})$$

Supposing that all masses are known, these equations determine the projection of $\tilde{p}_0$ along the two lepton directions, but are not sufficient to determine $\tilde{p}_0$ completely. Hence, we also need to use the $\tilde{q}$ relation $(C4)$:

$$M^2_\tilde{q} = (p_0 + p_2 + p_{11})^2 + p_{12}^2 + 2p_{02}(p_0 + p_2 + p_{11}) = M^2_\chi + 0 + 2p_{02}p_0 + 2(p_{12} + p_{11})p_{12}$$

$$\Rightarrow 2p_0p_{12} = M^2_\tilde{q} - M^2_\chi - 2(p_{12} + p_{11})p_{12} = 2E_{12}(E_0 - |p_0|\cos\theta_{120})$$

This allows to determine the third projection of $\tilde{p}_0$ and thus to reconstruct the neutralino momentum if all masses are known. But it still provides no constraint allowing to determine masses. Altogether, the equations can be put in the form:

$$|\tilde{p}_0|\cos\theta_{120} = \tilde{p}_0 \cdot \tilde{n}_2 = E_0 - \frac{M^2_\tilde{q} - M^2_0}{2E_{12}}$$

$$|\tilde{p}_0|\cos\theta_{110} = \tilde{p}_0 \cdot \tilde{n}_1 = E_0 - \frac{M^2_\chi - M^2_R - 2p_0p_1}{2E_{11}}$$

$$|\tilde{p}_0|\cos\theta_{120} = \tilde{p}_0 \cdot \tilde{n}_q = E_0 - \frac{M^2_\tilde{q} - M^2_\chi - 2(p_{12} + p_{11})p_{12}}{2E_{12}}$$

where $\tilde{n}_2$, $\tilde{n}_1$, and $\tilde{n}_q$ are unit vectors along the direction of $p_{12}$, $p_{11}$ and $p_{12}$, respectively. The equations can be solved for $\tilde{p}_0$ by matrix inversion and iteration or, more explicitly, by defining $\tilde{m}_2 = \tilde{n}_2 - \tilde{n}_q$ and

Figure 33: Long chain decay of a gluino.
\( \vec{m}_1 = \vec{n}_1 - \vec{n}_q \) the first two equation yield

\[
\begin{align*}
\vec{p}_0 \cdot \vec{m}_2 &= \frac{M_Q^2 - M_X^2 - 2(p_{l2} + p_{l1})p_{q2}}{2E_{q2}} 
\quad \frac{M_R^2 - M_0^2}{2E_{l2}} \equiv f_2 \\
\vec{p}_0 \cdot \vec{m}_1 &= \frac{M_Q^2 - M_X^2 - 2(p_{l2} + p_{l1})p_{q2}}{2E_{q2}} 
\quad \frac{M_R^2 - 2p_{l1}p_{l2}}{2E_{l1}} \equiv f_1
\end{align*}
\]

or

\[
\begin{align*}
p_{0x}m_{2x} + p_{0y}m_{2y} &= -p_{0z}m_{2z} + f_2 \\
p_{0x}m_{1x} + p_{0y}m_{1y} &= -p_{0z}m_{1z} + f_1
\end{align*}
\]

which allow to solve for \( p_{0x} \) and \( p_{0y} \) as a function of \( p_{0z} \). The determinant is \( \det = m_{2x}m_{1y} - m_{2y}m_{1x} \) and

\[
\begin{align*}
p_{0x} &= A_xp_{0z} + B_x \quad \text{with } A_x = \frac{m_{2y}m_{1z} - m_{2z}m_{1y}}{\det} 
B_x = \frac{f_{2}m_{1y} - f_{1}m_{2y}}{\det} \\
p_{0y} &= A_yp_{0z} + B_y \quad \text{with } A_y = \frac{m_{2z}m_{1x} - m_{2x}m_{1z}}{\det} 
B_y = -\frac{f_{2}m_{1x} - f_{1}m_{2x}}{\det}
\end{align*}
\]

Introduced in the third equation, we get

\[
\begin{align*}
p_{0x}n_{qx} + p_{0y}n_{qy} + p_{0z}n_{qz} &= (A_xn_{qz} + A_yn_{qy} + n_{qz})p_{0z} + (B_xn_{qz} + B_yn_{qy}) \\
eq A_zp_{0z} + B_z &= E_0 - \frac{M_Q^2 - M_X^2 - 2(p_{l2} + p_{l1})p_{q2}}{2E_{q2}}
\end{align*}
\]

which can be solved for \( p_{0z} \)

\[
E_0 = A_zp_{0z} + B_z + \frac{M_Q^2 - M_X^2 - 2(p_{l2} + p_{l1})p_{q2}}{2E_{q2}} \equiv A_zp_{0z} + D
\]

squaring both sides and using \( E_0^2 = |\vec{p}_0|^2 + M_Q^2 \).

\[
\begin{align*}
M_Q^2 + p_{0x}^2 + p_{0y}^2 + p_{0z}^2 &= M_Q^2 + (A_x^2 + A_y^2 + 1)p_{0z}^2 + 2(A_xB_x + A_yB_y)p_{0z} + B_x^2 + B_y^2 \\
&= A_zp_{0z}^2 + 2A_zDp_{0z} + D^2
\end{align*}
\]

which leads to the second order equation

\[
(A_x^2 + A_y^2 + 1 - A_z^2)p_{0z}^2 + 2(A_xB_x + A_yB_y - A_zD)p_{0z} + (M_Q^2 + B_x^2 + B_y^2 - D^2) = 0
\]

Hence there are two solutions for the neutralino 3-momentum.

In conclusion, if we have a decay chain starting from a squark, it is possible to determine the whole kinematics (but with generally two solutions) on an event by event basis, provided we know all sparticle masses.

### 18.2 Reconstruction of the \( \tilde{\chi}_1^0 \) momentum from a gluino decay chain

If the decay chain starts from a gluino, it provides an alternative way to compute the neutralino kinematics.

Using equation (C5) leads to

\[
\begin{align*}
M_Q^2 &= M_Q^2 + 0 + 2p_{q1}p_0 + 2(p_{l2} + p_{l1} + p_{q2})p_{q1} \\
&= 2p_0p_{q1} = M_Q^2 - M_Q^2 - 2(p_{l2} + p_{l1} + p_{q2})p_{q1} = 2E_{q1}(E_0 - |\vec{p}_0|\cos\theta_{q10})
\end{align*}
\]

and can be used, instead of the similar equation for the squark, to determine the \( \tilde{\chi}_1^0 \) momentum.

In this case, \( \vec{n}_q \) is the direction of \( p_{q1} \) (rather than \( p_{q2} \)), the functions \( f_2 \) and \( f_1 \) are defined as

\[
\begin{align*}
f_2 &= \frac{M_Q^2 - M_Q^2 - 2(p_{l2} + p_{l1} + p_{q2})p_{q1}}{2E_{q1}} - \frac{M_R^2 - M_0^2}{2E_{l2}} \\
f_1 &= \frac{M_Q^2 - M_Q^2 - 2(p_{l2} + p_{l1} + p_{q2})p_{q1}}{2E_{q1}} - \frac{M_R^2 - 2p_{l1}p_{l2}}{2E_{l1}}
\end{align*}
\]

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the form of the equations for \( p_{0z} \) and \( p_{0y} \) are unchanged but \( p_{0z} \) is the solution of
\[
 p_{0z} n_{qz} + p_{0y} n_{qy} + p_{0z} n_{qz} = (A_x n_{qz} + A_y n_{qy} + n_{qz}) p_{0z} + (B_x n_{qz} + B_y n_{qy}) p_{0z} n_{qz} + (B_x n_{qz} + B_y n_{qy}) p_{0z} n_{qz} \\
= E_0 - \frac{M_Q^2 - M_X^2 - 2(p_{12} + p_{11} + p_{q2}) p_{q1}}{2E_q}
\]
which can be solved in a similar manner as for the squark case. It gives again two solutions, but in general only one of them is common to the squark and the gluino case, provided the sparticle masses are correct. This allows in principle to solve the ambiguity.

18.3 Reconstruction of the sparticle masses

If the decay chain starts from a gluino, the equation (C5) provides an additional constraint which can be used to determine also the sparticle masses. As every event brings in one constraint, 5 events are needed to solve for all 5 unknown sparticle masses.

The above solutions point to a (pedestrian) way to evaluate the masses. To simplify the notation, let us write as \( p_1 \) and \( p_2 \) the modulus of the \( \chi_i^1 \) momentum obtained from the two calculations. For the true values of the sparticle masses and infinite measurement precision we should have \( p_1 = p_2 \).

The measurement uncertainties will introduce a spread on these quantities which can be estimated by propagating the errors on the quark energy. From the preceding section, it is seen that the neutralino momentum depends on the quark momentum only through the expression
\[
 R = \frac{M_Q^2 - M_X^2 - 2(p_{12} + p_{11}) p_{q2}}{2E_q} \\
= \frac{M_Q^2 - M_X^2}{2E_q} - \frac{2E_q[(E_{12} + E_{11}) - |\vec{p}_{12} + \vec{p}_{11}| \cos \theta_{12}]}{2E_q} \\
= \frac{M_Q^2 - M_X^2}{2E_q} - [(E_{12} + E_{11}) - |\vec{p}_{12} + \vec{p}_{11}| \cos \theta_{12}]
\]
in the case of the decay of a squark, with energies and momentum in the laboratory frame and assuming \( m_{q2} = 0 \). The derivative is then
\[
 \frac{\partial R}{\partial E_{q2}} = -\frac{M_Q^2 - M_X^2}{2E_{q2}^2}
\]
Furthermore
\[
 \frac{\partial f_1}{\partial E_{q2}} = \frac{\partial f_2}{\partial E_{q2}} = \frac{\partial R}{\partial E_{q2}}
\]
and
\[
 \frac{\partial B_x}{\partial E_{q2}} = \frac{1}{\det} \left[ m_{1y} \frac{\partial f_2}{\partial E_{q2}} - m_{2y} \frac{\partial f_1}{\partial E_{q2}} \right] = \frac{m_{1y} - m_{2y}}{\det} \frac{\partial R}{\partial E_{q2}} = \frac{n_{1y} - n_{2y}}{\det} \frac{\partial R}{\partial E_{q2}} \\
\frac{\partial B_y}{\partial E_{q2}} = -\frac{1}{\det} \left[ m_{1x} \frac{\partial f_2}{\partial E_{q2}} - m_{2x} \frac{\partial f_1}{\partial E_{q2}} \right] = -\frac{m_{1x} - m_{2x}}{\det} \frac{\partial R}{\partial E_{q2}} = -\frac{n_{1x} - n_{2x}}{\det} \frac{\partial R}{\partial E_{q2}} \\
\frac{\partial B_z}{\partial E_{q2}} = n_{py} \frac{\partial B_x}{\partial E_{q2}} + n_{py} \frac{\partial B_y}{\partial E_{q2}} = \frac{n_{py}(n_{1y} - n_{2y}) - n_{py}(n_{1x} - n_{2x})}{\det} \frac{\partial R}{\partial E_{q2}}
\]
From equation (585) it follows that

![Mathematical equation image]

we get

![Mathematical equation image]

From equation (585) it follows that

![Mathematical equation image]

which leads to the derivative

![Mathematical equation image]

It remains to compute the derivative of \( p_0 \) which is a solution of equation (586) of the form

![Mathematical equation image]

where only \( b \) and \( c \) depend on \( E_{q_2} \)

![Mathematical equation image]

The solution to the second order equation is

![Mathematical equation image]

hence

![Mathematical equation image]

From this, finally, the error propagation on \( |\bar{p}_0| \) can be done using

![Mathematical equation image]

The above equations apply to the calculation using the quarks from squark decay. For the quark from the gluino decay, the expressions remain the same, except that

![Mathematical equation image]
and thus
\[ \frac{\partial R}{\partial E_{q_1}} = -\frac{M_G^2 - M_Q^2}{2E_{q_1}} \]  

(606)

As the errors on the two quark jets are uncorrelated, the error on the neutralino momentum entering the chi-squared is obtained from
\[ \Delta p_i^2 = \left( \frac{\partial \bar{p}_{q_2}}{\partial E_{q_2}} \right)^2 \Delta E_{q_2}^2 + \left( \frac{\partial \bar{p}_{q_1}}{\partial E_{q_1}} \right)^2 \Delta E_{q_1}^2 \]  

(607)

Note that the chi-squared written in (590) is equivalent to expressing it as
\[ \chi^2 = \sum_i \left[ \left( \frac{p_{1i} - \bar{p}_i}{\sigma_{1i}} \right)^2 + \left( \frac{p_{2i} - \bar{p}_i}{\sigma_{2i}} \right)^2 \right] \]  

(608)

where \( \bar{p}_i \) is the solution for the neutralino momentum of event \( i \). The minimization of the chi-squared with respect to \( \bar{p}_i \) leads to the linear equations
\[ \frac{\partial \chi^2}{\partial \bar{p}_i} = -2 \frac{p_{1i} - \bar{p}_i}{\sigma_{1i}} - \frac{p_{2i} - \bar{p}_i}{\sigma_{2i}} \]  

(609)

from which
\[ \bar{p}_i = \frac{1}{\sigma_{1i}^2 + \sigma_{2i}^2} \frac{\sigma_{1i}^2 p_{1i} + \sigma_{2i}^2 p_{2i}}{\sigma_{1i}^2 + \sigma_{2i}^2} = \frac{p_{1i} p_{2i}}{\sigma_{1i}^2 + \sigma_{2i}^2} \]  

(610)

showing that \( \bar{p}_i \) is the weighted mean of the two solutions. Introducing this value in the above chi-squared gives
\[ \chi^2 = \sum_i \frac{1}{\sigma_{1i}^2 + \sigma_{2i}^2} \left( \frac{p_{1i} - \bar{p}_i}{\sigma_{1i}} \right)^2 \left( p_{1i} - p_{2i} \right)^2 + \frac{1}{\sigma_{2i}^2} \left( \frac{p_{2i} - \bar{p}_i}{\sigma_{2i}} \right)^2 \left( p_{2i} - p_{1i} \right)^2 = \sum_i \left( \frac{p_{1i} - p_{2i}}{\sqrt{\sigma_{1i}^2 + \sigma_{2i}^2}} \right)^2 \]  

(611)

which is indeed the expression used in (590).

In practice, some difficulties arise when performing the numerical calculation. These are:

1. when the two leptons are nearly collinear in the \((x,y)\) plane, the determinant \( det \) can become exceedingly small and lead to numerical problems. This can be improved by choosing as \((x,y)\) the plane in which \( det \) takes its largest value.

2. even when the pulls for the individual distributions of \( p_{1i} \) and \( p_{2i} \) show an r.m.s. of 1, the r.m.s. for the pulls of \( p_{1i} - p_{2i} \) are typically around 0.9. This is due to the uncertainties which tend to shift both \( p_{1i} \) and \( p_{2i} \) into the same direction.

3. in addition to the above problem, the fact that the neutralino momentum has two solutions in each case implies that a choice has to be made among the four possible cases. This can be taken as the pair giving the smallest difference \( |\bar{p}_{q_1} - \bar{p}_{q_2}| \). When measurement uncertainties are taken into account, this leads sometimes to a wrong choice which, necessarily, gives a smaller \( p_{1i} - p_{2i} \) than for the right choice. This further reduces the r.m.s. of the pull to typically 0.8. It is possible to correct for this effect by rescaling the errors such as to produce a pull distribution with an r.m.s. of 1. However, the distribution of pulls shows also long non-Gaussian tails which make the estimate of uncertainties difficult to control.

### 18.4 Minimization of the chi-squared for mass determination

Once the chi-squared has been computed as a function of a sparticle mass, its minimum, and hence the best mass value, can be estimated by least squares minimization. Near the minimum the chi-squared can be approximated by a parabola
\[ y = ax^2 + bx + c = \sum_{a=1}^{3} \alpha_a f_a(x) \]  

(612)
linear in the parameters $a, b, c$. The chisquared is

$$\chi^2 = \sum_i \left( \frac{y_i - y}{\sigma_i} \right)^2$$  \hspace{1cm} (613)$$

where the sum runs over the points used to determine the minimum. Computing the derivatives

$$\frac{\partial \chi^2}{\partial \alpha_a} = -2 \sum_i \frac{y_i - \sum_{b=1}^{3} \alpha_b f_b(x_i)}{\sigma_i^2} f_a(x_i)$$  \hspace{1cm} (614)$$

the minimum is obtained for

$$\sum_i \frac{y_i f_a(x_i)}{\sigma_i^2} - \sum_{b=1}^{3} \alpha_b \sum_i \frac{f_b(x_i) f_a(x_i)}{\sigma_i^2} = 0$$  \hspace{1cm} (615)$$

and

$$\frac{\partial^2 \chi^2}{\partial \alpha_b \partial \alpha_a} = 2 \sum_i \frac{f_b(x_i) f_a(x_i)}{\sigma_i^2}$$  \hspace{1cm} (616)$$

The variance $V_{ab}$ is

$$V_{ab}^{-1} = H_{ab} = \frac{1}{2} \frac{\partial^2 \chi^2}{\partial \alpha_b \partial \alpha_a} = \sum_i \frac{f_b(x_i) f_a(x_i)}{\sigma_i^2}$$  \hspace{1cm} (617)$$

To simplify the notation, we can write

$$U_a = \frac{y_i f_a(x_i)}{\sigma_i^2}$$  \hspace{1cm} (618)$$

so that the minimization equation reads

$$U_a - \sum_{b=1}^{3} \alpha_b H_{ab} = 0$$  \hspace{1cm} (619)$$

which has the solution

$$\alpha_a = U_b H_{ab}^{-1}$$  \hspace{1cm} (620)$$

For the above parabola, the $H_{ab}$ matrix and $U_a$ are

$$H_{11} = \sum_i \frac{x_i^4}{\sigma_i^2} , \quad H_{22} = \sum_i \frac{x_i^2}{\sigma_i^2} , \quad H_{33} = \sum_i \frac{1}{\sigma_i^2}$$

$$H_{12} = H_{21} = \sum_i \frac{x_i^3}{\sigma_i^2} , \quad H_{13} = H_{31} = \sum_i \frac{x_i^2}{\sigma_i^2} , \quad H_{23} = H_{32} = \sum_i \frac{x_i}{\sigma_i^2}$$

$$U_1 = \sum_i \frac{y_i x_i^2}{\sigma_i^2} , \quad U_2 = \sum_i \frac{y_i x_i}{\sigma_i^2} , \quad U_3 = \sum_i \frac{y_i}{\sigma_i^2}$$  \hspace{1cm} (621)$$

The inverse of the matrix is given by the minors as $H_{ab}^{-1} = M_{ab}/\text{det}$, where the determinant is

$$\text{det} = H_{11} H_{22} H_{33} + 2H_{12} H_{13} H_{23} - H_{13}^2 H_{22} - H_{12}^2 H_{33} - H_{23}^2 H_{11}$$  \hspace{1cm} (622)$$

and the minors

$$M_{11} = H_{22} H_{33} - H_{23}^2 , \quad M_{22} = H_{11} H_{33} - H_{13}^2 , \quad M_{33} = H_{11} H_{22} - H_{12}^2$$

$$M_{12} = H_{13} H_{23} - H_{12} H_{33} , \quad M_{13} = H_{12} H_{23} - H_{13} H_{22} , \quad M_{23} = H_{12} H_{13} - H_{23} H_{11}$$  \hspace{1cm} (623)$$

From these expressions, the solutions for the parameters and their error matrix can be computed. The value of the mass at the minimum is then given by

$$x_m = -\frac{b}{2a}$$  \hspace{1cm} (624)$$
and the errors can be propagated using

\[ \delta x_m = -x_m \left( \frac{\delta a}{a} - \frac{\delta b}{b} \right) \]  

which gives

\[
\Delta x_m^2 = x_m^2 \left( \frac{M_{11}}{a^2} + \frac{M_{22}}{b^2} - 2 \frac{M_{12}}{ab} \right) \frac{1}{\det} 
= \frac{1}{4a^2} \left( \frac{b^2}{a^2} M_{11} + M_{22} - 2 \frac{b}{a} M_{12} \right) \frac{1}{\det}
\]  

(626)

References


[13] F. Heinemann, *The discovery potential of the $\tilde{\tau}_2^0$ in mSUGRA in the $\tau$-channel at high $\tan\beta$ at the LHC*, Diploma thesis, ETH Zurich, Switzerland, March 2004.


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