Supergravities in Diverse Dimensions

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1 Introduction

Supergravity was initially envisaged to be an elementary field theory which will be free of UV divergences and thus unite all the forces with the gravity. Nowadays it's viewed as an effective theory of a more fundamental theory, the only candidate for which is superstring theory, or rather M-theory, in the framework of which all the supergravity and superstring theories are connected to each other.

In this paper we are going to discuss the general structure of the massless fields which are in the same multiplet as the graviton, called supergravity multiplet. To do that we'll introduce spinors in general dimensions, how they work, what possibilities one has, which will give us some ground to continue to supersymmetry algebras discussion in section two, which will include the discussion of the R-symmetry groups in the various dimensions. Then we'll discuss the supergravity multiplets from the group-theoretical point of view and classify all possible supergravity theories. Doing that we'll go to specific cases and discuss in detail the structure of $d = 11$ theory, and also outline the main properties of the three $d = 10$ theories. To help the reader to understand $d < 10$ theories in his/her future, we then introduce non-linear sigma models, and discuss them in the case of IIB theory.

We conclude with a very brief discussion of non Lorentz scalar "central" charges (it's an abuse of notation to call them central, since that would seem to imply that they commute with Lorentz, which obviously won't be the case if they are not scalars), and a couple of references that the reader may find useful for a more powerful immersion into the field of supergravity.

2 Spinors in diverse dimensions

Let us start our discussion of spinors with a brief introduction of Clifford algebras. In Appendix A we define Clifford algebras more rigorously, which will hopefully make the reader more comfortable with them (especially that we all know Clifford algebra $\mathcal{C}_1$ in pseudo-Euclidean space - it's just the same as algebra of complex numbers).

The way the Clifford algebra is usually introduced in physics is - *think of it in terms of gamma matrices.* That will be almost good enough for our discussion. To bring the way of thinking of Clifford algebras in terms of gamma matrices closer to what Clifford algebras are, let us introduce the following matrices. Let's for simplicity work in a d-dimensional Euclidean space $\mathbb{R}^d$, $d = 2n$ being even, and introduce main ingredients of Clifford algebra $\mathcal{C}_d$ in $\mathbb{R}^d$. Let us also consider $2^n \times 2^n$-dimensional matrices in M_{2^n} (note that the dimension of M_{2^n} is $(2^n)^2 = 2^d$!).

1. Let $\mathbb{1}$ be the identity in M_{2^n}.

2. Let $\gamma^a,\ a = 1, \ldots, d$ be our gamma matrices, which satisfy $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$, and are hermitian. Clearly this implies that $(\gamma^a)^2 = 1$.

3. Let $\gamma^{a_1 a_2 \cdots a_k} = \frac{1}{k!}\gamma^{[a_1}\gamma^{a_2}\cdots\gamma^{a_k]}$.

In fact these matrices can be thought of as a basis of Clifford algebra representation in terms of matrices. I.e. any combination of gamma matrices (sum or product, since this is an algebra) can be written in a form $b\mathbb{1} + b_a\gamma^a + \cdots + b_{a_1a_2\cdots a_d}\gamma^{a_1a_2\cdots a_d}$, where $b, b_a, \ldots, b_{a_1a_2\cdots a_d}$ are some real numbers.

Doing simple counting we see that the number of basis elements in Clifford algebra, hence the dimension is $\binom{d}{0} + \binom{d}{1} + \cdots + \binom{d}{d} = (1+1)^d = 2^d (= dim(M_{2^n}))$. For d odd, it turns out that the Clifford algebra $\mathcal{C}_d$ is the direct product of two subalgebras, both isomorphic to $\mathcal{C}_{d-1}$.

Another interesting fact is that matrix representation of Clifford algebra, forms also a representation of the rotation group, which is actually a fundamental representation of the rotation group, and is called *spinor representation*! But the map (homomorphism) from matrix representation of Clifford algebra to spinor representation of the rotation group is not $1-1$ (an isomorphism), rather it's $2-1$, i.e. for every element in the rotation group there are two elements in the Clifford algebra which map to it. The elements of the vector space on which the matrices of spinor representation of Clifford algebra act are called *Dirac spinors* and naturally have dimension $2^{\lfloor \frac{d}{2} \rfloor}$ (recall that for d odd, $\mathcal{C}_n = \mathcal{C}_{n-1} \oplus \mathcal{C}_{n-1}$, it reduces to the even case).

Let's now consider more specifically all of the above for spinor representations of $SO(t,s)$, where t and s denote the number of plus and minus signs correspondingly in the signature of the metric η_{ab}, and let $d = t + s$ denote the dimension of our space. So for instance a d-dimensional Minkowski space would correspond to $SO(1, d-1)$. Condition 2 for gamma matrices then becomes:

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}, \tag{1}$$

where $a = 1, \ldots, d$, and γ^a are hermitian for $a = 1, \ldots, t$ and antihermitian for $a = t+1, \ldots, d$. As before the dimension of the gamma matrices is $2^{\lfloor \frac{d}{2} \rfloor} \times 2^{\lfloor \frac{d}{2} \rfloor}$. An explicit representation of gamma matrices can be constructed as tensor products of Pauli matrices and the 2×2 unit matrix and is given in [1].

Consider now the matrix

$$\gamma_5 = (-1)^{\frac{1}{4}(s-t)}\gamma^1\gamma^2\cdots\gamma^d \propto \gamma^{a_1a_2\cdots a_d} \tag{2}$$

(the volume form). It's should be clear that $(\gamma_5)^2 = \mathbb{1}$, $\{\gamma_5, \gamma^a\} = 0$ and that it's hermitian, by just using (1). Using γ_5 one can define *Weyl spinors* with positive (or negative) chirality by their γ_5 eigenvalues:

$$\gamma_5\psi_+ = \psi_+ \qquad (or\ \ \gamma_5\psi_- = -\psi_-) \tag{3}$$

In odd dimensions γ_5 is proportional to the identity, so it doesn't help us to reduce the spinors (this is in perfect accordance with Dynkin diagram for SO(2n), which has two branching dots on one end which correspond to two inequivalent spinor representations, as well as with Dynkin diagram for SO(2n+1), which has a short root on one end, corresponding to one spinor representation).

But this is not all of the reducing power we have! It turns out that Clifford algebra has a fundamental anti-automorphism under which $\gamma^a \to -\gamma^a$ for dimensions $s - t \equiv 0, 1, 2 \bmod 8$ (the number 8 comes out here, and will later too, because it can be shown that representations of Clifford algebras are periodic with period 8). In a sense this means that $\exists B_-$, such that:

$$-(\gamma^a)^* = B_-\gamma^a B_-^{-1}, \qquad with\ \ B_-^*B_- = \mathbb{1} \tag{4}$$

The matrix B_- naturally defines a *charge conjugation* of a spinor:

$$\psi^c = B_-^{-1}\psi^* \tag{5}$$

It can then be used to impose *Majorana condition*, which is a reality constraint:

$$\psi^c = \psi, \tag{6}$$

which identifies the charge conjugated spinor with itself.

Another lucky fact for us is that Clifford algebra also has another fundamental anti-automorphism under which $\gamma^a \to -\gamma^a$ for dimensions $s - t \equiv 2, 3, 4 \bmod 8$ (there are just these two, in case you're wondering about more). I.e. $\exists B_+$, such that:

$$(\gamma^a)^* = B_+ \gamma^a B_+^{-1}, \qquad \text{with } B_+^* B_+ = \mathbb{1}, \tag{7}$$

charge conjugation being defined by:

$$\psi^c = B_+^{-1} \psi^* \tag{8}$$

and the *pseudo-Majorana condition* being:

$$\psi^c = \psi \tag{9}$$

Recall that the reason the above defined charged conjugation is called so, is because the charge conjugated spinor satisfied Dirac equations with opposite electric charge (in case of B_+ conjugation the mass term changes it's sign too). Note also that charge conjugation can be defined in terms of Dirac conjugate:

$$\begin{aligned} &\bar{\psi} = \psi^\dagger A, \qquad A = \gamma^1 \gamma^2 \cdots \gamma^t \\ &\psi^c = C_+ \bar{\psi}^T \qquad \textit{or} \qquad \psi^c = C_- \bar{\psi}^T \qquad \text{with } C_\pm = B_\pm^{-1} (A^{-1})^T, \end{aligned} \tag{10}$$

where $C_\pm$ are unitary matrices. For more details we refer the reader to [1].

Taking into account that in general, when a spinor ψ has a chirality + (-), the charge conjugated spinor ψ^c has a chirality $(-1)^{\frac{1}{2}(s-t)}$ $(-(-1)^{\frac{1}{2}(s-t)})$, we can see that (pseudo) Majorana-Weyl spinors are possible only if ψ has the same chirality as ψ^c, i.e. when $s - t \equiv 0 \bmod 8$. In Minkowski space where $t = 1$ and $s = d - 1$, (pseudo)Majorana-Weyl spinors exist if $d \equiv 2 \bmod 8$.

There is one more final type of spinors we'll be dealing with. In cases when we cannot impose (pseudo)Majorana condition because $(\psi^c)^c = -\psi$ and we have Dirac spinors (or Weyl spinors in even dimensions). Then we could introduce a fermion doublet consisting of even number of fermions ψ^i, $i = 1, \ldots, 2d$, and impose symplectic condition:

$$\psi^i = \Omega^{ij} (\psi^j)^c, \tag{11}$$

where Ω^{ij} is a constant antisymmetric matrix. Such spinors are called symplectic (pseudo) Majorana spinors. They clearly don't reduce the dimension of the spinors, since we double the number of fermions, and then impose a condition that halves the number back, but they can be more convenient than Dirac spinors, if the theory has a symplectic symmetry.

Let us summarize our results for a particular case of d-dimensional Minkowski space in a table.

3 Superalgebras

We restrict ourselves in this section to flat Minkowski spacetime of dimension d. The generators of the super-Poincaré algebra consist of supercharges $Q^{\alpha i}$ (we'll often omit the spinor index α if it'll be unambiguous what we mean), transforming as spinors under the Lorentz group, generators of translations P_a, generators of the Lorentz group M_{ab} and possibly additional generators (*central charges*) that commute with the supercharges, which we'll ignore for now.

So we have supercharges that commute with translations and are spinors under Lorentz transformations:

$$[P_a, Q^i] = 0, \qquad [M_{ab}, Q^i] = \frac{1}{2} \gamma_{ab} Q^i \tag{12}$$

and the form of these relations is independent on what type of spinor are the supercharges. But the anti-commutation relation between the supercharges do depend on the type of the spinor, thus the dimension of the space. Let's recall though that for a single supercharge they are:

$$\{Q_\alpha, \bar{Q}_\beta\} = (\gamma^a)_{\alpha\beta} P_a \tag{13}$$

Type	d mod 8	n
W	2, 4, 6, 8	$2^{\lfloor \frac{d}{2} \rfloor}$
M	2, 3, 4	$2^{\lfloor \frac{d}{2} \rfloor}$
pM	2, 8, 9	$2^{\lfloor \frac{d}{2} \rfloor}$
MW	2	$2^{\lfloor \frac{d}{2} \rfloor -1}$
pMW	2	$2^{\lfloor \frac{d}{2} \rfloor -1}$
sM	7	$2^{\lfloor \frac{d}{2} \rfloor +1}$
spM	5	$2^{\lfloor \frac{d}{2} \rfloor +1}$
sMW	6	$2^{\lfloor \frac{d}{2} \rfloor}$

Table 1: Possible types of spinors in d-dimensional Minkowski space ($t = 1$, $s = d - 1$). W, M, pM, MW, pMW, sM, spM, sMW denote Weyl, Majorana, pseudo Majorana, Majorana-Weyl, pseudo Majorana-Weyl, symplectic Majorana, symplectic pseudo Majorana, symplectic Majorana-Weyl spinors respectively. Second column entries list the possible spacetime dimension congruency classes for each type of spinors, and the last column n is the number of independent real components of the spinor.

Let's defer the discussion of anti-commutation relations for a moment and discuss an important concept of the *automorphism group* H_R. To do that we consider not 1 but N spinor charges which transform reducibly under the Lorentz group and comprise N irreducible spinors. For Weyl we can consider N_+ positive chirality and N_- negative chirality spinors. In all these case there exist a group H_R acting on the supercharges, which commutes with the Lorentz group and leaves the super-Poincaré algebra invariant. This group is also referred to as the *R-symmetry group*, and is formally defined as *the largest automorphism group of the supersymmetry algebra that commutes with the Lorentz group.* As an example of H_R consider case of 1 supercharge, $N = 1$, then clearly (13) is invariant under $U(1)$ transformations:

$$Q_\alpha \to e^{i\theta} Q_\alpha \implies \{Q_\alpha, \bar{Q}_\beta\} \to e^{i\theta} e^{-i\theta} \{Q_\alpha, \bar{Q}_\beta\} = \{Q_\alpha, \bar{Q}_\beta\} \tag{14}$$

We will encounter more complicated examples of H_R when we start discussing possible superalgebras in different dimensions.

Another important concept we have to introduce before we start enumerating dimensions and algebras is one which we skipped in the beginning - the concept of *central charges* Z^{ij}. It turns out that one can have such charges which commute with Lorentz group, and supercharges, but have R-charge, i.e. transform under the automorphism group H_R. They will also appear in general in the supercharge anticommutation relations. Actually we are cheating here a little bit, since one can have a more general form of central charges, which transform under Lorentz group, the discussion of which we'll skip for now.

So combining all we said, the invariant (under change of dimension) part of the super-Poincaré algebra is:

$$\begin{aligned} &[M_{ab}, Q^i] = \frac{1}{2}\gamma_{ab} Q^i, \qquad [T^A, Q^i] = (t^A)^i{}_j Q^j, \\ &[T^A, Z^{ij}] = (t^A)^i{}_k Z^{kj} + (t^A)^j{}_k Z^{ik}, \qquad [T^A, T^B] = f^{AB}{}_C T^C, \end{aligned} \tag{15}$$

where t^A are representation matrices and $f^{AB}{}_C$ are the structure constants of the Lie algebra of the automorphism group H_R.

The possible automorphism groups and forms of the anticommutator $\{Q, Q\}$ are listed in the Appendix B.

4 Supergravity multiplets

Particles that appear in supergravity theories belong to irreducible representations of super-Poincaré algebras. We are interested here in massless states only.

As far as it's known today (June 10, 2003) there are no consistent *interacting* theories of particles with helicity ≥ 2, and the only known consistent interacting theory of particle with helicity 2 is general relativity, i.e. a theory where helicity 2 particle couples to the energy-momentum tensor (for details see [2]). This puts a limitation on the possible spacetime dimension a supergravity theory can have.

Consider spacetime $d = 12$. And let's suppose that we have just one supercharge, i.e. $N = 1$. The supercharge is then Weyl or Majorana (but not both!), depending on the conditions one imposes. The number of independent real components it has is $2^{\lfloor \frac{12}{2} \rfloor} = 64$ (using Table 1). First we note that central charges vanish, $Z^{ij} = 0$, for massless states [3]. Then going to light-cone coordinates the supersymmetry algebra becomes just creation and annihilation operators algebra with half of the operators represented trivially (assuming the number of real components of the supercharges is even, which is the case for $D \geq 4$, for discussion of $D = 2$ and $D = 3$ cases see [4]). Then we're left with 32 creation and annihilation operators, thus 16 creation operators, each of which will increase the helicity of the state it acts on by 1/2. Thus we'll clearly have to have helicity 4 states, acting on helicity -4 state with 16 creation operators each of which carries helicity 1/2, will give helicity $-4 + \frac{16}{2} = 4$ state. It should be clear that increasing the spacetime dimension d or the number of supercharges would worsen this, thus this puts a limit on the spacetime dimension to be 11! Just as a check let's see that $N = 1$ in 11 dimensions doesn't have helicity > 2 particles. The supercharges in $d = 11$ are Majorana, thus the number of real components is $2^{\lfloor \frac{11}{2} \rfloor} = 32$. Halving for massless states, we get 8 creation operators and 8 annihilation operators, which will allow you going from helicity -2 states to helicity 2 states, thus we will stay in the consistent regime!

Having noticed the dimensional constraint there is limited number of possibilities we have to consider. Let's concentrate on the $d = 11$ case and introduce the concepts that are needed to classify possible supergravity multiplets. The main concept we need is the concept of the *little group*, which is the subgroup of Lorentz transformations that leaves the momentum invariant in the light-cone coordinates. It should be clear that for $SO(1, d-1)$ the little group is $SO(d-2)$ (we are just chopping off time and one space space coordinate, whose linear combinations, $x^0 + x^1$ and $x^0 - x^1$, are fixed in light-cone coordinates). Recalling that in $d = 11$ there are 16 independent spinor components, which would thus transform in the spinor representation of $SO(16)$, we want to see how would that spinor representation decompose under the embedding of the little group $SO(9)$, to figure the helicity content of the bosonic and fermionic states. This is similar to 4 dimensions, where we consider decomposition under the $SU(2)$ subgroup to determine what the helicities of the particles are.

It then turns out that out of two spinor representations of $SO(16)$ (two Weil spinors), one branches into:

$$\mathbf{128} \to \mathbf{44} + \mathbf{84}, \tag{16}$$

while the second one transforms irreducibly:

$$\mathbf{128}(SO(16)) \to \mathbf{128}(SO(9)). \tag{17}$$

This may look quite strange but it's actually very similar to $N = 1$ supergravity in 4 dimensions. The $\mathbf{44}$ is the veilbein ${e_\mu}^a$ (or the graviton), the $\mathbf{128}$ is the Rarita-Schwinger representing a spin $\frac{3}{2}$ particle ψ_μ (gravitino), but also there is an additional to 4 dimensional case particle - a bosonic 3-form $B_{\mu\nu\rho}$ which has 84 components, which is an object similar to A^μ in electromagnetism.

Some remarks we would like to add here are the following. First it's easy to see that bosonic and fermionic degrees of freedom are the same. This is of course a general fact for supersymmetry. The second remark is that the way we got the graviton and gravitino, i.e. by finding the irreducible representations under the little group embedding, makes them automatically satisfy the so called γ-traceless condition, which arises if one considers starting with vector and spinor representations and tensoring them together to get spin $\frac{3}{2}$ particle, but since tensor product is always reducible one also gets a residual spin $\frac{1}{2}$ particle to get rid of which γ-traceless condition is imposed:

$$\gamma^\mu \psi_\mu = 0, \tag{18}$$

and similarly for the veilbein (for more discussion see [2]).

One can do this procedure for all dimensions and all the possibilities including not only supergravity multiplets (those containing graviton and thus gravitino), but also matter and Yang-Mills multiplets are listed in [4]. We list here only the possible supergravity multiplets in Table 2.

The numbers of the physical degrees of freedom for each field are:

$$
\begin{aligned}
e_\mu{}^a \quad &: \quad \frac{1}{2}(d-2)(d-1)-1, \qquad \text{symmetric traceless tensor} \\
B_{\mu\nu\rho\sigma} \quad &: \quad \binom{d-2}{4} = \frac{1}{24}(d-2)(d-3)(d-4)(d-5), \qquad \text{4-form} \\
B_{\mu\nu\rho} \quad &: \quad \binom{d-2}{3} = \frac{1}{6}(d-2)(d-3)(d-4), \qquad \text{3-form} \\
B_{\mu\nu} \quad &: \quad \binom{d-2}{2} = \frac{1}{2}(d-2)(d-3), \qquad \text{2-form} \\
B_\mu \quad &: \quad \binom{d-2}{1} = d-2, \qquad \text{1-form/vector} \\
\phi \quad &: \quad 1, \qquad \text{real scalar} \\
\psi_\mu \quad &: \quad \frac{1}{2}(d-2-1)2^{\lfloor \frac{d}{2} \rfloor}, \qquad \text{Rarita-Schwinger spin } \tfrac{3}{2} \text{ spinor} \\
\lambda \quad &: \quad \frac{1}{2}2^{\lfloor \frac{d}{2} \rfloor}, \qquad \text{spin } \tfrac{1}{2} \text{ spinor},
\end{aligned} \tag{19}
$$

where the -1 for graviton and gravitino comes from the above mentioned γ-traceless condition.

5 Supergravities in higher dimensions

In this section we will discuss supergravities in $d = 11$ and $d = 10$ in some detail, and at the end briefly discuss some lower dimensional theories. We assume from the reader some knowledge of the $d = 4$ minimal supergravity theory, a short review of which is in [1], and a longer one in [3].

5.1 $d = 11$ supergravity

The field content of this theory as we discussed earlier contains the graviton $e_\mu{}^a$, a gravitino ψ_μ and a 3-form $B_{\mu\nu\rho}$. The Lagrangian has the following form [5]:

$$
\begin{aligned}
\mathcal{L} \quad = \quad & -\frac{1}{4}eR - \frac{1}{2}ie\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{1}{48}eF_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} + \frac{1}{96}e(\bar{\psi}_\mu\gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\bar{\psi}^\alpha\gamma^{\beta\gamma}\psi^\delta)F_{\alpha\beta\gamma\delta} \\
& + \frac{2}{144^2}\epsilon^{\alpha_1\cdots\alpha_4\beta_1\cdots\beta_4\mu\nu\rho}F_{\alpha_1\cdots\alpha_4}F_{\beta_1\cdots\beta_4}B_{\mu\nu\rho} + \ \text{(4-fermi terms)},
\end{aligned} \tag{20}
$$

where $F_{\mu\nu\rho\sigma} = 4\partial_{[\mu}B_{\nu\rho\sigma]}$ is the field strength of the 3-form, covariant derivative of gravitino is a spin connection covariant derivative, i.e. $D_\nu\psi_\rho = (\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\psi_\rho$. This Lagrangian is invariant under the local supertransformation

$$
\begin{aligned}
& \delta e_\mu{}^a = -i\bar{\epsilon}\gamma^a\psi_\mu, \qquad \delta B_{\mu\nu\rho} = \frac{3}{2}\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]}, \\
& \delta\psi_\mu = D_\mu\epsilon + \frac{i}{144}(\gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\gamma^{\beta\gamma\delta}\delta^\alpha_\mu)\epsilon F_{\alpha\beta\gamma\delta} + \ \text{(3-fermi terms)},
\end{aligned} \tag{21}
$$

in addition to general coordinate and local Lorentz transformations. It is also invariant up to surface terms under the local gauge transformations of the 3-form $B \to B + d\Lambda$, where Λ is a 2-form.

As usual supersymmetry algebra closes *on shell* [5]. We emphasize this, because since we don't include auxiliary fields the supersymmetry algebra doesn't close off-shell. In the present status of theory it is possible to actually include auxiliary fields and close the algebra off-shell only for $d = 4$, $N = 1$.

d	N or (N_+, N_-)	dimensional reduction?	fields	Q_{irr}	n
11	1	—	$e_\mu{}^a$, ψ_μ, $B_{\mu\nu\rho}$	32	128
10	(1,1)	$N=1$	$e_\mu{}^a$, $\psi_{+\mu}$, $\psi_{-\mu}$, $B_{\mu\nu\rho}$, $B_{\mu\nu}$, B_μ, λ_+, λ_-, ϕ	32	128
	(2,0)	—	$e_\mu{}^a$, $2\psi_{+\mu}$, $B^{(+)}_{\mu\nu\rho\sigma}$, $2B_{\mu\nu}$, $2\lambda_-$, 2ϕ	32	128
	(1,0)	—	$e_\mu{}^a$, $\psi_{+\mu}$, $B_{\mu\nu}$, λ_-, ϕ	16	64
9	2	$(1,1)$ and $(2,0)$	$e_\mu{}^a$, $2\psi_\mu$, $B_{\mu\nu\rho}$, $2B_{\mu\nu}$, $3B_\mu$, 4λ, 3ϕ	32	128
	1	$(1,0)$	$e_\mu{}^a$, ψ_μ, $B_{\mu\nu}$, B_μ, λ, ϕ	16	56
8	2	$N=2$	$e_\mu{}^a$, $2\psi_\mu$, $B_{\mu\nu\rho}$, $3B_{\mu\nu}$, $6B_\mu$, 6λ, 7ϕ	32	128
	1	$N=1$	$e_\mu{}^a$, ψ_μ, $B_{\mu\nu}$, $2B_\mu$, λ, ϕ	16	48
7	4	$N=2$	$e_\mu{}^a$, $4\psi_\mu$, $5B_{\mu\nu}$, $10B_\mu$, 16λ, 14ϕ	32	128
	2	$N=1$	$e_\mu{}^a$, $2\psi_\mu$, $B_{\mu\nu}$, $3B_\mu$, 2λ, ϕ	16	40
6	(4,4)	$N=4$	$e_\mu{}^a$, $4\psi_{+\mu}$, $4\psi_{-\mu}$, $5B_{\mu\nu}$, $16B_\mu$, $20\lambda_+$, $20\lambda_-$, 25ϕ	32	128
	(4, 2)	—	$e_\mu{}^a$, $4\psi_{+\mu}$, $2\psi_{-\mu}$, $5B^{(+)}_{\mu\nu}$, $B^{(-)}_{\mu\nu}$, $8B_\mu$, $10\lambda_+$, $4\lambda_-$, 5ϕ	24	64
	(2, 2)	$N=2$	$e_\mu{}^a$, $2\psi_{+\mu}$, $2\psi_{-\mu}$, $B_{\mu\nu}$, $4B_\mu$, $2\lambda_+$, $2\lambda_-$, ϕ	16	32
	(4, 0)	—	$e_\mu{}^a$, $4\psi_{+\mu}$, $5B^{(+)}_{\mu\nu}$	16	24
	(2, 0)	—	$e_\mu{}^a$, $2\psi_{+\mu}$, $B^{(+)}_{\mu\nu}$	8	12
5	8	$(4,4)$	$e_\mu{}^a$, $8\psi_\mu$, $27B_\mu$, 48λ, 42ϕ	32	128
	6	$(4,2)$	$e_\mu{}^a$, $6\psi_\mu$, $15B_\mu$, 20λ, 14ϕ	24	64
	4	$(2,2)$ and $(4,0)$	$e_\mu{}^a$, $4\psi_\mu$, $6B_\mu$, 4λ, ϕ	16	24
	2	$(2,0)$	$e_\mu{}^a$, $2\psi_\mu$, B_μ	8	8
4	8	$N=8$	$e_\mu{}^a$, $8\psi_\mu$, $28B_\mu$, 56λ, 70ϕ	32	128
	6	$N=6$	$e_\mu{}^a$, $6\psi_\mu$, $16B_\mu$, 26λ, 30ϕ	24	64
	5	—	$e_\mu{}^a$, $5\psi_\mu$, $10B_\mu$, 11λ, 10ϕ	20	32
	4	$N=4$	$e_\mu{}^a$, $4\psi_\mu$, $6B_\mu$, 4λ, 2ϕ	16	16
	3	—	$e_\mu{}^a$, $3\psi_\mu$, $3B_\mu$, λ	12	8
	2	$N=2$	$e_\mu{}^a$, $2\psi_\mu$, B_μ	8	4
	1	—	$e_\mu{}^a$, ψ_μ	4	2

Table 2: Supergravity multiplets. d is the spacetime dimension, N (or (N_+, N_-)) the number of supercharges (of positive/negative chirality), third column indicates whether the theory comes from the dimensional reduction of a theory one dimension higher, fourth column is the field content of the theory, Q_{irr} is the total number of real irreducible components of the supercharges and n denotes the number of real bosonic = fermionic physical degrees of freedom. The subscripts $\pm$ on spinor fields denote chiralities. The superscripts $(\pm)$ on k-forms mean that they are (anti-)self-dual.

5.2 $d = 10$ supergravities

In 10 dimensions there are 3 supergravity theories - $(1, 1)$ supergravity [11], which is a massless sector of the type IIA superstring theory, $(2, 0)$ supergravity [6, 7, 8], which is a chiral (left-right asymmetric) and is a massless sector of the type IIB superstring theory, and finally $(1, 0)$ supergravity [9, 10], which is also chiral and is a massless sector of the type I superstring theory.

Let's start from type IIA theory. As indicated in Table 2 it's a dimensional reduction of the $d = 11$ theory. To see how it happens consider as an example the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{22}$$

where $\mu, \nu = 0, \ldots, 10$. Now consider parameterizing the 10th dimension as follows

$$ds^2 = g_{ij} dx^i dx^j + e^\phi (dx^{10} + V_i dx^i)(dx^{10} + V_j dx^j), \tag{23}$$

where $i, j = 0, \ldots, 9$. If we now consider the x^{10}-independent terms only, we'll get a metric of 1 dimension less, a vector field, and a scalar field. One can see more features of this, specifically that V_μ is gauge invariant. By making coordinate transformations $x^{10} \to x^{10} - \xi(x)$ and $x^i \to x^i$, where $\xi(x)$ is an arbitrary function of the first 10 spacetime coordinates, we get gauge invariance for the vector field:

$$V_\mu(x) \to V_\mu(x) + \partial_\mu \xi(x). \tag{24}$$

Similarly the other fields can be reduced under dimensional reduction, and the gravitino will decompose into two gravitinos of the lower dimension (two because we have Weyl decomposition into chiral fields), and the extra degrees of freedom from the last dimension will realize in terms of two chiral spin $\frac{1}{2}$ particles λ_+ and λ_-. Finally the 3-form will decompose into a 3-form of lower dimension and the 10th dimension degrees of freedom will become a 2-form - $B_{ij} \equiv B_{10ij}$ (these are all the extra degrees of freedom a 3-form has since it's totally antisymmetric).

We include the Lagrangian for this theory, without comments, to demonstrate how bad things can look like in supergravity theories:

$$\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{4} e^{(10)} R(e, \omega) - \frac{1}{16} e\phi^{9/4} F_{\mu\nu} F^{\mu\nu} + \frac{9}{32} e(\partial_\mu \phi/\phi) \partial^\mu \phi/\phi - \frac{1}{48} e\phi^{3/4} F'_{\mu\nu\lambda\rho} F'^{\mu\nu\lambda\rho} \\
& + \frac{1}{12} e\phi^{-3/2} F_{\mu\nu\lambda} F^{\mu\nu\lambda} + [3/2(12)^3] \epsilon^{\mu_1 \cdots \mu_{10}} F_{\mu_1 \cdots \mu_4} F_{\mu_5 \cdots \mu_8} B_{\mu_9 \mu_{10}} - \frac{1}{2} ie\bar{\psi}_\mu \gamma^{\mu\nu\lambda} D_\nu \left[\frac{1}{2}(\omega + \hat{\omega})\right] \psi_\lambda \\
& + \frac{1}{2} ie\bar{\lambda} \gamma^\mu D_\mu \left[\frac{1}{2}(\omega + \hat{\omega})\right] \lambda - \frac{3}{8} i\sqrt{2} e\bar{\psi}_\mu [(\partial\!\!\!/\phi + \hat{D\!\!\!/}\phi)/2\phi] \gamma^\mu \gamma_\| \lambda - \frac{1}{32} ie\phi^{9/8} (\bar{\psi}_\mu \gamma^{\mu\nu\lambda\rho} \gamma_\| \psi_\lambda \\
& - 2\bar{\psi}^\nu \gamma_\| \psi^\rho + \frac{3}{2} \sqrt{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} \lambda - 3\sqrt{2} \bar{\psi}^\nu \gamma^\rho \lambda - \frac{5}{4} \bar{\lambda} \gamma^{\nu\rho} \gamma_\| \lambda)(F + \hat{F})_{\nu\rho} \\
& + \frac{1}{192} e\phi^{3/8} (\bar{\psi}_\sigma \gamma^{\mu\nu\lambda\rho\sigma\kappa} \psi_\kappa + 12 \bar{\psi}^\mu \gamma^{\nu\lambda} \psi^\rho + \frac{3}{4} \bar{\lambda} \gamma^{\mu\nu\lambda\rho} \lambda - \frac{1}{3} \sqrt{2} \bar{\psi}_\sigma \gamma^{\mu\nu\lambda\rho\sigma} \gamma_\| \lambda \\
& + 2\sqrt{2} \bar{\psi}^\mu \gamma^{\nu\lambda\rho} \gamma_\| \lambda)(F + \hat{F})_{\mu\nu\lambda\rho} + \frac{1}{48} e\phi^{-3/4} (\bar{\psi}_\rho \gamma^{\rho\mu\nu\lambda\sigma} \gamma_\| \psi_\sigma + 6 \bar{\psi}^\mu \gamma^\nu \gamma_\| \psi^\lambda \\
& - \sqrt{2} \bar{\psi}_\rho \gamma^{\mu\nu\lambda} \gamma^\rho \lambda)(F + \hat{F})_{\mu\nu\lambda} + \mathcal{L}_4,
\end{aligned} \tag{25}$$

with the 4-fermi terms given by:

$$\begin{aligned}
\mathcal{L}_4 \;=\; & -\frac{1}{64} e \Big[\frac{1}{108} \sqrt{2} \bar{\lambda} \gamma^{\mu\nu\lambda} \lambda (\bar{\psi}^\rho \gamma_{\mu\nu\lambda\rho} \gamma_\| \lambda - 37 \bar{\psi}_\mu \gamma_{\nu\lambda} \gamma_\| \lambda) + \frac{1}{3} \sqrt{2} \bar{\psi}_\rho \gamma^{\mu\nu\lambda\rho} \gamma_\| \lambda (2\bar{\psi}_\mu \gamma_\nu \psi_\lambda + \frac{1}{3} \sqrt{2} \bar{\psi}_\mu \gamma \nu \lambda \gamma_\| \lambda) \\
& + \frac{4}{9} \bar{\lambda} \gamma^{\mu\nu} \gamma_\| \lambda (2\bar{\psi}^\lambda \gamma_{\mu\nu\lambda\rho} \gamma_\| \psi^\rho + 2\bar{\lambda} \gamma_{\mu\nu} \gamma_\| \lambda - \frac{1}{2} \sqrt{2} \bar{\psi}^\lambda \gamma_{\mu\nu\lambda} \lambda + \frac{44}{3} \sqrt{2} \bar{\psi}_\mu \gamma_\nu \lambda) \\
& + \frac{16}{3} \sqrt{2} \bar{\psi}^\mu \gamma^\nu \lambda (\bar{\psi}^\lambda \gamma_{\mu\nu\lambda\rho} \gamma_\| \psi^\rho - \frac{1}{3} \sqrt{2} \bar{\psi}^\lambda \gamma_{\mu\nu\lambda} \lambda - \frac{4}{3} \sqrt{2} \bar{\psi}_\nu \gamma_\mu \lambda) - \frac{16}{3} \sqrt{2} \bar{\psi}^\mu \gamma_\| \psi^\nu \bar{\psi}^\lambda \gamma_{\mu\nu\lambda} \lambda \\
& + \frac{2}{3} \bar{\psi}_\mu \gamma^{\mu\nu} \gamma_\| \lambda (24 \bar{\psi}_\nu \gamma_\| \lambda - 4\sqrt{2} \bar{\psi}^\lambda \gamma_\lambda \psi_\nu - \frac{1}{3} \bar{\psi}^\lambda \gamma_{\lambda\nu} \gamma_\| \lambda) + \frac{128}{9} (\bar{\psi}^\mu \gamma_\mu \lambda)^2
\end{aligned}$$

$$+\frac{1}{3}\sqrt{2}\bar{\psi}^{\mu}\gamma^{\nu\lambda}\gamma_{||}\lambda(\bar{\psi}^{\rho}\gamma_{\rho\mu\nu\lambda\sigma}\psi^{\sigma}+4\bar{\psi}_{\nu}\gamma_{\mu}\psi_{\lambda}+\frac{1}{3}\sqrt{2}\bar{\psi}_{\nu}\gamma_{\mu\lambda}\gamma_{||}\lambda)\Big]! \tag{26}$$

The next theory we want to discuss is the I theory. It's not a dimensional reduction of $d = 11$ theory, but it can be obtained from the $d = 11$ theory by an operation called truncation. Morally truncation get's rid of the say negative chirality fields. Doing it more consistently is achieved by putting

$$\begin{aligned} &e_{\mu}{}^{a} = \begin{pmatrix} e_{i}{}^{A} & 0 \\ 0 & e_{10}{}^{10} \end{pmatrix}, \qquad B_{ijk} = 0, \\ &\psi_{-i} \equiv \frac{1}{2}(1-\gamma_5)\psi_i = 0, \qquad \psi_{+10} \equiv \frac{1}{2}(1+\gamma_5)\psi_{10} = 0, \end{aligned} \tag{27}$$

where the latin indices run over 10 dimensions, and greek ones - over 11. Under this truncation the leftover fields would be the 10-dimensional graviton, a scalar from the graviton truncation the 2-form part of the 3-form B_{10ij}, a gravitino from $\psi_{+\mu}$ truncation and an additional negative chirality spin $\frac{1}{2}$ field from the $\psi_{-\mu}$ truncation, thus the field content presented in Table 2. The action for this theory has a fairly simple form compared to IIA theory action and is presented in [9].

Probably the most interesting case is the type IIB theory. It doesn't arise neither from dimensional reduction nor from truncation of fields. We can see that it has two scalars, which makes it quite different from the previous two cases, since the scalars are the coordinates of our manifold and thus will contain interesting features, which we'll discuss later when we talk about non-linear sigma models. The other thing to notice is that it has a self-dual 4-form. The self-duality should be clear by just counting the number of indices, the field strength of a 4-form is a 5-form, thus since we're in 10 dimensions the dual form would again be a 5-form, and thus the dual field - a 4-form. One can show that there is no covariant Lagrangian for such a theory [14, 12]. It is also notable that the IIA and IIB theories relate to each other by so called *T-duality* [12], when one compactifies them on circles with radii R_A and R_B such that $R_A R_B = 1$.

5.3 $d < 10$ supergravities

Supergravities in lower dimensions are most easily obtained in a similar way to the way one obtains IIA and I theories from $d = 11$ theory, i.e. by dimensional reduction or by truncation. Their field content is give in Table 2. For small N ($N \leq 4$) theories there are also superspace formulations.

Notice that we did not include the case $d \leq 3$, the reason for that being that one has no dynamical degrees of freedom in those dimensions for the graviton.

A general structure for $d < 10$ theories that we will be interested for the next section is the scalar field structure. When they are present the theory has a rigid non-compact symmetry G, but also the scalars transform locally under the maximal compact subgroup $H \subset G$, so to get the physical degrees of freedom one has to mod out by H.

6 Non-linear sigma models

Scalar fields appearing in supergravities are described by G/H non-linear sigma models, where G is a non-compact Lie group and H is a maximal compact subgroup of G. What this means is that one can extend the symmetry of the scalar fields to get a linear G-symmetric model of scalars, but then to get back our the original theory one has to quotient the additional local gauge transformations on the scalars described by H.

Rather than presenting the mechanism for a general case, we will do it for a particular case of IIB theory discussed previously, which will nevertheless include almost all of the tricks of non-linear sigma models. For a more general treatment see [13].

Let us consider the case $G = SL(2,\mathbb{R}) \sim SU(1,1)$ and $H = SO(2) \sim U(1)$. This sigma model appears in IIB theory and also in $d = 4$, $N = 4$ theory. The $SU(1,1)$-valued scalar field $V(x)$ is parameterized by two complex scalar fields $\phi_0(x), \phi_1(x)$ as

$$V(x) = \begin{pmatrix} \phi_0(x) & \phi_1^*(x) \\ \phi_1(x) & \phi_0^*(x) \end{pmatrix}, \qquad , |\phi_0|^2 - |\phi_1|^2 = 1. \tag{28}$$

The rigid $SU(1,1)$ transformations are given by:

$$V(x) \to gV(x), \quad g = \begin{pmatrix} a & b^* \\ b & a^* \end{pmatrix} \in SU(1,1), \tag{29}$$

where $|a|^2 - |b|^2 = 1$. The local $H = U(1)$ transformations act from the right:

$$V(x) \to V(x)h^{-1}(x), \quad h(x) = \begin{pmatrix} e^{i\theta(x)} & 0 \\ 0 & e^{-i\theta(x)} \end{pmatrix} \in U(1). \tag{30}$$

To construct an invariant Lagrangian that will satisfy the demanded properties of global G and local H invariances one has to look at the decomposition of the Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{n}, \tag{31}$$

where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{n}$ is its orthogonal complement in $\mathfrak{g}$. The orthogonality is defined with respect to a trace in a certain representation: $\mathrm{tr}(\mathfrak{h}\mathfrak{n}) = 0$. Orthogonality then implies

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{n}] \subset \mathfrak{n}. \tag{32}$$

The $\mathfrak{g}$-valued field $V^{-1}\partial_\mu V$ is decomposed as

$$V^{-1}\partial_\mu V = Q_\mu + P_\mu, \qquad Q_\mu \in \mathfrak{h}, \quad P_\mu \in \mathfrak{n}. \tag{33}$$

For our case of $SU(1,1)/U(1)$ we have

$$\begin{aligned} Q_\mu &= (\phi_0^*\partial_\mu\phi_0 - \phi_1^*\partial_\mu\phi_1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ P_\mu &= \begin{pmatrix} 0 & (\phi_0\partial_\mu\phi_1 - \phi_1\partial_\mu\phi_0)^* \\ \phi_0\partial_\mu\phi_1 - \phi_1\partial_\mu\phi_0 & 0 \end{pmatrix}. \end{aligned} \tag{34}$$

Since we know how V transforms under local H transformations (30) and have the orthogonality conditions (32) we can figure out the transformation laws for Q_μ and P_μ under H:

$$\begin{aligned} Q_\mu &\to hQ_\mu h^{-1} + h\partial_\mu h^{-1}, \\ P_\mu &\to hP_\mu h^{-1}. \end{aligned} \tag{35}$$

We see that Q_μ transforms as an H gauge field, and thus can be used as a connection to define a covariant derivative. In fact using (33) we have:

$$P_\mu = V^{-1}(\partial_\mu V - VQ_\mu) \equiv V^{-1}D_\mu V, \tag{36}$$

where D_μ is the H-covariant derivative on V.

Finally noting that Q_μ and P_μ are invariant under the rigid G transformations, we can now finally construct an action that would be invariant under the global G and local H. In particular the kinetic term of the scalar field is:

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}(P_\mu P^\mu) = |\phi_0\partial_\mu\phi_1 - \phi_1\partial_\mu\phi_0|^2 = \frac{\partial_\mu z \partial^\mu z^*}{(1-|z|^2)^2} \qquad \left(z \equiv \phi_1^*(\phi_0^*)^{-1}\right) \tag{37}$$

This is the final result we wanted to achieve! We have an action which has only two real scalars, as in IIB theory, the variable z is $U(1)$ invariant and the action has a non-linearly realized $SU(1,1)$ symmetry as

$$z \rightarrow \frac{az + b^*}{bz + a^*} \tag{38}$$

Similar sigma models can be constructed for other supergravities which have scalars. The full list of G and H groups is given in [1].

7 Remarks and Conclusion

There are many issues in supergravities that we left out in this paper like massive supermultiplets, coupling of supergravity multiplets with other multiplets (if there are any depending on the dimension [4]), duality symmetries and the constraints they impose on the possible (or impossible as we saw for IIB) Lagrangians.

We would like to address here something else that we left out in the beginning - the non Lorentz scalar central charges. If one introduces to our supergravities external p-dimensionally extended objects, called p-branes (0-branes are point-particles, 1-branes - strings, etc.) which could couple to supermultiplet fields, one can write down a general Lagrangian in spacetime dimension d, describing interaction of a graviton, a k-form and a scalar with a p-brane, and general electrically charged p-brane solutions were found [16], as well as "solitonic" solutions which exhibit topological magnetic charge satisfying Dirac quantization rule. In the cases where the solutions are supersymmetric, those charges actually *are* the central charges! And the supersymmetry algebra takes the approximate form:

$$\{Q_\alpha, Q_\beta\} \propto \sum_p (\gamma^{\mu_1 \cdots \mu_p} C)_{\alpha\beta} Z_{\mu_1 \cdots \mu_p}, \tag{39}$$

where the sum is over the possible p-branes or alternatively central charges. Of course the possible types of branes have been classified and are given as a nice graph in [1].

As a conclusion we would like to say that in this paper we barely scratched the surface of the supergravity area, which is already quite old (in modern time-scales), but which still has new aspects to be discovered and open problems to be solved. For those who want to study it more deeply we would recommend reviews [13, 1, 15].

Appendix

A Clifford algebras

We wish to define Clifford algebras in this section and list some useful facts about them. The way we do it here is probably a little outdated, but is nevertheless quite illuminating. We follow [17].

Consider real Euclidean space of n dimensions, $\mathbb{R}^n$. Consider also scalars a, vectors a^i, antisymmetric tensors of rank 2 (or 2-forms) a^{ij}, 3-forms a^{ijk}, etc. Obviously we can't have more indices than the dimension of the space n, otherwise we'll get trivially zero.

Consider now new objects, which we'll call *aggregates. A collection consisting of a scalar, vector, 2-form, 3-form,..., n-form, will be called an aggregate* $\mathbf{A}$:

$$\mathbf{A} = (a, a^i, a^{ij}, \ldots, a^{i_1 i_2 \cdots i_n}) \tag{40}$$

The numbers appearing in (40) are called the coordinates of $\mathbf{A}$ in the given basis.

We wish to demonstrate that with certain operations of summation and multiplication aggregates may be regarded as elements of a certain algebra over the field of real numbers (which will of course be *Clifford algebra*).

First of all the operation of addition is defined trivially:

$$\mathbf{A}(a, a^i, \ldots, a^{i_1 \cdots i_n}) + \mathbf{B}(b, b^i, \ldots, b^{i_1 \cdots i_n}) = \mathbf{C}(a+b, (a+b)^i, \ldots, (a+b)^{i_1 \cdots i_n}). \tag{41}$$

Note that since by summing k-forms we get back k-forms this is an invariant definition. Multiplication by a number is also defined trivially:

$$b\mathbf{A} = \mathbf{A}b = \mathbf{A}(ba, ba^i, \ldots, ba^{i_1 \cdots i_n}). \tag{42}$$

With the risk of introducing confusion we introduce the following convention:

$$\begin{aligned} &a \equiv \mathbf{A}(a, 0, \ldots, 0), \\ &\mathbf{a} \equiv \mathbf{A}(0, 0, \ldots, a^{i_1 \ldots i_k}, \ldots, 0) \end{aligned} \tag{43}$$

Choosing then an orthonormal basis $\mathbf{e}_i$ and denoting $\mathbf{e}_{i_1 \ldots i_k} \equiv \frac{1}{k!}\mathbf{e}_{[i_1} \cdots \mathbf{e}_{i_k]}$, we have:

$$\mathbf{A} = a + \sum_{i_1} a^{i_1}\mathbf{e}_{i_1} + \sum_{i_1 < i_2} a^{i_1 i_2}\mathbf{e}_{i_1 i_2} + \cdots + a^{12\ldots n}\mathbf{e}_{12\ldots n} \tag{44}$$

We now define *multiplication* of aggregates, and demand it to have the following properties.
(a) It should be distributive:

$$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}, \quad \mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA} + \mathbf{CB}, \tag{45}$$

(b) associative

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}), \tag{46}$$

(c) if $\mathbf{A} = (a, 0, \ldots, 0) \equiv a$, then

$$\mathbf{AB} = \mathbf{BA} = a\mathbf{B}, \tag{47}$$

(d) if $\mathbf{a} \equiv (0, a^i, 0, \ldots, 0)$ then

$$\mathbf{aa} = |\mathbf{a}|^2, \tag{48}$$

where $|\mathbf{a}|^2$ should be regarded as $(|\mathbf{a}|^2, 0, \ldots, 0)$.
(e) And finally for arbitrary vectors $\mathbf{p_1}, \ldots, \mathbf{p_k}$ regarded as aggregates the following should hold:

$$\mathbf{p}_{[1} \cdots \mathbf{p}_{k]} = \mathbf{p}_1 \wedge \cdots \wedge \mathbf{p}_k, \tag{49}$$

where on the left hand-side is aggregate multiplication, while on the right hand-side wedging of vectors, the result of which is treated as an aggregate.

The algebra of aggregates obeying the above written laws of summation and multiplication is called a *Clifford algebra* $\mathcal{C}_n$.

After this probably confusing definition we want to show that this implies the rules we know. Using property (d) of the product it's not hard to show that *orthogonal vectors anticommute*, in particular:

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2\delta_{ij}, \tag{50}$$

which is the defining property of the gamma matrices as we know! In fact by fundamental theorem of Clifford algebras, the forms $\mathbf{e}_{i_1\cdots i_k}$ can be put into 1-1 correspondence with $\gamma^{i_1\cdots i_k}$, which are the antisymmetrizations of Gamma matrices as defined in Section 2.

We already discussed some of the aspects of Clifford algebras in Section 2, so let us just give an example of the simplest Clifford algebra. Consider instead of real Euclidean space, 1-dimensional pseudo-Euclidean space, this just means that the basis vector squares to -1 instead of 1 - $\mathbf{e}_1^2 = 1$. Then the aggregates have the form:

$$\mathbf{A} = a + a^1\mathbf{e}_1, \tag{51}$$

where a, a^1 are arbitrary real numbers. This is just the algebra of complex numbers!

Similarly one can consider Clifford algebra on 2-dimensional pseudo-Euclidean space (where both vectors square to -1), and get the algebra of quaternions.

For a more modern definition of Clifford algebras and there geometrical interpretation we refer the reader to the extensive literature available at [18].

B Superalgebras in different dimensions

We list possible automorphism (R-symmetry) groups H_R and the form of the anticommutators $\{Q, Q\}$, depending on the spinor type of Q^i, thus the spacetime dimension.

(a) $d = 4, 8$ **mod** 8

The supercharges are Weyl spinors with positive chirality Q^i_+ $(i = 1, \ldots, N)$. Their charge conjugations have negative chirality $(Q^i_+)^c = Q_{-i}$, where the charge conjugation matrix $C = C_-$ $(C = C_+)$ is used for $d = 4$ $(d = 8)$ mod 8. The automorphism group $H_R = U(N)$. Anticommutators of the supercharges are:

$$\begin{aligned}
\{Q^i_+, Q^T_{-j}\} &= \frac{1}{2}(1+\gamma_5)\gamma^a C P_a \delta^i_j, \\
\{Q^i_+, Q^{jT}_+\} &= \frac{1}{2}(1+\gamma_5) C Z^{ij},
\end{aligned} \tag{52}$$

where Z^{ij} is antisymmetric for $d = 4$ mod 8 and symmetric for $d = 8$ mod 8. **(b)** $d = 10$ **mod 8**

The supercharges are Majorana-Weyl spinors with positive chirality Q^i_+ $(i = 1, \ldots, N_+)$ and Majorana-Weyl spinors with negative chirality Q^i_- $(i = 1, \ldots, N_-)$. The automorphism group $H_R = SO(N_+) \times SO(N_-)$ (note that it's SO group instead of U as in previous case, because we have Majorana spinors, and thus have a reality condition). Anticommutators of the supercharges are:

$$\begin{aligned}
\{Q^i_+, Q^{jT}_+\} &= \frac{1}{2}(1+\gamma_5)\gamma^a C_- P_a \delta^{ij}, \\
\{Q^i_-, Q^{jT}_-\} &= \frac{1}{2}(1-\gamma_5)\gamma^a C_- P_a \delta^{ij}, \\
\{Q^i_+, Q^{jT}_-\} &= \frac{1}{2}(1+\gamma_5) C_- Z^{ij}.
\end{aligned} \tag{53}$$

(c) $d = 6$ **mod 8**

The supercharges are symplectic Majorana-Weyl spinors with positive chirality Q^i_+ $(i = 1, \ldots, N_+)$ and symplectic Majorana-Weyl spinors with negative chirality Q^i_- $(i=1,\dots,N_-)$. They satisfy $\Omega^{ij}_\pm (Q^j_\pm)^c = Q^i_\pm$, where $\Omega^{ij}_\pm$ are antisymmetric matrices. The numbers N_+ and N_- must be even (simply by the definition of symplectic spinors - they are *doublets*). The automorphism group is $H_R = USp(N_+)\times USp(N_-)$, where $USp(N)$ is the group of unitary symplectic $N\times N$ matrices. Anticommutators of the supercharges are:

$$
\begin{aligned}
\{Q^i_+, Q^{jT}_+\} &= \frac{1}{2}(1+\gamma_5)\gamma^a C_- P_a \Omega^{ij}_+, \\
\{Q^i_-, Q^{jT}_-\} &= \frac{1}{2}(1-\gamma_5)\gamma^a C_- P_a \Omega^{ij}_-, \\
\{Q^i_+, Q^{jT}_-\} &= \frac{1}{2}(1+\gamma_5) C_- Z^{ij}.
\end{aligned} \tag{54}
$$

(d) $d = 9, 11$ **mod 8**
The supercharges are pseudo Majorana spinors for $d=9$ mod 8 and Majorana spinors for $d=11$ mod 8 - Q^i $(i=1,\dots,N)$. The automorphism group is $H_R = SO(N)$. Anticommutators of the supercharges are:

$$
\{Q^i, Q^{jT}\} = \gamma^a C P_a \delta^{ij} + C Z^{ij}, \tag{55}
$$

where $C = C_+$, Z^{ij} is symmetric for $d=9$ mod 8 and $C=C_-$, Z^{ij} is antisymmetric for $d=11$ mod 8.
(e) $d = 5, 7$ **mod 8**
The supercharges are symplectic pseudo Majorana spinors for $d=5$ mod 8 and symplectic Majorana spinors for $d=7$ mod 8 - Q^i $(i=1,\dots,N)$. They satisfy $\Omega^{ij}(Q^j)^c = Q^i$, where Ω^{ij} is an antisymmetric matrix. The number N is even. The automorphism group is $H_R = USp(N)$. Anticommutators of the supercharges are:

$$
\{Q^i, Q^{jT}\} = \gamma^a C P_a \Omega^{ij} + C Z^{ij}, \tag{56}
$$

where $C=C_+$, Z^{ij} is antisymmetric for $d=5$ mod 8 and $C=C_-$, Z^{ij} is symmetric for $d=7$ mod 8.

References

[1] Y. Tanii, *Introduction to Supergravities in Diverse Dimensions*, arXiv:hep-th/9802138.

[2] S. Weinberg, *The Quantum Theory of Fields, vol. 3*, Cambridge Univ. Press, 2000.

[3] J. Wess and J. Bagger, *Supersymmetry and Supergravity, 2nd edition*, Princeton Univ. Press, 1992.

[4] J. Strathdee, *Extended Poincaré Supersymmetry*, Int. J. Mod. Phys. **A2** (1987) 273.

[5] E. Cremmer, B. Julia and J. Scherk, *Supergravity Theory in 11 Dimensions*, Phys. Lett. **B76** (1978) 409.

[6] J.H. Schwarz and P.C. West, *Symmetries and transformations of chiral $N=2$ $D=10$ supergravity*, Phys. Lett. **B126** (1983) 301;

[7] J.H. Schwarz, *Covariant field equations of chiral $N=2$ $D=10$ supergravity*, Nucl. Phys. **B226** (1983) 269;

[8] P. Howe and P.C. West, *The complete $N=2$, $d=10$ supergravity*, Nucl. Phys. **B238** (1984) 181.

[9] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, *Ten-dimensional Maxwell-Einstein supergravity, its currents, and the issue of its auxiliary fields*, Nucl. Phys. **B195** (1982) 97;

[10] G. Chapline and N.S. Manton, *Unification of Yang-Mills theory and supergravity in ten-dimensions*, Phys. Lett. **B120** (1983) 105.

[11] M. Huq and M.A. Namazie, *Kaluza-Klein supergravity in ten dimensions*, Class. Quantum Grav. **2** (1985) 293.

[12] J. Polchinski, *String Theory, vol. 2*, Cambridge Univ. Press, 1998.

[13] B. de Wit, *Supergravity*, arXiv:hep-th/0212245.

[14] N. Marcus and J.H. Schwarz, *Field theories that have no manifestly Lorentz-invariant formulation*, Phys. Lett. **115B** n. 2 (1982) 111

[15] A. Van Proeyen, *Structure of supergravity theories*, arXiv:hep-th/0301005.

[16] M.J. Duff and J.X. Lu, *Black and super p-branes in diverse dimensions*, arXiv:hep-th/9306052.

[17] P.K. Raševskiĭ, *The theory of spinors*, Amer. Math. Soc. Transl. **v.6** (1957) 1

[18] *International Clifford Algebra Society homepage*, www.clifford.org.