1 Introduction

String theory, to say the least, gets even more interesting when dimensions become compactified. In closed string theory, new physics emerge due to the fact closed strings can wrap around circular dimensions as well as live on top of them. This displays a remarkable symmetry: if you invert the radius of the compact dimension from $R \rightarrow \frac{\alpha'}{R}$, and exchange the number of times the string wraps around the dimension with its quantized momentum; both pictures become indistinguishable. This is T(toroidal)-duality and the above claims will be laid out in Section 2. The rest of this paper will be on T-duality of open strings and D-branes. In open string theory, the endpoints interchange Neumann and Dirichlet boundary conditions under T-duality for each compact dimension of spacetime. This alters the dimension of the brane they live on. For example, if the string has Neumann boundary conditions on a Dp-brane wrapped in a circular dimension, in the dual world the string would live on a D(p-1)-brane that is located at a specific point on that circle. It turns out that this position is set by Wilson line of the Dp-brane covering the compactified dimension. T-duality reveals some extremely remarkable (and cool) properties about electromagnetic fields on D-branes. In section 6.1, we discover there is a relationship between the strength of a constant electric field on the Dp-brane and the velocity of the dual D(p-1)-brane traveling around the circular dimension. Since the brane must travel less than the speed of light, there is a cutoff maximum electric field strength which happens to equal the tension of the string. We conclude from this that D-branes must be govern by Born-Infeld electrodynamics. In section 6.2, we will replace the electric field with a constant magnetic field and see that, in the dual world, our brane gets rotated instead of boosted with the tilting angle determined by the magnetic field strength. The D(p-1)-brane, extended in the orthogonal flat dimension, will continually wind around the compact dimension like a candy cane stripe. A periodicity emerges in the flat dimension from the distance the brane takes to make a full lap around the circle dimension. We will take this length and wrap it into a circle to form a donut-shaped 2-torus and observe that the magnetic flux on the torus is quantized. Lastly, we will study the T-duality of a homologically equivalent configuration which will give us a very stringy interpretation of the quantized flux.
2 T-duality of closed strings

2.1 Introduction to Compactification

The bosonic string requires a spacetime critical dimension of $D=26$. The idea of circular (or compactified) extra dimensions has been a property of unification theories since the work of Kaluza-Klein in 1926 [1]. We will start by compacting one of the 25 spacial dimensions into a circle with radius $R$, beginning our toroidal construction.

$$x^{25} = x^{25} + 2\pi R$$  \hspace{1cm} (1)

The periodicity of the dimension will spur out some interesting physics; first being that closed strings can now live on the compact dimension as well as wrapped around it. The amount of times the closed string can be wrapped around is quantified by the integer $m$ called ”the winding number.” The string lives on the dimension when $m=0$.

$$X(\tau, \sigma + 2\pi) = X(\tau, \sigma) + m(2\pi R)$$  \hspace{1cm} (2)

The winding number is a topological property that describes the mapping between the string parameter $\sigma \in [0, 2\pi)$ to the coordinates in $x^{25}$ [2]. The winding number must be conserved,. For example, a $m=0$ string can split into a $m=+1$ and $m=-1$ as shown in Fig [1]. The negative winding states just mean the string is mapped along the negative $x^{25}$ direction.

Figure 1: This is a good illustration of closed strings in a circular dimension given by Polchinski [3]. The integers are the winding numbers and the bottom picture depicts the string $m=0$ splitting into $m=+1$ and $m=-1$ strings.
The second interesting feature is the quantized momentum, which can be understood by examining the translation operator inducing translations in a full circle $e^{-ipa} = e^{-ip(a+2\pi R)} = e^{-ipa}e^{-ip2\pi R}$, thus, $|e^{-ip2\pi R}| = 1 \implies p = \frac{n}{R}$.

It will be useful to define a new term called the winding, $\omega = \frac{mR}{\alpha'}$, since the presence of the compacted dimension introduces a new term to our string's coordinates,

$$X(\tau, \sigma) = x_0 + \alpha' p\tau + \alpha' \omega \sigma + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (\alpha_n e^{-in\sigma} + \tilde{\alpha}_n e^{in\sigma}); \quad (3)$$

$\tilde{\alpha}_n$ corresponds to right moving oscillations and $\alpha_n$ correspondence to left moving oscillations, with $n > 0$ annihilating excitations and $0 > n$ creating excitations. The additional term in (Eq.3) comes from that fact that the zero modes may not be equal, $\tilde{\alpha}_0 - \alpha_0 = \sqrt{2\alpha'}\omega$.

### 2.2 spectrum

For non-compact dimensions, $\omega$ is always zero and $\alpha_0 = \tilde{\alpha}_0 = \sqrt{\frac{\alpha'}{2}}P$. We define our momentum for the periodic dimension as follows.

$$P_L = \sqrt{\frac{2}{\alpha'}\alpha_0} = \frac{n}{R} + \frac{mR}{\alpha'} \quad (4a)$$

$$P_R = \sqrt{\frac{2}{\alpha'}\tilde{\alpha}_0} = \frac{n}{R} - \frac{mR}{\alpha'} \quad (4b)$$

The Virasoro generators are

$$L_0^\perp = \frac{\alpha' P_L^2}{4} + \sum_{n=1}^{\infty} \alpha_{-n} \alpha_n \quad (5a)$$

$$\tilde{L}_0^\perp = \frac{\alpha' P_R^2}{4} + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \tilde{\alpha}_n. \quad (5b)$$

Before compactification, the condition $L_0^\perp - \tilde{L}_0^\perp = 0$ forced $N^\perp - N^\perp = 0$, where $N^\perp$ is just the number operator or the second term in (Eq. 5) in disguise. If the first condition is to hold true, that would mean $N^\perp - \tilde{N}^\perp = 0$ only if the string has zero momentum, winding number, or both. Otherwise,

$$N^\perp - \tilde{N}^\perp = nm. \quad (6)$$

We calculate the mass$^2$ of the string to be

$$M^2 = p^2 + \omega^2 + \frac{2}{\alpha'}(N^\perp - \tilde{N}^\perp - 2) = \frac{n^2}{R^2} + \left(\frac{mR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N - \tilde{N} - 2). \quad (7)$$
There is a beautiful symmetry of this spectrum. We can take $R \to \frac{\alpha'}{R}$, and since $n$ and $m$ are just integers, we can just send $n \to m$ and $m \to n$ which leaves the mass spectrum invariant. In other words, physics remains indistinguishable between compacted radii $R$ and its dual, $(\tilde{R} = \frac{\alpha'}{R})$, upon the exchange of momentum and winding. It is pretty remarkable a dimension with an extremely large radius is equivalent to one with an extremely small radius. There is a few other aspects we need to touch on inorder to complete our full transition into the dual realm. It went without saying that our string coordinate in (Eq. 3) was constructed by adding the left and right components of the string; $X(\tau, \sigma) = X_L(\tau + \sigma) + X_R(\tau - \sigma)$. Our ”dual” string coordinate is instead defined by:

$$\tilde{X}(\tau, \sigma) = X_L(\tau + \sigma) - X_R(\tau - \sigma).$$

(8)

So that the momentum and winding number are now switched

$$\tilde{X}(\tau, \sigma) = q_0 + \alpha' \omega \tau + \alpha' p \sigma + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} e^{-in\tau} (\alpha_n e^{-in\sigma} - \tilde{\alpha}_n e^{in\sigma}).$$

(9)

The dual momentum is defined as

$$\tilde{P}_\tau = \frac{1}{2 \pi \alpha'} \frac{\partial \tilde{X}}{\partial \tau} = T(\dot{X}_L - \dot{X}_R),$$

(10)

where $T = \frac{1}{2 \pi \alpha'}$ is the string’s tension. Since $p$ and $w$ are switched, the commutation relations are adjusted to $[q_0, \omega] = i$ and $[q_0, p] = 0$. T-duality is a full quantum symmetry.

### 3 Interlude I: Wilson line

Before I carry on with my discussion of T-duality, it’s important to discuss the new properties that arise among Maxwell potentials on compact dimensions. This will allow us to study open string T-duality since their end points live on branes which carry gauge fields. This usual gauge transformation leaves Maxwell equations invariant,

$$A_x \to A_x + \partial_x \chi.$$  

(11)

But not the Schrodinger equation for a charge particle as described,

$$H \psi = i \frac{\partial \psi'}{\partial t} = \frac{1}{2m} (\nabla - q \vec{A})^2 \psi + q \Phi \psi.$$  

(12)

We must introduced a U(1) phase factor to the wave function to recover invariance,

$$U(x) = e^{i \alpha(x)}; \psi' = U \psi,$$  

(13)

which will be the gauge parameter of our theory instead of $\chi(x)$. This can be physically seen from the Aharonov-Bohm effect, where the wave function of a particle becomes affected
to a phase by the gauge potential in the absence of EM field. At first, it was thought of as very bizarre that quantum mechanics should be effected by a potential which was considered unphysical. Compactified dimensions are also effected by the gauge potential; before we see this lets first take a closed curve over a constant electric field in ordinary space.

\[
\oint_{\Gamma} \vec{E} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = 0. \tag{14}
\]

However, lets say \( \Gamma \) runs around a compact dimension. It is closed, yet there is no surface which \( \Gamma \) traces out a boundary, so therefore

\[
\oint_{\Gamma} \vec{E} \cdot d\vec{l} \neq 0. \tag{15}
\]

We will study electric fields over compact dimensions in Section 6, for now lets consider the gauge potential \( A_x \) which runs along the compact dimension. By integrating over the potential,

\[
\omega = q \oint dx A_x, \tag{16}
\]

Using the transformation in (Eq.11), (Eq.16) becomes,

\[
\omega' \rightarrow q \oint dx (A_x + \frac{\partial \chi}{\partial x}) = \omega + q(\chi(x_0 + 2\pi R) - \chi(x_0)), \tag{17}
\]

where \( x_0 \) is just some arbitrary point on the circle. The last term in (Eq. 17) does not vanish but equals \( 2\pi m \) since \( U(x) \) is our gauge parameter and must be periodic \( U(x+2\pi R) = U(x) \). This concludes that \( \omega \) must be periodic: \( \omega' = \omega + 2\pi m \). More specifically taking the values \([0, 2\pi]\), we can look at it as an angle that traces along the unit circle. This angle is used to calculate the holonomy of the gauge field from a Wilson line defined as

\[
W = e^{i\theta} = e^{i\omega} = e^{iq \oint dx A_x}. \tag{18}
\]

The Wilson line is a gauge invariant quantity and becomes gauge trivial with shifts of that are \( 2\pi \) periodic. We can fix the gauge \( \chi \) to be linear in \( x \),

\[
q\chi = (2\pi m) \frac{x}{2\pi R} = \frac{mx}{R}. \tag{19}
\]

The gauge transformation in (Eq.11) now becomes,

\[
qA_x(x) \rightarrow qA_x(x) + \frac{m}{R}. \tag{20}
\]

When considering a Wilson line with constant \( A_x \), we see that

\[
qA_x = \frac{\theta}{2\pi R}, \tag{21}
\]

where \( \theta \) now represents the value \( \omega \) in (Eq. 16) which we will see takes on a new and very important interpretation in the world of open strings.
4 Open strings

4.1 T-duality

We now have the tools to study T-duality of open strings. Let’s first assume we have a D25-brane covering the entire universe. Thus our open string coordinate has only Neumann boundary conditions \( \partial_\sigma X(\tau, \sigma)|_{\sigma=\pi} = \partial_\sigma X(\tau, \sigma)|_{\sigma=0} = 0 \), given by

\[
X(\tau, \sigma) = x_0 + \sqrt{2\alpha'}\alpha_0\tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n}{n} \cos n\sigma e^{-in\tau}.
\]  

(22)

We are considering \( x^{25} \) to be compact and this brane wraps completely around the dimension. In order to examine the effects of T-duality, we must break up our string into left and right movers:

\[
X_L = \frac{1}{2}(x_0 + q_0) + \sqrt{\frac{\alpha'}{2}}\alpha_0(\tau + \sigma) + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n}{n} e^{-in(\tau+\sigma)}
\]  

(23a)

\[
X_R = \frac{1}{2}(x_0 - q_0) + \sqrt{\frac{\alpha'}{2}}\alpha_0(\tau - \sigma) + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n}{n} e^{-in(\tau-\sigma)}
\]  

(23b)

We have defined the T-dual coordinate in (Eq. 8) to be \( \tilde{X} = X_L - X_R \). For open strings this gives us,

\[
\tilde{X}(\tau, \sigma) = X_L - X_R = q_0 + \sqrt{2\alpha'}\alpha_0\sigma + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n}{n} \sin(n\sigma)e^{-in\tau}.
\]  

(24)

Notice that this is the just coordinate of the string with Dirichlet boundary conditions \( \partial_\tau X(\tau, \sigma)|_{\sigma=\pi} = \partial_\tau X(\tau, \sigma)|_{\sigma=0} = 0 \). First, let’s clearly establish the mathematical implications,

\[
\partial_\tau \tilde{X} = X'_R - X'_L = \partial_\sigma X
\]  

(25a)

\[
\partial_\tau X = X'_R + X'_L = \partial_\sigma \tilde{X},
\]  

(25b)

we see that the boundary conditions switch during the transformation to the dual world. This means that the D25 brane wrapped around dimension \( x^{25} \) with radius \( R \) in the first world cannot exist in its dual world. The boundary becomes Dirichlet and our D25-brane reduces down to a D24-brane occupying a single point on the circle with radius \( \tilde{R} = \frac{\alpha'}{R} \).

\[
(D(p); R) \rightarrow (D(p - 1); \frac{\alpha'}{\tilde{R}}).
\]  

(26)

If our brane is wrapped around all \( k \) circular dimensions that exist, the duality would logically follow: \( D(p)-brane \rightarrow D(p-k)-brane \), with the new dual radii for each corresponding dimension. Likewise if a string lives on a brane only in flat spacial dimension that
is orthogonal to \( k \) compact dimensions, then in the dual world the string will live on a D\((p+k)\)-brane wrapped up in the \( k \) compact dimensions.

The open string, which is a free scalar field in directions normal to the brane, can wind around the compact dimension if the boundary conditions are Dirichlet just like the closed string. We can take the endpoints of Eq. (23) to see,

\[
\tilde{X}(\tau, \pi) - \tilde{X}(\tau, 0) = \sqrt{2\alpha'\alpha_0}(\pi - 0) = 2\pi\alpha'p = 2\pi\frac{\alpha'}{R}n = 2\pi Rn.
\] (27)

If the end points possess Neumann boundary conditions, the string would still be able to wrap around the dimension but since they are able to travel anywhere along this dimension, the string would lose its nice periodic structure.

What is of great interest-and the motivation behind section 2- is when a Maxwell gauge field carrying brane that is wrapped in the curled dimension. From the Hamiltonian in (Eq. 12), we see the the coupling is now tacked to the momentum term, \( p \rightarrow p - qA \). Using our result from (Eq. 21), the momentum along the periodic direction of our string becomes

\[
\frac{n}{R} \rightarrow \frac{n}{R} - \frac{\theta}{2\pi R}.
\] (28)

We interpret \( \theta \) as the position of the dual D\((p-1)\)brane along the compact dimension. When the both the string’s end points are attached to the same brane they possess the same value for \( \theta \), but have opposite charge so the effect cancels out \( p - qA_{x25} + qA_{x25} = p \).

In order to study this new feature we will consider 2 branes, which curl around each other in the original world. We examine Wilson line for each brane which gives us a value \( \theta_1 \) for
the first brane and $\theta_2$ for the second brane. The mass squared formula for the open string is given by:

$$M^2 = p^2 + \frac{1}{\alpha'}(N^\perp - 1).$$

(29)

With the momentum transformation,

$$p - qA^1_{x^{25}} + qA^2_{x^{25}} = \frac{n}{R} - \frac{\theta_1}{2\pi R} + \frac{\theta_2}{2\pi R}.$$  

(30)

In the dual world, both branes can only occupy a single point along the circle. If we consider $\theta_1$ to be the dual position of the first brane and $\theta_2$ to be the position of the second brane, then the interpretation follows that $\theta_2 - \theta_1 = \Delta \theta$ is the length which the string is stretched between the two branes, as shown in fig.[2]. The mass$^2$ in (Eq. 29) becomes

$$M^2 = \left(\frac{n}{R} + \frac{\theta_2 - \theta_1}{2\pi R}\right)^2 + \frac{1}{\alpha'}(N^\perp - 1).$$

(31)

For $n=0$, we recover the expression for a string stretched between two branes in flat dimensions,

$$M^2 = \left(\frac{\bar{X}_2 - \bar{X}_1}{2\pi \alpha'}\right)^2 + \frac{1}{\alpha'}(N^\perp - 1),$$

(32)

providing strong support for this interpretation. It is noteworthy to add that the separating overlapping branes, plays the same role as symmetry breaking in quantum field theory. N overlapping branes can be expressed as a U(N) gauge field. If N=2, we get U(2)=SU(2)XU(1), giving us the 4 massless bosons responsible for the weak interaction. As we saw in (Eq. 31/32), separating the D-branes gives mass to string. String theory interprets brane separation as the Higgs mechanism.

5 Interlude II: Open strings on electromagnetic background

T-duality leads to bizarre implications when background electromagnetic field permeate the Dp-brane, which will be discussed in the next section. This section will set the stage. The end points of open strings couple to Maxwell potentials the same way charge particles do:

$$S = \int d\tau d\sigma \mathcal{L}(\dot{X}, X') + \int d\tau A_n(X) \frac{dX^n}{d\tau} |_{\sigma=\pi} - \int d\tau A_n(X) \frac{dX^n}{d\tau} |_{\sigma=\pi},$$

(33)

"m" and "n" are brane world indices so they span from 0,...,p. Here we have $A_n = \frac{1}{2} F_{mn} X^m$, where $F_{mn}$ is the electromagnetic field tensor. In order to find the equations of motion for the string, we must vary the action and set it equal to zero, using the notation $P^\sigma_\mu = \partial \mathcal{L}/\partial X'^\mu$ and $P^\tau_\mu = \partial \mathcal{L}/\partial \dot{X}^\mu$, "$\mu$" is a spacetime index spanning from 0,...25. Making the following substitutions we arrive at

$$\delta S = \int d\tau d\sigma (P^\tau_\mu \partial_\tau \delta X'^\mu + P^\sigma_\mu \partial_\sigma \delta X^\mu) + \frac{1}{2} \int d\tau F_{mn}(\delta X^m \partial_\tau X^n + X^m \partial_\tau \delta X^n)|_{\sigma=\pi} = \ldots$$

(34)
To save room, it is only necessary to examine one of the end points since the $\sigma = 0$ term comes in symmetrically. We will now utilize the fact that the string satisfies the Klein Gordon wave equation $\partial_\tau P_\mu + \partial_\sigma P_\mu^\sigma = 0$. This way we can simplify things by writing out the total derivative.

$$\delta S = \int d\tau d\sigma \partial_\sigma (P_\mu^\sigma \delta X^\mu) + \int d\tau \delta X^m F_{mn} \delta_\tau X^n|_{\sigma = \pi} + ...$$ (35)

Another thing we can handle here is that $\delta X^a = 0$ where "$a$" spans the directions normal to the brane; $a = p+1, ..., 25$. Cleaning things up a bit we see that

$$\delta S = \int d\tau \delta X^m (P_\mu^m + F_{mn} \delta_\tau X^n)|_{\sigma = \pi} + ...$$ (36)

Now that we have everything nice and tidy under one integral. We evaluate $P_\mu^m + F_{mn} \partial_\tau X^n = 0$ for $\sigma = 0, \pi$. In static and light cone gauge our momentum takes the form $P_\mu^{\sigma} = -T \partial_\sigma X_\mu = -1/(2\pi \alpha') \partial_\sigma X_\mu$, so our boundary conditions become

$$\partial_\sigma X_m - 2\pi \alpha' F_{mn} \partial_\tau X^n = 0.$$ (37)

There are 3 types of boundary conditions the string can have. The first type is only Neumann, which occurs when $F_{mn} = 0$, since the only part left is $\partial_\sigma X_m = 0$. The next type of boundary conditions are mixed with Neumann and Dirichlet. This can occur in a situation where a string is attached to 2 branes with different dimensions. It is best to see how they come about in (Eq. 37) with an example. Suppose we only have a magnetic field $F_{34} = -F_{43} = B$. Our mix boundary conditions looks like

$$\partial_\sigma X_3 = 2\pi \alpha B \partial_\tau X^4,$$ (38a)  
$$\partial_\sigma X_4 = -2\pi \alpha' B \partial_\tau X^3.$$ (38b)

The last type is strictly Dirichlet and that can only happen if we crank our magnetic field up to a really large value (in fact infinity) such that the first terms in (Eq 38), are negligible, leaving $\partial_\tau X^{3\text{or}4} = 0$. In this case the field is so large that the end points in the $x^3$ and $x^4$ directions become stuck. We will view this specific situation in more detail. Having these boundary conditions, we are now able to continue our exploration of T-duality.

6 T-duality of D-branes

6.1 D-branes with electric fields

I propose that a D(p-1)-brane moving around a compact dimension with a velocity $v$ is T-dual to a Dp-brane wrapped on the circular dimension with an electric field pointed along that direction. In fact the velocity of the dual brane determines the strength of the electric field and vise versa. This will lead to profound implications. The proof might be a little hard to follow so I will give a quick run-down.
• collect electric field boundary conditions
• take the dual of $x^{25}$
• boost the D(p-1) brane in the dual world around the circle
• Show that the boundary conditions are equal to the Dp-brane world with an electric field

Well lets get started. The electric field around the compact, $x^{25}$, dimension is given by $F_{25,0} = -F_{0,25} = E$ which yields the boundary conditions,

$$\partial_\sigma X_0 - 2\pi\alpha' F_{0,25} \partial_\tau X^{25} = 0,$$
$$\partial_\sigma X_{25} - 2\pi\alpha' F_{25,0} \partial_\tau X^0 = 0. \tag{39a}$$

When we replace $X_0 = -X^0, E = 2\pi\alpha' E$, and $X^{25} = X$, we find

$$\partial_\sigma X^0 - E \partial_\tau X^{25} = 0 \tag{40a}$$
$$\partial_\sigma X - E \partial_\tau X^0 = 0. \tag{40b}$$

Now that we have both coordinates on the same footing, it would be nice to represent the boundary conditions in matrix form. $\partial_\tau$ and $\partial_\sigma$ will not be friendly towards this pursuit since there is no invertible linear relation between them. The partial derivatives we will be using are $\partial_+ = \frac{1}{2}(\partial_\tau + \partial_\sigma)$ and $\partial_- = \frac{1}{2}(\partial_\tau - \partial_\sigma)$. With a little algebraic manipulation, we can solve for $\partial_+ X^0$ and $\partial_+ X$.

$$\partial_+ \begin{bmatrix} X^0 \\ X \end{bmatrix} = \begin{bmatrix} 1 + \frac{\epsilon^2}{1 - \epsilon^2} & \frac{2\epsilon}{1 - \epsilon^2} \\ \frac{2\epsilon}{1 + \epsilon^2} & 1 + \frac{\epsilon^2}{1 - \epsilon^2} \end{bmatrix} \partial_- \begin{bmatrix} X^0 \\ X \end{bmatrix}. \tag{41}$$

This is a very attractive form. Similarly, we can write the Neumann boundary conditions, $\partial_+ X^0 = \partial_- X^0$, $\partial_+ X = \partial_- X$, in the form of a matrix; namely the identity.

$$\partial_+ \begin{bmatrix} X^0 \\ X \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \partial_- \begin{bmatrix} X^0 \\ X \end{bmatrix}. \tag{42}$$

Taking the dual of $X$, the boundary condition becomes Dirichlet, which we can write in terms of $\partial_+$

$$\partial_+ X = \partial_+ \bar{X}, \partial_- X = -\partial_- \bar{X}. \tag{43}$$

In matrix form

$$\partial_+ \begin{bmatrix} X^0 \\ \bar{X} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_- \begin{bmatrix} X^0 \\ \bar{X} \end{bmatrix}. \tag{44}$$
We are in the position to boost our brane along the circle to see if our proposition is true. We’ll refer to the rest frame of the circle as \( S \) and the rest frame of the brane as \( S' \), the relation between them is a Lorentz transformation with boosting parameter \( \beta = \frac{v}{c} \)

\[
\begin{align*}
X'^0 &= \gamma (X^0 - \beta X) \\
\dot{X}' &= \gamma (-\beta X^0 + \dot{X})
\end{align*}
\]  \( (45a) \) 

where \( \gamma = (1 - \beta^2)^{\frac{1}{2}} \). Putting the Lorentz transformation in matrix form we get

\[
\begin{bmatrix}
X'^0 \\
\dot{X}'
\end{bmatrix} = \gamma \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} X^0 \\
\dot{X} \end{bmatrix} = M \begin{bmatrix} X^0 \\
\dot{X} \end{bmatrix}
\]  \( (46) \)

We can combine our T-duality transformation with our Lorentz transformation to get our string boundary conditions in the \( S \) frame of reference

\[
\partial_+ \begin{bmatrix} X^0 \\
\dot{X} \end{bmatrix} = M^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} M \partial_- \begin{bmatrix} X^0 \\
\dot{X} \end{bmatrix}
\]  \( (47) \)

Using (Eq. 44), we can revert back to the original dual world with \( \partial_- \dot{X} \rightarrow -\partial_- X \) and \( \partial_+ X \rightarrow \partial_+ X \) in matrix form,

\[
\partial_+ \begin{bmatrix} X^0 \\
\dot{X} \end{bmatrix} = M^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} M \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_- \begin{bmatrix} X^0 \\
\dot{X} \end{bmatrix},
\]  \( (48) \)

which becomes

\[
\partial_+ \begin{bmatrix} X^0 \\
\dot{X} \end{bmatrix} = \begin{bmatrix}
\frac{1+\beta^2}{1-\beta^2} & \frac{2\beta}{1-\beta^2} \\
\frac{2\beta}{1-\beta^2} & \frac{1+\beta^2}{1-\beta^2}
\end{bmatrix} \partial_- \begin{bmatrix} X^0 \\
\dot{X} \end{bmatrix}.
\]  \( (49) \)

Comparing this with (Eq. 41), we see that

\[
\mathcal{E} = 2\pi \alpha' E = \beta,
\]  \( (50) \)

thus confirming our proposition. There is something interesting to be observed. According to the theory of special relativity, objects cannot have a speed greater than \( c \), so \( \beta < 1 \). This leads to the inequality,

\[
|E| = \frac{|\beta|}{2\pi \alpha'} < \frac{1}{2\pi \alpha'} = T,
\]  \( (51) \)

which means the maximum value an electric field can have equal to the tension of the string! We can think about this intuitively by considering the two forces that act on the end points of the string: the electric force and string’s tension. Including relativistic effects, the effective tension of the string becomes \( T_0 (1 - v_\perp^2) \), so for \( E < T_0 \) we can find a value for \( v_\perp \) which balances out the forces. For \( E = T_0 \), the end points stop moving.
Based on this claim, we find that Maxwell’s equations aren’t suitable to describe electric fields on the world volume of D-branes since the electron carries an infinite self energy. We look to a new theory of non-linear electrodynamics, Born-Infeld theory, which yields a finite electrostatic energy. The lagrangian density describing such a theory is given by

\[ \mathcal{L} = -b^2 \sqrt{-\text{det}(\eta_{\mu\nu} + \frac{1}{b} F_{\mu\nu}) + b^2}, \]  

(52)

where \( \eta_{\mu\nu} \) is the Minkowski metric. This is a good chance to weave in the fact that the string coupling also changes under T-duality, \( \tilde{g} = \sqrt{\frac{\alpha'}{2\pi}} g \). From section 2, the beauty of T-duality was that it left the mass of the closed string unchanged. What about the brane? Since the mass is equal to its tension multiplied by the volume, and the volume of the brane clearly changes, we must accommodate this by altering the tension.

\[ T_{p-1}(\tilde{g}) = 2\pi R T_p(g) \]  

(53)

The lagrangian for a relativistic point particle is given by

\[ L = -m \sqrt{1 - \frac{v^2}{c^2}}. \]  

(54)

We replace the mass of a particle with the mass of the brane,

\[ L = -V_{p-1} T_{p-1}(\tilde{g}) \sqrt{1 - \frac{v^2}{c^2}}. \]  

(55)

Taking the dual, we utilize Eq.50 and Eq. 53 to get

\[ L = -V_{p-1}(2\pi R) T_p(g) \sqrt{1 - (2\pi \alpha'E)^2}. \]  

(56)

We (partially) see that the lagrangian density is

\[ \mathcal{L} = -T_p(g) \sqrt{-\text{det}(\eta_{\mu\nu} + 2\pi \alpha' F_{mn})}. \]  

(57)

Comparing this with Eq. 52, we find that the normalization constant \( b = 1/2\pi \alpha' \), and the additive contribution is no longer relevant since the rest energy of our brane is needed. Thus Eq. 57 governs the physics of D-branes, which up until now, we have not considered independent dynamical objects. In the next section we will confirm that Eq. 57 is the correct lagrangian by considering magnetic fields.

6.2 Magnetic fields on D-branes

We will repeat a similar analysis with a magnetic field this time instead of an electric field. We will see that Dp-branes carrying magnetic fields correspond to rotations rather than...
boosts of the D(p-1) branes in the dual world. We have already computed the boundary conditions for a magnetic field, $F_{34} = -F_{43} = B$, in (Eq. 38). We will stick with these boundary conditions and just as we did for electric fields, we can re-write them in terms of $B = 2\pi \alpha' B$.

$$\partial_\sigma X^3 = B \partial \tau X^4$$  \hspace{1cm} (58a)  
$$\partial_\sigma X^4 = -B \partial \tau X^3$$  \hspace{1cm} (58b)

In terms of $\partial_+$ and $\partial_-$, our boundary conditions take a similar form as (Eq. 41) up to some sign differences:

$$\partial_+ \begin{bmatrix} X^3 \\ X^4 \end{bmatrix} = \begin{bmatrix} 1 - B^2 \\ + \frac{B^2}{1 + B^2} \end{bmatrix} \partial_- \begin{bmatrix} X^3 \\ X^4 \end{bmatrix}. \hspace{1cm} (59)$$

The $x^4$ direction is compacted and the magnetic field covers the cylinder that it forms with the $x^3$ direction. We take its dual so Dirichlet boundary conditions are imposed,

$$\partial_+ \begin{bmatrix} X^3 \\ X^4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \partial_- \begin{bmatrix} X^3 \\ X^4 \end{bmatrix}. \hspace{1cm} (60)$$

Here, $S'$ is the rotated frame of reference. The transformation from $S \rightarrow S'$ by angel $\theta$ is as follows:

$$\begin{bmatrix} X'^3 \\ X'^4 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} X^3 \\ X^4 \end{bmatrix} = R \begin{bmatrix} X^3 \\ X^4 \end{bmatrix}. \hspace{1cm} (61)$$

Recovering the boundary conditions from (Eq. 54),

$$\partial_+ \begin{bmatrix} X^3 \\ X^4 \end{bmatrix} = R_{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} R \partial_- \begin{bmatrix} X^3 \\ X^4 \end{bmatrix}, \hspace{1cm} (62)$$

and detting back from the dual ($\partial_- X^4 = -\partial_- \tilde{X}^4$)

$$\partial_+ \begin{bmatrix} X^3 \\ X^4 \end{bmatrix} = R_{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} R \begin{bmatrix} 1 \\ 0 \end{bmatrix} \partial_- \begin{bmatrix} X^3 \\ X^4 \end{bmatrix}. \hspace{1cm} (63)$$

We see the boundary conditions we started with in (Eq.60) are expressed in terms of rotations is given by

$$\partial_+ \begin{bmatrix} X^3 \\ X^4 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \partial_- \begin{bmatrix} X^3 \\ X^4 \end{bmatrix}. \hspace{1cm} (64)$$

We can solve for $B$ by looking at the diagonal terms first,

$$\frac{1 - B^2}{1 + B^2} = \cos 2\theta. \hspace{1cm} (65)$$
Solving this give us the relationship \( B = \pm \tan \theta \). By examining the off-diagonal components we see that the correct value is

\[
B = 2\pi \alpha' = -\tan \theta. \tag{66}
\]

It is interesting to look at this problem from the perspective of electromagnetic potentials that we studied in section 3. One possible configuration which we can extract our B field is to consider \( A_4 = Bx^3 \) and \( A_3 = 0 \). Since \( x^4 \) is periodic, we know from (Eq.20) \( A_4 \sim A_4 + \frac{n}{R_4} \), so the potential in that direction remains invariant under quantized leaps. However, the defined potential increases as you move along the cylinder in the \( x^3 \) direction,

\[
\Delta A_4 = B\Delta x^3 = \frac{n}{R_4}. \tag{67}
\]

In the dual world this becomes even more interesting; first we let \( n \to -n \) and see that

\[
\Delta x^3 = -\frac{n\hat{R}_4}{\alpha'B} = \frac{2\pi n\hat{R}_4}{2\pi \alpha'B} = \frac{2\pi n\hat{R}_4}{\tan \theta}. \tag{68}
\]

When \( n=1 \), \( \Delta x^3 \) represents the smallest area along the cylinder the brane wraps around \( x^4 \) once. Now we have this periodic structure along the \( x^3 \), which looks like candy cane stripes in this configuration. We now have the opportunity to enrich our toroidal structure by making \( x^3 \) periodic as well,

\[
2\pi R_3 = \Delta x^3 = \frac{2\pi \hat{R}_4}{\tan \theta}. \tag{69}
\]

We now have an expression for the angle,

\[
\tan \theta = n \frac{\hat{R}_4}{R_3}, \tag{70}
\]

which is quantized. We can argue this by showing the magnetic flux ,\( \Phi \), along our 2-torus is quantized. The magnetic flux in the original world is,

\[
\Phi = B\Delta A = B(2\pi R_3)(2\pi R_4). \tag{71}
\]

Writing this expression in terms of \( \hat{R}_4 \) and using our valued obtained from B we see that,

\[
\Phi = -\frac{\tan \theta}{2\pi \alpha'} \frac{2\pi \alpha'}{\hat{R}_4} 2\pi R_3 = -2\pi n. \tag{72}
\]

So far we have kept the magnetic field constant. If we consider a magnetic field that is uniform but not constant, in the dual picture the D(p-1)-brane will trace out a path that is curved instead of a straight line.
Figure 3: The figure to the left shows the D(p-1) brane wrapping around $x^4$ while traveling along $x^3$. As $a \rightarrow a'$, $b \rightarrow b'$, and $c \rightarrow c'$ we get the configuration to the right, illustrating a possible way they can deform into each other provided by Ref.[4]

A tilted D(p-1)-brane that winds around $x^4$ n times before it takes one lap around $x^3$ is homologically equivalent to one D(p-1)-brane that only wraps in the $x^3$ direction and n D(p-1)-branes that only wrap in the $x^4$ direction as shown in Fig.[3]. They are not physically equivalent but only through deformations. We can study its dual, noting that the $x^3$ does possess T-duality; we find the transformation sends the D(p-1)-brane wrapped in $x^3$ to a Dp-brane covering the cylinder, and the n D(p-1)-branes circling $x^4$ become n D(p-2)-branes that live on the world volume of the Dp-brane.

There is a nice physical picture to help us understand. We have just realized, for example, that a D2-brane with constant magnetic field of flux $2\pi n$ is deformation equivalent to a D2-brane with n D0-branes on its world volume. Of course to get back from the latter to the former we just have the D0-brane dissolve into the D2-brane. Physically speaking the D2-brane represents a vanishing magnetic field everywhere except at the positions of D0-branes, where the magnetic field becomes infinite with a finite flux.

Using the newly constructed 2-torus, we find the brane volume of our D(p-1)-brane stretched along the diagonal of the torus is just the volume $V_{p-2}$ multiplied by the diagonal length. Our lagrangian is just the negative rest energy of the brane.

$$L = V_{p-2} L_{\text{diag}} T_{p-1}(\tilde{g}) = -V_{p-2}\sqrt{(2\pi R_3)^2 + (2\pi \tilde{R}_4)^2 T_{p-1}(\tilde{g})} \tag{73}$$

Using (Eq.70), we find that $2\pi \alpha' B = -\frac{\tilde{R}_4}{\tilde{R}_3}$. Taking the T-dual of Eq. 73 we obtain,

$$L = -V_{p-2}(2\pi R_3)\sqrt{1 + (2\pi \alpha' B)^2}(2\pi R_4)T_p(g). \tag{74}$$

Since $(2\pi R_3)(2\pi R_4)$ is just the volume of the torus wrapped by the Dp-brane, we divide by the total volume to get out lagrangian density

$$\mathcal{L} = -T_p(g)\sqrt{1 + (2\pi \alpha' B)^2}, \tag{75}$$

confirming that Eq.72 is the correct Lagrangian density.
7 Conclusion and acknowledgements

We have discussed a lot of astonishing consequences of string theory. As a recap, in studying closed strings we saw a relationship emerge between compact dimensions with radius $R$ and dimensions with inverse quantity $\frac{1}{R}$. The boundary conditions for an open string interchange upon swapping its radius with its inverse. This means an open string free to move on a compact dimension becomes fixed at one point in the dual world. This point is determined by the gauge potential carried by the brane that's wrapped in the compact dimension. By considering a constant electric field along the brane in the compact dimension, the potential must vary with time, thus the position of the brane in the dual world must vary with time, giving rise to a velocity. The velocity is determined by the strength of the electric field and vice versa, thus a maximum velocity implies a maximum electric field. This fact led to the insight that branes can be studied on their own as a dynamical and gravitating objects, governed by the Born-Infeld theory of electrodynamics. A magnetic field along a compact dimension tilts the brane in the dual world by an angle determined by the magnetic field strength. The length which the brane travels before making a full loop around the circular dimension is periodic and by making it compact we find that the magnetic flux along the area is quantized.

Aside from the impressive role T-duality in bosonic string theory, its theoretical strength becomes more profound due to its ability to relate different types of superstring theories. For example, Type II A superstring theory and Type II B superstring theory become exchanged under T-duality. This relationship led to a more fundamental theory of nature called M-theory, which holds all 5 superstring theories by this relation \[2\]. There are many things I left out of the discussion, one of them is the self-dual radius when $R = \sqrt{\alpha'}$, which is the minimum length scale for string theory. This adds geometric structure to the background at that distance. Ref. [5] describes that non-commutative closed string theories are T-dual to a commutative closed string theory.

I would like to give special thanks to my mentor, Dr. K.V. Shajesh at Southern Illinois University-Carbondale, for encouraging and guiding my independent study of string theory during the 2017 fall semester of undergraduate studies. I would like to thank Dr. Savdeep Sethi, for giving me the privilege to sit in on his graduate level string theory course during my time in "limbo," and his graduate student, Christen Ferko, who was willing to take time out of his studies to answer any physics question I had with great enthusiasm.

8 References