Grading.
The maximum score on this problem set was 10 points · 6 problems = 60 points. (Here I count “problem 2” which includes Polchinski 2.8 - 2.10, as three separate problems.)

Email cferko@uchicago.edu with questions or corrections.

Exercise 1. Polchinski 2.3

The expectation value of a product of exponential operators on the plane is

\[ \prod_{i=1}^{n} : e^{ik_i X(z_i, z)} : = iC^X (2\pi)^D \delta^D \left( \sum_{i=1}^{n} k_i \right) \prod_{i,j=1}^{n} |z_{ij}|^{\alpha' k_i k_j}. \quad (1) \]

with \( C^X \) a constant. This can be obtained as the limit of the expectation value (6.2.17) on the sphere, which we will obtain by several methods in chapter 6.

(a) Show that the leading behavior as one vertex operator approaches another is in agreement with the OPE (2.2.14).

(b) As implied by eq (2.2.13), the expectation value is \( |z_{12}|^{\alpha' k_1 k_2} \) times a function that is smooth as \( z_1 \to z_2 \). For \( n = 3 \), work out the explicit expansion of this smooth function in powers of \( z_{12} \) and \( z_{12} \). From the ratio of the large-order terms find the radius of convergence.

(c) Give a general proof that the free field OPE (2.2.4) is convergent inside the dashed circle in figure 2.1

Solution 1.

(a) For reference, Polchinski’s (2.2.14) is

\[ : e^{ik_1 X(z, z)} : Z e^{ik_2 X(0, 0)} := |z|^{\alpha' k_1 k_2} : e^{i(k_1 + k_2) X(0, 0)} [1 + \mathcal{O}(z, \bar{z})] :. \quad (2) \]

Of course, we can multiply either side of (2) by a product of exponential operators and take expectation values to find

\[ \left< : e^{ik_1 X(z, z)} : e^{ik_2 X(0, 0)} \left( \prod_{i=3}^{n} e^{ik_i X(z_i, z_i)} : \right) \right> = \left< : e^{ik_1 X(z_1, z)} : e^{ik_2 X(0, 0)} [1 + \mathcal{O}(z, \bar{z})] \left( \prod_{i=3}^{n} e^{ik_i X(z_i, z_i)} : \right) \right>. \quad (3) \]

We wish to check whether the claimed expectation value (1) is consistent with (3) to leading order when \( |z| \to 0 \).

To facilitate comparison, we will change notation slightly in (1): call our first coordinate \( z_1 \equiv z \), \( \bar{z}_1 \equiv \bar{z} \), and take our second coordinate to lie at \( z_2 = \bar{z} = 0 \). Then (1) reads

\[ \left< : e^{ik_1 X(z, z)} : \prod_{i=3}^{n} e^{ik_i X(z_i, z_i)} : \right> = iC^X (2\pi)^D \delta^D \left( \sum_{i=1}^{n} k_i \right) \prod_{i,j=1}^{n} |z_{ij}|^{\alpha' k_i k_j}. \quad (4) \]

On the right side of (4), we extract the factor \( |z_2 - z_2|^{\alpha' k_1 k_2} \equiv |z|^{\alpha' k_1 k_2} \) which is leading:

\[ \left< : e^{ik_1 X(z, z)} : \prod_{i=3}^{n} e^{ik_i X(z_i, z_i)} : \right> = \delta z |z|^{\alpha' k_1 k_2} C^X (2\pi)^D \delta^D \left( \sum_{i=1}^{n} k_i \right) \prod_{i,j=1}^{n} |z_{ij}|^{\alpha' k_i k_j} \prod_{i=3}^{n} \left( |z_i - z|^{\alpha' k_i k_1} |z_i|^{\alpha' k_i k_2} \right). \quad (5) \]
In the limit as \( z_1 \to z_2 \), which in our conventions means \( z \to 0 \), the factor of \( |z_i - z|^\alpha k_i k_i \) in the second line of (5) can be replaced by \( |z_i|^\alpha k_i k_i \) to required order. But then equation (5) becomes

\[
\left\langle : e^{i k_1 X(z,z)} : e^{i k_2 X(0,0)} : \prod_{i=3}^{n} e^{i k_i X(z_i,z_i)} : \right\rangle = |z|^{\alpha' k_1 k_2}.
\]

\[
C^X (2\pi)^D \delta^D \left( \sum_{i,j=1}^{n} |z_{ij}|^{\alpha' k_i k_j} \prod_{i=3}^{n} |z_i - z|^\alpha k_i (k_i + k_2), \right)
\]

where we can again appeal to equation (1) to write the underbraced expression as an expectation value of exponential operators.

Interpreted as an operator statement, equation (6) tell us that we may treat

\[
\left\langle \prod_{i=1}^{3} e^{i k_i X(z_i,z_i)} : \right\rangle = i C^X (2\pi)^D \delta^D (k_1 + k_2 + k_3) |z_12|^{\alpha' k_1 k_2} |z_{13}|^{\alpha' k_1 k_3} |z_{23}|^{\alpha' k_2 k_3}.
\]

We would like to expand the right side of (8) in powers of \( z_{12} \) and \( z_{12} \) in the limit as \( z_1 \to z_2 \) (and hence \( z_{12} \to 0 \)). At the moment, this is complicated by the presence of \( z_{13} \) in (8). Thankfully, we may write

\[
z_{13} = z_1 - z_3 = (z_1 - z_2) + (z_2 - z_3) = z_{12} + z_{23},
\]

then extract a factor of \( z_{23} \) from \( |z_{13}| \) and use that \( |w| = \sqrt{ww} \) to convert the moduli into products as follows:

\[
|z_{12}|^{\alpha' k_1 k_2} |z_{13}|^{\alpha' k_1 k_3} |z_{23}|^{\alpha' k_2 k_3} = |z_{12}|^{\alpha' k_1 k_2} |z_{12} + z_{23}|^{\alpha' k_1 k_3} |z_{23}|^{\alpha' k_2 k_3}
\]

\[
= |z_{12}|^{\alpha' k_1 k_2} (1 + \frac{z_{12}}{z_{23}}) |z_{23}|^{\alpha' k_2 k_3 + \alpha' k_1 k_3}
\]

\[
= |z_{12}|^{\alpha' k_1 k_2 + \alpha' k_1 k_3} \left( 1 + \frac{z_{12}}{z_{23}} \right) |z_{23}|^{\alpha' k_1 k_3} \left( 1 + \frac{z_{12}}{z_{23}} \right) |z_{23}|^{\alpha' k_2 k_3 + \alpha' k_1 k_3}
\]

Next, we are asked to find the explicit expansions of the factors that look like \( (1 + \bullet)^n \) in powers of \( z_{12} \). The first term in this expansion is our familiar approximation

\[
(1 + \bullet)^n = 1 + n \bullet + \cdots,
\]

but the expression for the higher terms hidden in “...” may be less familiar (at least, they were to me). Wikipedia tells me one has the general expansion

\[
(x + y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k = x^r + r x^{r-1} y + r(r - 1) x^{r-2} y^2 + \frac{r(r - 1)(r - 2)}{3!} x^{r-3} y^3 + \cdots,
\]
where \( \binom{r}{k} \) is a generalization of the usual definition of the binomial coefficient when \( r \) and \( k \) are both integers, namely
\[
\binom{r}{k} = \frac{r(r-1) \cdots (r-k+1)}{k!} = \frac{(r)_k}{k!}.
\] (13)

I am told that \((r)_k\) is something called the Pochhammer symbol.\(^1\)

Using this result, one can write
\[
\left(1 + \frac{z_{12}}{z_{23}}\right)^{\alpha' k_1 k_3} = \sum_{k=0}^{\infty} \frac{(\alpha' k_1 \cdot k_3)_k}{k!} \left(\frac{z_{12}}{z_{23}}\right)^k,
\] (14)

and likewise for the factor involve \(z_{12}\). Thus the expectation value in equation (8) has the expansion
\[
\left\langle \prod_{i=1}^{3} e^{i k_i X(z_i, z_i)} : \right\rangle = i C_X (2\pi)^D \delta^D (k_1 + k_2 + k_3) [z_{12}]^{\alpha' k_1 k_2 + \alpha' k_1 k_3} [z_{23}]^{\alpha' k_2 k_3 + \alpha' k_1 k_3}.
\] (15)

Equation (15) gives the explicit expansion of the smooth function in powers of \(z_{12}\) and \(z_{12}\), as desired.

Finally, we would like to consider the ratio of the large-order terms in order to compute the radius of convergence. The ratio between the coefficient \(a_{k+1}\) of the \((k+1)\)-th term in the series and the coefficient \(a_k\) of the \(k\)-th term in the series is
\[
a_{k+1} = \left(\frac{(\alpha' k_1 k_3)_{k+1}}{(\alpha' k_1 k_3)_k}\right) \left(\frac{1}{z_{23}}\right)^{k+1} \left(\frac{1}{z_{23}}\right)^k
= \frac{1}{(k+1)z_{23}} \cdot [((\alpha' k_1 \cdot k_3) \cdots ((\alpha' k_1 \cdot k_3) - (k+1))] \cdot [((\alpha' k_1 \cdot k_3) \cdots ((\alpha' k_1 \cdot k_3) - k)]^{-1}
= \frac{1}{z_{23}} \frac{\alpha' k_1 \cdot k_3 - k - 1}{k+1}.
\] (16)

By the ratio test, we expect the series to converge when the ratio \(L = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| < 1\). But we see that
\[
L = \lim_{k \to \infty} \left| \frac{\alpha' k_1 \cdot k_3 - k - 1}{k+1} \right| = \frac{1}{|z_{23}|},
\] (17)

which means that the radius of convergence for the first sum is \(|z_{23}|\). By a completely analogous argument, the radius of convergence of the second sum is \(|z_{23}|\), so we conclude that the radius of convergence for the entire expectation value (15) is \(|z_{23}|\).

(c) Equation (2.2.4) is
\[
X^\mu \left(z_1, \tilde{z}_1\right) X^\nu \left(z_2, \tilde{z}_2\right) = -\frac{\alpha'}{2} \eta^{\mu \nu} \log \left(|z_{12}|^2\right)
+ \sum_{k=1}^{\infty} \frac{1}{k!} \left[ (z_{12})^k : X^\nu \partial^k X^\mu \left(z_2, \tilde{z}_2\right) : + (\tilde{z}_{12})^k : X^\nu \partial^k X^\mu \left(z_2, \tilde{z}_2\right) :ight]
\] (18)

To be somewhat more explicit, equation (18) is an operator statement. This means that it should hold inside any time-ordered correlation function with an arbitrary string of operators tacked on.

\(^1\)This is rather fun to say, so I will use the notation \((r)_k\) freely in what follows.
at the end:

$$
\langle X^\mu (z_1, \bar{z}_1) X^\nu (z_2, \bar{z}_2) \mathcal{O}_1(w_1) \cdots \mathcal{O}_n(w_n) \rangle = -\frac{\alpha'}{2} \eta^{\mu\nu} \log \left( |z_{12}|^2 \right) \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \\
+ \sum_{k=1}^{\infty} \frac{1}{k!} \left( (z_{12})^k \langle : X^\nu \partial^k X^\mu (z_2, \bar{z}_2) : \mathcal{O}_1(w_1) \cdots \mathcal{O}_n(w_n) \rangle \\
+ (\bar{z}_{12})^k \langle : X^\nu \partial^k X^\mu (z_2, \bar{z}_2) : \mathcal{O}_1(w_1) \cdots \mathcal{O}_n(w_n) \rangle \right).
$$

Equation (19) is a series expansion in powers of $z_{12}$ and $\bar{z}_{12}$ whose coefficients are expectation values involving only $z_2$, $\bar{z}_2$, and the $w_i$. By rotation and scale invariance, we know that such a correlation function can only depend on the distances between the points:

$$
\langle : X^\nu \partial^k X^\mu (z_2, \bar{z}_2) : \mathcal{O}_1(w_1) \cdots \mathcal{O}_n(w_n) \rangle = f_k (|w_i - z_2|, |w_i - w_j|).
$$

In fact, we can constrain the functional form of $f$ a bit more. Suppose the operators $\mathcal{O}_i$ have weights $\hat{h}_i, \bar{\hat{h}}_i$, and define their scaling dimensions $\Delta_i = h_i + \bar{h}_i$. Under a re-scaling of coordinates $z \to \lambda z, \bar{z} \to \lambda \bar{z}$, the operators therefore transform as $\mathcal{O}_i \to \mathcal{O}_i = \lambda^{-\Delta_i} \mathcal{O}_i$. The right side of (20) respects this scaling property only if

$$
f_k (|w_i - z_2|, |w_i - w_j|) = \prod_{i,j \neq i} |w_i - z_2|^{-k - \Delta_i} |w_i - w_j|^{-\Delta_i - \Delta_j} g_k(w_i, z_2),
$$

where $g(w_i, z_2)$ is some conformally invariant function, and $k$ is the scaling dimension of the operator $X^\nu \partial^k X^\mu$.

For instance, when there are three operators $\mathcal{O}_i$, this $g_k$ can be any function of the “conformal cross ratios” $u$ and $v$:

$$
u = \frac{|w_1 - z_2|^2 |w_3 - w_2|^2}{|w_1 - w_3|^2 |w_2 - w_2|^2}.
$$

For the case at hand, though, we are only asked to prove convergence of the OPE in the situation of figure 2.1.

In the figure, there are only four operator insertions total, so our functions $f_k$ can only be functions of $|w_1 - z_2|$ and $|w_2 - z_2|$. As there are no conformally invariant cross ratios in the three-point case\(^2\), the functions $f_k$ are fixed up to constants as

$$
f_k (|w_i - z_2|, |w_i - w_j|) = c_k |w_1 - z_2|^{-k - \Delta_1} |w_2 - z_2|^{-k - \Delta_2} |w_2 - w_1|^{-\Delta_2 - \Delta_1}.
$$

Now we can repeat the argument of part (b). Ignoring the piece of the coefficients in (19) that are constant in $k$, the large-order terms are of the form

$$
(z_{12})^k |w_i - z_2|^{-k} = \left( \frac{z_{12}}{|w_i - z_2|} \right)^k.
$$

\(^2\) I suspect that the arguments of this section can be extended to show that an arbitrary OPE converges with radius of convergence equal to the minimum distance to another operator insertion, but I can’t see how to do it directly without knowing more about the functions $g_k$. 

PHYS 483: String Theory I Problem Set 2 Solutions
The limit of the norm of the ratio between successive terms, as \( k \) grows to infinity, will therefore be less than 1 so long as

\[
    z_{12} < \min \left( |w_1 - z_2|, |w_2 - z_2| \right).
\]

This is the statement that the OPE converges within the dashed line of figure 2.1, as desired.

As an aside, there is a completely different proof of OPE convergence (which is more general than the one above, since it works for an arbitrary number of operator insertions) outlined in section 2.9 of Polchinski, and summarized in a somewhat easier-to-parse way in section 2 this. The idea is to cut out a sphere around the middle operator, separated from the other insertions, apply the state-operator correspondence, and expand the states you get out of this correspondence in a basis of energy eigenstates. Convergence of the OPE then follows from convergence of scalar products in Hilbert spaces.

**Exercise 2.** Polchinski 2.8-2.10

(a) (Pol 2.8) What is the weight of \( f_{\mu \nu} : \partial X^\mu \partial X^\nu e^{ikX} : \)? What are the conditions on \( f_{\mu \nu} \) and \( k_\mu \) in order for it to be a tensor?

(b) (Pol 2.9) Derive the central charges for the linear dilaton, \( b, c \), and \( \beta \gamma \) CFTs by working out the TT OPEs.

(c) (Pol 2.10) Consider now a patch of world-sheet with boundary. For convenience suppose that the patch lies in the upper half-\( w \)-plane, with the real axis being the boundary. Show that the expectation value of a normal-ordered operator, say \( : \partial X^\mu(z) \partial X^\nu(z) : \), although finite in the interior, diverges as \( z \) approaches the boundary (represent the effect of the boundary by an image charge). Define boundary normal ordering \( :: \), which is the same as \( :: \) except that the contraction includes the image charge piece as well. Operators on the boundary are finite if they are boundary normal ordered.

In a general CFT on a manifold with boundary, the interior operators and the boundary operators are independent. Label the basis \( \mathcal{A} \) and \( \mathcal{B} \) respectively, and define the former by \( :: \) and the latter by \( :: \). Each set has its own closed OPE: \( \mathcal{A} \mathcal{A} \to \mathcal{A} \) and \( \mathcal{B} \mathcal{B} \to \mathcal{B} \). In addition, the sets are related: \( \mathcal{A} \), as it approaches the boundary, can be expanded in terms of the \( \mathcal{B} \). Find the leading behaviors of

\[
    : e^{ik_1 X(y_1)} e^{ik_2 X(y_2)} : , \quad y_1 \to y_2 \ (y \text{ real}),
\]

\[
    : e^{ik X(z, z)} : , \quad \text{Im}(z) \to 0.
\]

The identity (2.7.14) is useful for the latter.

**Solution 2.**

(a) We will first compute the OPE of the operator \( V \equiv f_{\mu \nu} : \partial X^\mu \partial X^\nu e^{ikX} : \) with the free-field stress tensor. From this we can answer both questions: the weight \( h \) can be extracted from the \( z^{-2} \) term in the OPE, while the conditions for \( V \) to be a tensor will be chosen so that the OPE has no term of order \( z^{-3} \) or worse.

As a first step, we will compute the OPE between \( \partial X^\mu \) and \( e^{ikX(w)} \).

\[
    \partial X^\mu(z) e^{ikX(w)} = \sum_{n=1}^{\infty} \frac{1}{n!} \partial X^\mu(z) (ik_\nu X^\nu(w))^n.
\]

As per Wick’s theorem, we must consider all possible contractions of \( \partial X^\mu(z) \) with one of the factors \( X^\nu(w) \). In each term of the sum, there are of course \( n \) such ways to perform such a contraction, each of which generates

\[
    \partial X^\mu(z) X^\nu(w) = \partial_z \left( X^\mu(z) X^\nu(w) \right)
    = \partial_z \left( -\frac{\alpha'}{2} n^\mu \log(z - w) \right)
    = -\frac{\alpha'}{2} n^\mu \frac{z - w}{z - w}.
\]
Using this in equation (26), we find

\[ \partial X^\mu(z)e^{ikX(w)} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left( \frac{\alpha'}{2} \right) \left( \frac{i k_\nu \eta^{\mu\nu}}{z-w} \right) (ik_\nu X^\nu(w))^{n-1} \]
\[ = -\frac{i\alpha'}{2} k^\mu \left( \frac{1}{z-w} \right) e^{ik_\nu X^\nu(w)}. \quad (28) \]

With that warm-up complete, we consider the full OPE with the stress tensor:

\[ -2f_{\mu\nu} : \partial X^\mu(z) \partial X_\rho(z) :: \partial X^\mu(w) \bar{\partial} X^\nu(w) e^{ikX(w)} :. \quad (29) \]

There will be several types of terms:

1. One of the \( \partial X(z) \) contracted against \( \partial X(w) \).
2. One of the \( \partial X(z) \) contacted against \( e^{ikX(w)} \).
3. One of the \( \partial X(z) \) contracted against \( \partial X(w) \) and the other \( \partial X(z) \) contracted against \( e^{ikX(w)} \).
4. Both of the \( \partial X(z) \) contracted against the \( e^{ikX(w)} \).

Let us compute each contribution in turn.

There are 2 ways to obtain contraction (1) because of the two \( \partial X(z) \) factors, each of which yields \( \partial X^\rho(z) \partial X^\mu(w) = -\frac{\alpha' \eta^{\mu\rho}}{(z-w)^2} \), so from (1) we get

\[ -2f_{\mu\nu} \left( -\frac{\alpha'}{2} \right) \left( \frac{1}{(z-w)^2} \right) \eta^{\mu\rho} \partial_\rho X(z) \bar{\partial} X^\nu(w) e^{ikX(w)} \]
\[ = \frac{\alpha' f_{\mu\nu}}{(z-w)^2} \partial^\mu X(z) \bar{\partial} X^\nu(w) e^{ikX(w)}. \quad (30) \]

Using our intermediate result (28), the contribution of type (2) – which again comes with a combinatorial factor of 2, since we may choose either of the two \( \partial X^\rho(z) \) factors to contract – will give

\[ -2f_{\mu\nu} \left( -\frac{i\alpha'}{2} k^\rho \left( \frac{1}{z-w} \right) e^{ik_\nu X^\nu(w)} \right) \partial X_\rho(z) \partial X^\mu(w) \tilde{\partial} X^\nu. \quad (31) \]

The contractions of type (3) will be the most singular, since we will get a \( (z-w)^{-2} \) from the \( \partial X(z) \partial X(w) \) contraction and another factor \( (z-w)^{-1} \) from the \( \partial X(z) e^{ikX(w)} \) contraction. Explicitly, we find

\[ -2f_{\mu\nu} \left( -\frac{i\alpha'}{2} k_\rho \left( \frac{1}{z-w} \right) e^{ik_\nu X^\nu(w)} \right) \left( -\frac{\alpha' \eta^{\mu\rho}}{(z-w)^2} \right) \partial X^\nu(w) \]
\[ = i\alpha'^2 f_{\mu\nu} k^\nu \left( \frac{1}{(z-w)^3} \right) \partial X^\nu(w) e^{ik\alpha X^\alpha(w)}. \quad (32) \]

If the operator \( V \) is to be primary, this term (32) must vanish, which gives us the “polarization condition”

\[ k^\nu f_{\mu\nu} = 0. \quad (33) \]

Later we will see that this has the interpretation of removing transverse polarizations for the graviton, dilaton, and Kalb-Ramond field in the closed string sector (the operator \( V \), once integrated over the worldsheet, is associated with massless closed string states).

Finally, terms of type (4) will give contributions obtained by iterating our result (28). These look like

\[ \frac{f_{\mu\nu} \alpha' k^\rho k_\rho}{4(z-w)^2} \partial X^\mu(z) \bar{\partial} X^\nu(w) e^{ikX(w)}. \quad (34) \]
Combining the contributions of type (1) and type (4) gives the total term in the OPE proportional to \((z - w)^{-2}\), namely
\[
\frac{1 + \alpha' k^2}{(z - w)^2} V.
\]
(35)
From this we read off the weights of \(V\):
\[
(h, \bar{h}) = \left(1 + \frac{\alpha' k^2}{4}, 1 + \frac{\alpha' k^2}{4} \right).
\]
(36)
Indeed, we could have obtained the result (36) immediately from Polchinski’s equation (2.4.19), which tells us that a general product of an exponential with some derivatives,
\[
: \left( \prod_i \partial^{m_i} X^{\mu_i} \right) \left( \prod_j \bar{\partial}^{n_j} X^{\nu_j} \right) e^{i k \cdot X} :,
\]
(37)
has weight
\[
(h, \bar{h}) = \left( \frac{\alpha' k^2}{4} + \sum_i m_i, \frac{\alpha' k^2}{4} + \sum_j n_j \right).
\]
(38)
Our operator \(f_{\mu \nu} \partial X^{\mu} \bar{\partial} X^{\nu} e^{i k \cdot X}\) is a sum of terms of the form (37) so that \(\sum_i m_i = 1 = \sum_j n_j\).

(b) We begin with the linear dilaton. The stress tensor is given in Polchinski’s equation (2.5.1a) on page 49 as
\[
T(z) = \frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : + V_\mu \partial^2 X^\mu,
\]
(39)
and likewise for \(\bar{T}(\bar{z})\) (replacing all \(\partial\)'s with \(\bar{\partial}\)'s).

We are interested in the OPE for \(T(z)T(w)\), which has the form
\[
T(z)T(w) = \frac{1}{\alpha'^2} : \partial_\sigma X^\mu(\sigma)\partial_\tau X_\mu(\tau) : + \partial_\sigma X^\nu(\sigma)\partial_\tau X_\nu(\tau) : + \partial_\sigma X^\nu(\sigma)\partial_\tau X_\rho(\tau) : + \partial_\sigma X^\nu(\sigma)\partial_\rho X_\nu(\tau) : + \partial_\rho X^\nu(\sigma)\partial_\tau X_\rho(\tau) : + \partial_\sigma X^\nu(\sigma)\partial_\tau X_\rho(\tau) :
\]
(40)
where we will eventually take \(w \to 0\) for simplicity.

Polchinski’s equation (2.25) tells us how to convert from a string of time-ordered operators to a string of normal-ordered operators: we apply Wick’s theorem and sum over all ways of contracting pairs of operators and replacing each pair \(X^\mu(z)X^\nu(w)\) with the propagator \(\frac{1}{2} \eta^{\mu \nu} \log \left| z - w \right|^2\).

In (40), we will need three types of contractions.

When we contract the two normal-ordered terms : \(\partial_\sigma X^\mu(\sigma)\partial_\tau X_\mu(\tau) : \) and : \(\partial_\sigma X^\nu(\sigma)\partial_\tau X_\nu(\tau) : \) against each other, we recover the usual term from the free scalar stress tensor OPE:
\[
: \partial_\sigma X^\mu(\sigma)\partial_\tau X_\mu(\tau) :: \partial_\rho X^\nu(\sigma)\partial_\tau X_\nu(\tau) :: = 2 \left( -\frac{\alpha' \eta^{\mu \nu}}{2} \frac{1}{(z - w)^2} \right)^2 - 4 \left( \frac{\alpha'}{2} \frac{\partial_\sigma X^\mu(\sigma)\partial_\tau X_\mu(\tau) :}{2} \frac{1}{(z - w)^2} \right),
\]
(41)
since there are two ways to contract both pairs of operators and four ways to contract one pair of operators, leaving the second un-contracted.

When we contract the two terms containing second derivatives, we find
\[
(V_\mu \partial^2 X_\mu(\sigma)) (V_\nu \partial^2 X_\nu(\tau)) = V_\mu V_\nu \partial_\sigma \partial_\tau \left( -\frac{\alpha'}{2} \eta^{\mu \nu} \frac{1}{(z - w)^2} \right) = 3 \alpha' V_\mu V_\nu \frac{1}{(z - w)^4},
\]
(42)
where again we have used the known contraction
\[
\overbrace{\partial_z X^\mu(z)} \partial_w X^\nu(w) = -\frac{\alpha'}{2} \eta^{\mu
u} \frac{1}{(z-w)^2},
\]
(43)
and taken two more derivatives to arrive at (42).

The contraction we will need for the cross term is of the form
\[
: \partial_z X^\mu(z) \partial_z X_\mu(z) : \partial^2 X^\nu(w),
\]
(44)
which can be contracted in two ways: the \(\partial^2 X^\nu(w)\) either contracts against the first factor \(\partial_z X^\mu(z)\), or against the second factor \(\partial_z X_\mu(z)\). In either case, the result can be found by taking one more derivative in (43):
\[
\overbrace{\partial_z X^\mu(z) \partial^2_w X^\nu(w)} = \alpha' \eta^{\mu\nu} \frac{1}{(z-w)^3}.
\]
(45)
Applying this to the expression (40), we find
\[
T(z)T(w) = \frac{1}{\alpha'} \frac{D + 6\alpha' V^2}{z^4} + O(z^4),
\]
(46)
from which we conclude that the central charge is
\[
c = D + 6\alpha' V^2.
\]
(47)
Because the antiholomorphic component \(\tilde{T}(z)\) of the stress tensor has exactly the same form, after replacing \(\partial\)'s with \(\bar{\partial}\)'s, a completely symmetric calculation gives
\[
\tilde{c} = c = D + 6\alpha' V^2.
\]
(48)

Next we turn to the \(bc\) and \(\beta\gamma\) ghost CFTs. In fact, we can treat both of these systems in one fell swoop by leaving the statistics of the two particles as a free variable. More precisely, consider a pair of holomorphic fermions \(b\) and \(c\) with conformal dimensions \(\lambda\) and \(1 - \lambda\), respectively, and with the OPE
\[
c(z)b(w) = \frac{1}{z-w} + \cdots, \quad b(z)c(w) = \frac{\epsilon}{z-w} + \cdots.
\]
(49)
When \(\epsilon = 1\), we have \(c(z)b(w) = -b(w)c(z)\), so the two ghost fields are anticommuting and we recover the usual \(bc\) ghost system. When \(\epsilon = -1\), we have \(c(z)b(w) = b(w)c(z)\), which is the \(\beta\gamma\) system. Thus we can solve both problems at once by leaving \(\epsilon\) undetermined.

Note that, in either case, the products \(c(z)c(w)\) and \(b(z)b(w)\) are regular.

The stress tensors for the \(bc\) system is given by equations (2.5.11a) and (2.5.11b) in Polchinski,
\[
T(z) = : (\partial b) c : -\lambda \partial : (bc :),
\]
\[
\bar{T}(\bar{z}) = 0.
\]
(50)
while that of the \(\beta\gamma\) system is given in equations (2.5.23a) and (2.5.23b) as
\[
T(z) = : (\partial \beta) \gamma : -\lambda \partial : (\beta \gamma :),
\]
\[
\bar{T}(\bar{z}) = 0.
\]
(51)
In terms of \(\epsilon\), and using the product rule, equations (50) and (51) can be combined as
\[
T(z) = -\lambda : b(z) \partial c(z) : + \epsilon (\lambda - 1) : c(z) \partial b(z) :,
\]
\[
\bar{T}(\bar{z}) = 0.
\]
(52)
As usual, we will compute the \(T(z)T(w)\) OPE and focus on the \((z-w)^{-4}\) term to extract the central charge. One finds
\[
T(z)T(w) = [-\lambda : b(z) \partial c(z) : + \epsilon (\lambda - 1) : c(z) \partial b(z) :] [-\lambda : b(w) \partial c(w) : + \epsilon (\lambda - 1) : c(w) \partial b(w) :].
\]
(53)
Finally, we must compute the contraction of the $b \partial c$ term in the first set of brackets with the $b \partial c$ term in the second set, we get

\[
(-\lambda : b(z) \partial c(z) :) (-\lambda : b(w) \partial c(w) :) = \lambda^2 \left[ b(z) \left( \partial_w c(w) \right) \partial_z \left( \frac{1}{z-w} \right) + \partial_z c(z) b(w) \partial_w \left( \frac{\epsilon}{z-w} \right) \right. \\
+ \left. \partial_w \left( \frac{1}{z-w} \right) \partial_z \left( \frac{\epsilon}{z-w} \right) \right] + \cdots \\
= \lambda^2 \left[ b(z) \left( -\partial_w c(w) \right) \frac{1}{(z-w)^2} + \partial_z c(z) b(w) \frac{\epsilon}{(z-w)^2} \right. \\
- \left. \frac{\epsilon}{(z-w)^2} \right] + \cdots \\
= \frac{\lambda^2}{(z-w)^2} \left( eb(w) \partial c(z) - b(z) \partial c(w) \right) - \frac{\epsilon \lambda^2}{(z-w)^4} + \cdots .
\]

(54)

When the $b \partial c$ term in the first set of brackets contracts with the $c \partial b$ term in the second set of brackets, we must confront a subtlety that was not present in the bosonic theory: when performing contractions, we must first commute the fermions fields past one another until they are adjacent.

For instance, in the following calculation, we will perform contractions like $b(z) \partial c(z) : c(w) \partial b(w) :$. To contract the $c(z)$ against the $\partial b(w)$, we must first commute it past the intervening $c(w)$. But this introduces a factor of $-\epsilon$, since $c(z)c(w) = -\epsilon c(w)c(z)$ (recall that $\epsilon = 1$ for anticommuting fields and $\epsilon = -1$ for commuting fields).

This will introduce extra factors of $-\epsilon$ whenever we need to perform an odd number of swaps. Explicitly,

\[
[-\lambda : b(z) \partial c(z) :] [\epsilon (\lambda - 1) : c(w) \partial b(w) :] = \lambda \epsilon (1 - \lambda) \left[ \frac{\epsilon (\epsilon - \epsilon)}{z-w} \partial c(z) \partial b(w) + b(z) c(w) \partial_z \partial_w \left( \frac{\epsilon}{z-w} \right) \right. \\
+ \left. \frac{\epsilon}{z-w} \partial_z \partial_w \left( \frac{\epsilon}{z-w} \right) \right] + \cdots \\
= \lambda \epsilon (1 - \lambda) \left[ \frac{\epsilon \partial c(z) \partial b(w)}{z-w} + 2 \frac{\epsilon b(z)c(w)}{(z-w)^3} + \frac{2 \epsilon}{(z-w)^4} \right].
\]

(55)

Here we have used that $\epsilon^2 = 1$, since $\epsilon = \pm 1$.

The term which arises when the $c \partial b$ term in the first set of brackets contracts against the $b \partial c$ term in the first set of brackets is similar to (55), although with $z$ and $w$ reversed. Since we are interested only in the $(z-w)^{-4}$ term, which is symmetric under the interchange $z \leftrightarrow w$, this gives another contribution of $\frac{2 \lambda \epsilon (1-\lambda)}{(z-w)^4}$.

Finally, we must compute the contraction of the $c \partial b$ term in the first set of brackets against the $c \partial b$ term in the second set of brackets. This gives

\[
[\epsilon (\lambda - 1) : c(z) \partial b(z) :] [\epsilon (\lambda - 1) : c(w) \partial b(w) :] = \epsilon^2 (\lambda - 1)^2 \left[ (\partial b(z)) c(w) \partial_w \left( \frac{1}{z-w} \right) \right. \\
+ c(z) (\partial b(w)) \partial_z \left( \frac{\epsilon}{z-w} \right) + \partial_z \left( \frac{\epsilon}{z-w} \right) \partial_w \left( \frac{1}{z-w} \right) \left. \right] \\
= \epsilon^2 (\lambda - 1)^2 \left[ \frac{\partial b(z)c(w)}{(z-w)^2} - \frac{\epsilon c(z) \partial b(w)}{(z-w)^2} - \frac{\epsilon}{(z-w)^4} \right].
\]

(56)

Since $\epsilon = \pm 1$, we have $\epsilon^2 = 1$ so the $(z-w)^{-4}$ contribution arising from (56) is simply $-\frac{\epsilon(\lambda-1)^2}{(z-w)^4}$. 

9
Adding the \((z - w)^{-4}\) contributions from the three flavors of contractions, we find

\[
T(z)T(w) = \frac{-\epsilon \lambda^2 + 4 \epsilon \lambda (1 - \lambda) - \epsilon (\lambda - 1)^2}{(z - w)^4} + \cdots
\]

\[
= \frac{-6 \epsilon \lambda^2 + 6 \epsilon \lambda - \epsilon}{(z - w)^4}.
\]  

We recall that the central charge is defined by \(T(z)T(w) = \frac{c/2}{(z - w)^2} + \cdots\), so with the correct factor of 2, our central charge is

\[
c = -2 \epsilon (6 \lambda^2 - 6 \lambda + 1)
\]

while the anti-holomorphic central charge vanishes, since \(\bar{T}(\bar{z}) = 0\):

\[
\bar{c} = 0.
\]  

More explicitly, letting \(\epsilon = 1\) for the \(bc\) system or \(\epsilon = -1\) for the \(\beta\gamma\) system, we find the two central charges

\[
c_{bc} = -2 (6 \lambda^2 - 6 \lambda + 1),
\]

\[
c_{\beta\gamma} = 2 (6 \lambda^2 - 6 \lambda + 1).
\]  

(c) First we aim to show that the expectation value of a normal-ordered operator, like \(\partial X^\mu(z)\overline{\partial X^\nu(\bar{z})}\), on the upper-half plane diverges near the real axis.

Physically, we are considering the worldsheet of an open string. This worldsheet can be most naturally presented as a strip, say \(-\infty < \tau < \infty\) and \(0 \leq \sigma < \pi\), but the strip can be conformally mapped onto the upper half-plane. After applying this conformal map, the endpoints of the string at \(\sigma = 0\) and \(\sigma = \pi\) are sent to the negative real axis and positive real axis, respectively.

For a string propagating in a spacetime with no branes, we expect the endpoints to satisfy Neumann boundary conditions. In the upper-half plane picture, this translates to asking the normal derivative of the \(X^\mu\) to vanish on the real axis.

Said differently, we would like to find a new two-point function \(\langle X^\mu(z, \bar{z})X^\nu(w, \bar{w}) \rangle\) for the upper half-plane, which still satisfies Polchinski’s equation (2.1.19),

\[
\partial \overline{\partial} \langle X^\mu(z, \bar{z})X^\nu(w, \bar{w}) \rangle = -\pi \alpha' \eta^{\mu\nu} \delta^2(z - w, \bar{z} - \bar{w}),
\]  

but now with Neumann boundary conditions at \(z = \bar{z}\).

Because equation (61) has the structure of the Poisson equation in electrodynamics, this problem is completely analogous to that of finding the electric potential in the upper half-plane with vanishing normal electric field on the real axis. We have seen that the way to answer this question is by placing an “image charge” at the point \(\bar{w}\) below the real axis which mirrors the real “charge” at \(w\).

Doing this, the new two-point function is

\[
\langle X^\mu(z, \bar{z})X^\nu(w, \bar{w}) \rangle_{\text{II}} = -\frac{\alpha'}{2} \eta^{\mu\nu} \left( \log(|z - w|^2) + \log(|z - \bar{w}|^2) \right),
\]  

where \(\text{II}\) means that this is the appropriate definition for the upper half-plane. We recall that our standard definition of normal-ordering subtracts off the expectation value using the definition for the entire plane, i.e.

\[
: X^\mu(z, \bar{z})X^\nu(w, \bar{w}) := X^\mu(z, \bar{z})X^\nu(w, \bar{w}) - \left( -\frac{\alpha'}{2} \eta^{\mu\nu} \log(|z - w|^2) \right).
\]  

Then we see that this definition of normal-ordering does not subtract off “all of the divergence” in the upper half-plane, as

\[
\langle : X^\mu(z, \bar{z})X^\nu(w, \bar{w}) : \rangle_{\text{II}} = -\frac{\alpha'}{2} \log(|z - \bar{w}|^2).
\]
Suppose we are near the real axis, so \( z \approx x \approx \bar{w} \) with \( x \) real; then we see that the logarithm in (64) is going to zero, so we still have a divergence near the real axis.

Motivated by this, we might define a new boundary normal ordering which subtracts off the divergence from the image charge:

\[
: X^\mu(z, \bar{z}) X^\nu(w, \bar{w}) : = X^\mu(z, \bar{z}) X^\nu(w, \bar{w}) - \left( -\frac{\alpha'}{2} \eta^{\mu\nu} \left( \log(|z-w|) + \log(|z-\bar{w}|) \right) \right). \tag{65}
\]

Near the real axis, where \( z = x \) and \( w = y \) are real, this becomes

\[
: X^\mu(x) X^\nu(y) : = X^\mu(x) X^\nu(y) + \alpha \eta^{\mu\nu} \log((x-y)^2). \tag{66}
\]

To compute the limiting behavior of a product of boundary normal ordered operators, like

\[
\exp(\alpha k_1 X(y_1) ; \exp(\alpha k_2 X(y_2)).
\]

I see two ways to proceed:

1. We can attempt to brute-force Wick’s theorem, using the modified contraction (66), or
2. We can appeal to the definition of ordering in terms of an exponential of functional derivatives in equation (2.7.14) of Polchinski.

Let’s try the second approach. Equation (2.7.14) tells us that, if we have two kinds of ordering,

\[
[X^\mu(z, \bar{z}) X^\nu(w, \bar{w})]_1 = [X^\mu(z, \bar{z}) X^\nu(w, \bar{w})]_2 + \eta^{\mu\nu} \Delta(z, \bar{z}, x, w, \bar{w}),
\]

then we can convert between orderings of a general operator \( \mathcal{F} \) via

\[
[\mathcal{F}]_1 = \exp \left( \frac{1}{2} \int d^2 z d^2 \bar{w} \delta \left( \frac{\delta}{\delta X^\mu(z, \bar{z})} \frac{\delta}{\delta X^\nu(w, \bar{w})} \right) \right) [\mathcal{F}]_2. \tag{69}
\]

This is especially easy to apply in the case \( \mathcal{F} = \exp(\alpha k_1 X(y_1)) \), since vertex operators are eigenfunctions of the functional derivative. Applying equation (69) between “boundary normal ordering” and “no normal ordering” gives us

\[
\exp(\alpha k_1 X(y_1) ; \exp(\alpha k_2 X(y_2)) = \exp \left( \alpha' k_1 \cdot k_2 \log((y_1-y_2)^2) \right) \exp(\alpha k_1 X(y_1) ; \exp(\alpha k_2 X(y_2)), \tag{70}
\]

while applying it to find the difference between “boundary normal ordering” and “usual normal ordering” gives

\[
\exp(\alpha' k z \cdot z) = \exp \left( \frac{\alpha'}{2} k^2 \log(|z|^2) \right) \exp(\alpha' k z \cdot z), \tag{71}
\]

Exercise 3. Polchinski 3.2

(a) Show that an \( n \)-index traceless symmetric tensor in two dimensions has exactly two independent components. The traceless condition means that the tensor vanishes when any pair of indices is contracted with \( g^{ab} \).

(b) Find a differential operator \( P_n \) that takes \( n \)-index traceless symmetric tensors into \( (n+1) \)-index traceless symmetric tensors.

(c) Find a differential operator \( P_n^T \) that does the reverse.

(d) For \( u \) an \( (n+1) \)-index traceless symmetric tensor and \( v \) and \( n \)-index traceless symmetric tensor, show that with appropriate normalization \( \langle u, P_n P_n^T v \rangle = \langle P_n^T u, v \rangle \). The inner product of traceless symmetric tensors of any rank is

\[
\langle t, t' \rangle = \int d^2 \sigma |g|^{1/2} t \cdot t', \tag{72}
\]

with a dot denoting contraction of all indices.
Solution 3.

(a)

Proof 1.

Consider a traceless symmetric tensor $S_{\mu_1 \cdots \mu_n}$ in two dimensions. Each of the indices $\mu_i$ can only take the values 0 or 1, so a general tensor would have $2^n$ components. However, since $S$ is symmetric under interchange of any two indices, we can swap any entry $S_{\mu_1 \cdots \mu_n}$ to put all the zeros at the beginning and all the ones at the end. So $S_{00 \cdots 0}$ is one element, $S_{00 \cdots 01}$ is another, and so on; a given element is uniquely specified by the number of 1’s, and thus there are $n$ of them.

Now let’s address the tracelessness. We need to impose $S^a{}_{b \mu_1 \cdots \mu_n} = 0$ when we contract any pair of indices. This means $\gamma^{ab} S_{ab \mu_1 \cdots \mu_n} = 0$. But this is a tensor equation with $(n-2)$ symmetric indices, which therefore has $(n-2)$ independent components. Each of these is a constraint which reduces the number of degrees of freedom in $S$ by 1. So we conclude that the number of degrees of freedom is $n - (n-2) = 2$, which is a constant independent of $n$.

Proof 2.

Another possibility is to prove this by induction. When $n = 1$, a “traceless symmetric tensor” is simply a vector $A$ with components $A_0$ and $A_1$.

Suppose that the claim holds when $n = k$. A $(k+1)$-index traceless symmetric tensor then has 2 degrees of freedom when the final index is zero, and another 2 when the final index is one. We subtract one degree of freedom by imposing that swapping the first and last indices leaves the component unchanged (symmetry under all other swaps then follows). Likewise, we demand that contracting the final index against the first one yields zero, which then gives tracelessness for contraction involving any pair of indices. This subtracts another degree of freedom, so the $(k+1)$-tensor also has two degrees of freedom.

The claim follows by induction on $n$.

(b) Our goal is to find a differential operator which maps symmetric traceless $n$-tensors into symmetric traceless $(n+1)$-tensors.

In order to be a “differential operator”, our guess had better involve a derivative somewhere. The most straightforward guess is to differentiate our symmetric $n$-tensor, which adds an index, and then symmetrize over the new index:

$$ P_n S_{\mu_1 \cdots \mu_n} = \frac{2}{n+1} \partial_{(\mu_{n+1}} S_{\mu_1 \cdots \mu_n)} = \partial_{(\mu_{n+1}} S_{\mu_1 \cdots \mu_n)}. $$

However, we can see that this ansatz will not be traceless in general. Indeed, we can compute the trace

$$ g^{\mu_{n+1}} = \partial_{(\mu_{n+1}} (\partial_{\mu_{n+1}} S_{\mu_1 \cdots \mu_n}) + \partial_{\mu_n} S_{\mu_{n+1}} S_{\mu_2 \cdots \mu_{n-1}} + \cdots + \partial_{\mu_n} S_{\mu_2 \cdots \mu_{n+1}}), $$

(74)

where on the right I have summed over all cyclic permutations of the indices $(\mu_{n+1}, \cdots, \mu_n)$; other permutations are equivalent by the symmetry of $S$.

For simplicity, I will now assume that $\partial_{\mu} g^{\alpha \beta} = 0$ so that we can move the metric factor in (74) inside the derivatives. You may justify this either by assuming that $\partial$ is really the covariant derivative $\nabla$, so that this assumption is merely metric compatibility, or by assuming that we are on the worldsheet and have spent our reparameterization invariance to put $g^{ab} = \eta^{ab}$.

In either case, this assumption allows us to discard terms of the form

$$ g^{\mu_{n+1}} \partial_{\mu_n} S_{\mu_{n+1}} S_{\mu_2 \cdots \mu_{n-1}} = \partial_{\mu_n} (g^{\mu_{n+1}} S_{\mu_{n+1}} S_{\mu_2 \cdots \mu_{n-1}}) = 0, $$

(75)

where the factor in parentheses vanishes by the tracelessness of $S$.

Then (74) becomes

$$ g^{\mu_{n+1}} \partial_{(\mu_{n+1}} S_{\mu_1 \cdots \mu_n)} = \frac{2}{n+1} \partial_{\mu_{n+1}} S_{\mu_2 \cdots \mu_n}. $$

(76)
Thus all we need to do in order to patch up our woefully un-traceless ansatz (73) is to subtract a symmetric tensor whose trace is given by (76). The simplest thing to do is to multiply (76) by the metric and then symmetrize over the resulting indices. This gives
\[ \frac{n}{n+1} \partial\nu S^\nu_{(\mu_1\cdots\mu_{n-1}g_{\mu_n\mu_{n+1}})}, \]  
(77)
where we have divided by 2 to correct for the factor of $g_{\mu\nu} g^{\mu\nu} = 2$ when we take the trace, and multiplied by $n$ to correct for the symmetrization.

All in all, this gives us our final proposal for a differential operator mapping symmetric traceless $n$-tensors to symmetric traceless $(n+1)$-tensors:
\[ P_n S_{\mu_1\cdots\mu_n} = \partial(\mu_{n+1} S_{\mu_1\cdots\mu_n}) - \frac{n}{n+1} \partial\nu S^\nu_{(\mu_1\cdots\mu_{n-1}g_{\mu_n\mu_{n+1}})} \]  
(78)

c) The most obvious thing to try is simply taking the divergence:
\[ P^T_n S_{\mu_1\cdots\mu_n} \equiv \partial^\alpha S_{\mu_1\cdots\mu_n}. \]  
(79)
The result is symmetric in its remaining $n-1$ indices and still traceless, since
\[ g^{\mu_2\mu_3} \partial\mu S_{\mu_1\cdots\mu_n} = \partial\mu_1 (g^{\mu_2\mu_3} S_{\mu_1\cdots\mu_n}) = 0, \]  
(80)
assuming that either the metric is constant or that we are using the covariant derivative so that $\partial_\mu g^{\alpha\beta} = 0$, which is the assumption we have been making.

However, we will see in part (d) that making the naive choice (79) will cause our desired inner product relationship will not hold – instead it has an extra minus sign. With some foresight, then, we will correct this here by instead defining
\[ P^T_n S_{\mu_1\cdots\mu_n} = -\partial^\mu S_{\mu_1\cdots\mu_n}. \]  
(81)

d) With $u$ and $v$ traceless symmetric tensors, we consider
\[ (u, P_n v) = \int d^2\sigma \sqrt{g} u^{\mu_1\cdots\mu_{n+1}} \left( \partial(\mu_{n+1} v_{\mu_1\cdots\mu_n}) - \frac{n}{n+1} \partial\nu v^\nu_{(\mu_1\cdots\mu_{n-1}g_{\mu_n\mu_{n+1}})} \right) \]  
(82)
Since $u$ is already symmetric in all of its upstairs indices, we can drop the symmetrizers in the terms in parenthesis. The second term of (82) then becomes
\[ u^{\mu_1\cdots\mu_{n+1}} \partial\nu v^\nu_{\mu_1\cdots\mu_{n-1}g_{\mu_n\mu_{n+1}}} = u^{\mu_1\cdots\mu_{n+1}} \partial\nu v^\nu_{\mu_1\cdots\mu_{n-1}}, \]  
(83)
where we have used the metric $g_{\mu\nu}$ to take the trace of $u$. Hence this term vanishes because $u$ is traceless. We are left with
\[ (u, P_n v) = \int d^2\sigma \sqrt{g} u^{\mu_1\cdots\mu_{n+1}} \partial_{\mu_{n+1}} v_{\mu_1\cdots\mu_n}, \]  
(84)
which we may integrate by parts (assuming suitable decay at infinity) to find
\[ (u, P_n v) = - \int d^2\sigma \sqrt{g} \left( \partial_{\mu_{n+1}} u^{\mu_1\cdots\mu_{n+1}} \right) v_{\mu_1\cdots\mu_n} = (P^T_n u, v). \]  
(85)

Now we see that the extra minus sign in part (b) was necessary to cancel that coming from integration by parts.

**Exercise 4.**

The linear dilaton background is an example that we have not considered in detail but is very nice.

(i) Using the energy-momentum tensor defined on p.49, explicitly compute the $T(z)T(0)$ OPE, and determine the central charge.
(ii) Compute the OPE of \( J^\mu(z) = \frac{i}{\alpha'} \partial_z X^\mu \) with \( T(z) \). What does the result teach you about the charge

\[
\int \frac{dz}{2\pi i} J^\mu(z)
\]
on a genus \( g \) Riemann surface?

(iii) Check that this background satisfies the \( \beta \)-function equations of motion (p. 111) with a varying dilaton, and a non-critical \( D \). What can you conclude about string perturbation theory in such a background?

**Solution 4.**

(i) We have already computed the \( T(z)T(w) \) OPE for the linear dilaton theory in problem 2, where we found the central charge

\[
c = D + 6\alpha' V_\mu V^\mu \equiv \tilde{c}.
\]

(ii) Using the given definition \( J^\mu = \frac{i}{\alpha'} \partial_z X^\mu(z) \) and the linear dilaton stress tensor \( T(z) = -\frac{1}{\alpha'} : \partial X^\nu \partial X_\nu : + V_\mu \partial^2 X^\nu \), we compute the OPE

\[
T(z)J^\mu(w) = \left( -\frac{1}{\alpha'} : \partial_z X^\nu(z) \partial_z X_\nu(z) : + V_\mu \partial^2 X^\nu(z) \right) \left( \frac{i}{\alpha'} \partial_w X^\mu(w) \right).
\]

We only need to consider two kinds of contractions. When \( \partial_z X^\nu(z) \) contracts against \( \frac{i}{\alpha'} \partial_w X^\mu(w) \), which can occur in 2 ways because of the two choices of \( \partial_z X^\nu(z) \) factor, we get a factor of

\[
\left( -\frac{i}{\alpha'^2} \right) \left( \frac{\alpha' \eta^{\mu\nu}}{2 (z-w)^2} \right) \partial_z X_\nu(z) = \frac{i \partial_z X^\mu}{\alpha'(z-w)^2}.
\]

When \( V_\mu \partial^2 X^\nu(z) \) contracts against \( \frac{i}{\alpha'} \partial_w X^\mu \), we get

\[
\frac{i}{\alpha'} X^\nu \partial^2_w \left( -\frac{\alpha'}{2} \log(z-w) \right) = -i V^\mu \frac{1}{(z-w)^3}.
\]

To express this in the usual OPE form, we replace \( \partial_z X^\mu \) in (89) using \( \partial_z X^\mu(z) = \frac{\alpha'}{\alpha'} J^\mu(z) \), then convert this to a function of \( w \) using

\[
J^\mu(z) = J^\mu(w) + \frac{\partial J^\mu(w)}{z-w} + \cdots,
\]

from which we find the OPE

\[
T(z)J^\mu(w) = -\frac{i V^\mu}{(z-w)^3} + \frac{J^\mu(w)}{(z-w)^2} + \frac{\partial J^\mu(w)}{z-w} + \cdots.
\]

Remarkably, the presence of a \( (z-w)^{-3} \) term means that \( \partial X^\mu \) is not primary in the linear dilaton theory – in other words, our standard \( U(1) \) chiral current \( J^\mu = \frac{i}{\alpha'} X^\mu \) has become anomalous!

There are a few perspectives that offer some insight on what this means. I’ll discuss three ways to think about it.

**First Way: Action with Background Fields**

This result can be interpreted from the linear dilaton worldsheet action, which couples the \( X^\mu \) to the scalar curvature as

\[
S = \frac{1}{4\pi \alpha'} \int d^2 \sigma \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X^\mu + \frac{1}{4\pi} \int d^2 \sigma \sqrt{-\sigma} R^{(2)} V_\mu X^\mu.
\]

---

3None of these interpretations was required for full credit on the problem set, but I find them super-interesting and thus include them here for the sake of spiritual enlightenment.
Because the $U(1)$ current $J^\mu$ was associated with invariance under translations of the form $X^\mu \rightarrow X^\mu + a^\mu$, we now see that we should not have expected the current to be conserved in the linear dilaton background – indeed, under such a shift $\delta X^\mu$, the action changes as

$$\delta S = \frac{1}{4\pi} \int d^2x \sqrt{-g} R^{(2)} V_\mu a^\mu = V_\mu a^\mu \chi,$$  \hspace{1cm} (94)

where $\chi$ is the Euler characteristic of the worldsheet.

It also follows directly from the action (93) that the equation of motion for $X^\mu$ is modified to

$$\partial_\mu X^\mu = \frac{\alpha'}{2} V^\mu R^{(2)},$$  \hspace{1cm} (95)

which means that $\delta J^\mu = i V^\mu R^{(2)}$. In particular, the current $J^\mu$ is no longer chirally conserved since $\partial V^\mu \neq 0$; said differently, $J^\mu = \partial X^\mu$ is no longer a holomorphic field.

This gives us one way of understanding the charge associated with our anomalous $J^\mu$, namely

$$Q = \oint \frac{dz}{2\pi i} J^\mu.$$  \hspace{1cm} (96)

As the radius of the bounding circle in the contour integral (96) grows very large, we can trade the result for an area integral of the divergence of the current over the entire plane. But this divergence integrand will be proportional to $\sqrt{-g} R^{(2)}$ by (95). As we integrate this quantity over the entire plane, we recover something proportional to the Euler characteristic $\chi$ by the Gauss-Bonnet theorem.

Thus on a worldsheet of genus $g$, the charge $Q$ “at infinity” will be proportional to the corresponding Euler characteristic $\chi = 2(1 - g)$. But we can make the analogy with a charge at infinity more precise using the Coulomb gas formalism, which we turn to next.

**Second Way: Background Charge at Infinity**

The strange $(z - w)^{-3}$ in the $TJ^\mu$ OPE means that the transformation $J^\mu$ under an infinitesimal conformal map $z \rightarrow z + \epsilon(z)$ will have an extra term that does not depend on $J^\mu$ itself. Using that a general operator varies under such a conformal transformation as

$$\delta O(w) = -\text{Res} \left[ \epsilon(z) T(z) O(w) \right],$$  \hspace{1cm} (97)

and assuming that $\epsilon(z)$ can be expanded as

$$\epsilon(z) = \epsilon(w) + \epsilon'(w)(z - w) + \frac{1}{2} \epsilon''(w)(z - w)^2 + \cdots,$$  \hspace{1cm} (98)

we find that

$$\delta J^\mu = -\text{Res} \left[ \epsilon(z) T(z) O(w) \right]$$

$$= -\text{Res} \left[ \epsilon(w) + \epsilon'(w)(z - w) + \frac{1}{2} \epsilon''(w)(z - w)^2 \right] \left( \frac{-i V^\mu}{(z - w)^3} + J^\mu(w) \frac{\partial J^\mu(w)}{z - w} + \cdots \right)$$

$$= +i V^\mu \epsilon''(w) - \epsilon'(w) \partial J^\mu(w) - \epsilon(w) J^\mu(w).$$  \hspace{1cm} (99)

The extra term proportional to $\epsilon''(w)$ is morally quite similar to the $-\frac{\alpha'}{2\pi} \epsilon''''(w)$ term which appears in the transformation law for the stress tensor $T(w)$. In both cases, the extra term is independent of the operator ($J^\mu$ or $T$) and thus gives the same contribution when evaluated on any state. For the stress tensor, this central charge $c$ represented a zero mode contribution to the energy. For the current, this term is a “background charge” associated with the $J^\mu$.

Let’s make the above statement slightly more precise. What do we mean by “charge under the $U(1)$ current $J^\mu$”? We define the charge operator $Q^\mu$ by

$$Q^\mu = \frac{1}{2\pi i} \oint dz \: J^\mu(z),$$  \hspace{1cm} (100)
where again \( J^\mu = \frac{1}{2\alpha'} \partial X^\mu \). Then we say that an operator \( \mathcal{O}(w) \) has charge \( q^\mu \) under \( J^\mu \) if

\[
[Q^\mu, \mathcal{O}(w)] = q^\mu \mathcal{O}(w).
\]  

(101)

In terms of the operator product expansion with \( J^\mu \), the definition (101) can be alternatively written as

\[
J^\mu(z)\mathcal{O}(w) = q^\mu \frac{\mathcal{O}(w)}{z-w} + \cdots.
\]  

(102)

To get some intuition for this definition, let’s take a brief aside and return to free field theory.

One might ask: what is the corresponding charge of the vertex operator \( \exp(ik \cdot X) \) in the free scalar theory? We recall from equation (4.26) of Tong’s notes that

\[
\partial X(z) : \exp(ik \cdot X(w)) : = -\frac{i\alpha'k^2}{2} \partial \frac{\exp(ik \cdot X(w))}{z-w},
\]  

(103)

so the operator \( \exp(ik \cdot X) \) has “charge” \( \frac{k^2}{2} \) in the free theory.

You may also recall from problem (2.3) on this problem set that the expectation value of a product of exponentials,

\[
\langle \prod_{i=1}^n \exp(ik_i \cdot X(z_i, \bar{z}_i)) \rangle,
\]  

(104)

is proportional to a delta function,

\[
\delta^D \left( \sum_{i=1}^n k_i \right).
\]  

(105)

But we have just learned that the \( k_i \) are, up to a numerical constant, the charges of the vertex operators under the \( U(1) \). For this reason, we sometimes speak about this momentum-conserving delta function as a charge cancellation condition: that is, the expectation value of a string of vertex operators is zero unless the sum of their charges cancels.

Now let’s return to the linear dilaton theory. Since we have seen that the field \( \partial X^\mu \) is no longer primary, one might worry whether the vertex operators \( \exp(ik \cdot X) \) are still primary. Thankfully, it turns out that they are:

\[
T(z) : \exp(ik \cdot X(w)) : = \left( -\frac{1}{\alpha'} \partial X^\nu \partial X^\nu : + V_\nu \partial^2 X^\nu : \right) \exp(ik \cdot X(w)) : = \frac{i\alpha'k^2}{4} + \frac{i\alpha'k \cdot V}{2} \frac{1}{(z-w)^2} + \partial \frac{\exp(ik \cdot X(w))}{z-w} + \cdots.
\]  

(106)

Since the vertex operators are primary, we can ask when the correlator of some product of exponentials \( \langle \prod_{i=1}^n \exp(ik_i \cdot X(z_i, \bar{z}_i)) : \rangle \) will be non-vanishing. In the linear dilaton case, I know of two ways to answer this question:

1. Study the path integral in the linear dilaton theory running over the zero mode of the scalars \( X^\mu \). See exercise 4.42 on page 123 of Elias Kiritsis’ “String Theory in a Nutshell” for this.

2. Use the Ward identity associated with the \( U(1) \) symmetry \( X^\mu \rightarrow X^\mu + a^\mu \). Under this shift, the action changes by a total derivative, since \( \int \sqrt{g} R^{(2)} = 4\pi \chi \) is proportional to the Euler characteristic; account for this term in the Ward identity appropriately.

I will not derive the result carefully here, but the result is that the expectation value for a string of exponential operators \( \exp(ik_i \cdot X(z_i, \bar{z}_i)) : \) is non-vanishing in the linear dilaton theory if

\[
\sum_i k_i^\mu = -iV^\mu \chi.
\]  

(107)

Thus we see that the “momentum conservation” or “charge cancellation” condition is modified in a way which depends on the Euler characteristic \( \chi \) of the worldsheet.
Heuristically speaking, the effect of the dilaton coupling in the action has been to add a new vertex operator at infinity, so that a correlator is only non-vanishing if it cancels the background charge associated with this vertex operator.

Also note that the right side of (107) is complex, so the momenta $k^\mu$ need also pick up imaginary parts. One can also see this from (106), which requires the momenta to be complex in order to get real conformal weights.

**Way Three: Central Extension of $T$, $J^\mu$ Algebra**

In the free field case, (108) indeed vanishes after a few lines of algebra which rely on the fact that $\partial X^\mu$ is a holomorphic field.

Motivated by this, we might try to be more general and study the commutator of $Q$ with an arbitrary Virasoro mode $L_n$ in the linear dilaton theory. We have seen in lecture that the operator product expansion expresses the same information as the algebra between Fourier modes of an arbitrary Virasoro mode

$$[\mathcal{H}(z, \bar{z}), Q(w)] = \left[ L_0(z) + \bar{L}_0(\bar{z}), Q(w, \bar{w}) \right]. \quad (108)$$

In the free-field case, (108) indeed vanishes after a few lines of algebra which rely on the fact that $\partial X^\mu$ is a holomorphic field.

The answer is yes. First we will assume that the current $J^\mu = \frac{\lambda}{\alpha^\prime} \partial X^\mu$ can be expanded in Fourier modes $J_n^\mu$ as

$$J_n^\mu = \oint \frac{dw}{2\pi i} w^n J^\mu(w). \quad (111)$$

I think that this expansion is only guaranteed to exist on the plane/cylinder. If so, this means that, in the present section, we must specialize to the case of a genus zero worldsheet. If anyone knows of a way to make this argument work on a genus $g$ worldsheet, please let me know.

Ignoring this subtlety, let’s compute the correlator between the stress tensor mode $L_m$ and the current mode $J_n^\mu$. This is

$$[L_m, J_n^\mu] = \oint \frac{dz}{2\pi i} z^{m+1} [T(z), J_n^\mu]$$

$$= \left( \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} - \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \right) z^{m+1} w^n T(z) J^\mu(w) \quad (112)$$

In the first step, we have used that the modes of the stress tensor are given by $L_m(z) = \frac{1}{2\pi i} \oint dz T(z) z^{m+1}$. In the second step, we have used the definition of $J^\mu(w)$ in terms of its modes.

As we have seen in lecture, operator products only have convergent expansions when the operators are radially ordered. Thus the commutator in (112) should be interpreted as a difference of two integration contours, the first with $|z| > |w|$ and the second with $|z| < |w|$ (this is what is meant
by the difference of integrals in the final line of (112)). But this is equivalent to a single contour integration over $w$ for a fixed value of $z$, i.e.

$$\left[L_m, J^\mu_n\right] = \oint \frac{dw}{2\pi i} \text{Res}_{z=w} \left[w^{m+1} w^n T(z) J^\mu(w)\right]$$

$$= \oint \frac{dw}{2\pi i} \text{Res}_{z=w} \left[w^{m+1} w^n \left(- \frac{iV^\mu}{(z-w)^3} + \frac{J^\mu(w)}{(z-w)^2} + \frac{\partial J^\mu(w)}{z-w} + \cdots\right)\right]. \quad (113)$$

Of course, we need to replace

$$z^{m+1} = w^{m+1} + (m+1)w^m(z-w) + \frac{1}{2}m(m+1)w^{m-1}(z-w)^2 + \cdots,$$  \quad (114)

in order to evaluate the residue. Using this, we find

$$\left[L_m, J^\mu_n\right] = \oint \frac{dw}{2\pi i} \text{Res}_{z=w} \left[w^{m+1} + (m+1)w^m(z-w) + \frac{1}{2}m(m+1)w^{m-1}(z-w)^2 + \cdots\right]$$

$$w^n \left(- \frac{iV^\mu}{(z-w)^3} + \frac{J^\mu(w)}{(z-w)^2} + \frac{\partial J^\mu(w)}{z-w} + \cdots\right)$$

$$= \oint \frac{dw}{2\pi i} \left[w^n (\frac{1}{2}m(m+1)w^{m-1}) + (m+1)w^m(w^n) J^\mu + w^{m+1} w^n \partial J^\mu\right]. \quad (115)$$

The first term in the integrand, proportional to $w^{n+m-1}$, gives a non-vanishing contribution to the contour integral when $n = -m$, in which case it yields $\frac{-iV^\mu}{2}m(m+1)$. The second term, which involves an integral of $w^{m+n}J^\mu$, picks out the mode $J_{m+n}$.

The third term can be integrated by parts, which will give another term that picks out the $J_{m+n}$ mode, but here is where I think the derivation must be modified on a worldsheet of arbitrary genus. Morally, this integration by parts should introduce an extra term due to the non-vanishing of $\partial J$, which presumably introduces dependence on the worldsheet Euler characteristic $\chi$.

I will ignore this issue for now and assume that we can integrate by parts freely, dropping the boundary term. Overall, we find

$$\left[L_m, J^\mu_n\right] = -nJ^\mu_{m+n} + \frac{(-iV^\mu)}{2}m(m+1)\delta_{m-n}. \quad (116)$$

This is exactly the sort of central extension that appeared in the $TT$ OPE. Interestingly, although the stress tensor modes are Hermitian ($L^\mu_n = L_{-m}$), the algebra (116) modifies the Hermiticity condition on the current modes to

$$(J^\mu_m)^\dagger = J^\mu_{-m} + iV^\mu \delta_{m,0}, \quad (117)$$

which you may be interested in showing as an exercise (see exercise 4.43 on page 123 of Kiritsis).

The punchline is that this modifies Hermiticity condition gives us a third way to think about the non-conservation of the charge $Q$ in the linear dilaton theory. Indeed, if we consider an early-time vacuum state with zero charge under the $U(1)$ current — that is, a state $|0\rangle$ with $J_0|0\rangle = 0$ — then daggering both sides shows us that the late-time vacuum state has non-zero charge:

$$|0\rangle J_0 = (J_0|0\rangle)^\dagger = iV^\mu \langle 0|.$$  \quad (118)

Said differently: an initially uncharged state in our linear dilaton theory picks up charge later. This gives a third way of understanding the behavior of $Q = \oint \frac{dw}{2\pi i} J^\mu(z)$ in this background.

(iii) The beta function for the dilaton is given on page 111 of Polchinski as

$$\beta^\Phi = \frac{D-26}{6} \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\nu \Phi \nabla^\nu \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda}. \quad (119)$$
In the background that we have been considering in this problem, we have a flat metric, vanishing Kalb-Ramond field strength, and a dilaton which depends linearly on the spacetime coordinates:

\[ G_{\mu\nu}(X^\alpha) = \eta_{\mu\nu}, \]

\[ H_{\mu\nu\lambda}(X^\alpha) = 0, \]

\[ \Phi(X^\alpha) = V_{\alpha}X^\alpha. \]  \hspace{1cm} (120)

Here \( V^\mu \) is the vector which appeared in our action and stress tensor above.

In order for the worldsheet theory to be conformally invariant, we need that the beta functions vanish, which requires

\[ D = 26 - 6\alpha' V^2. \]  \hspace{1cm} (121)

Indeed, this means that the equations of motion are satisfied for a non-critical dimension – i.e. a dimension other than \( D = 26 \) – which depends on the size of the dilaton gradient \( V^2 \) in units of \( \frac{1}{\alpha'} \). I am told that people have studied examples where \( V^2 \) is cranked up to the point that we are left with \( D = 1 \) or \( D = 2 \), which give simple models with low-dimensional target spacetimes.

Recall that, when we do string perturbation theory, we expand in powers of the string coupling constant

\[ g_s = e^\Phi, \]  \hspace{1cm} (122)

where \( \Phi \) is the dilaton. In most cases, we assume that the dilaton is a constant \( \Phi = \Phi_0 \), which means that the contribution of the dilaton to the worldsheet action,

\[ S_\Phi = \frac{1}{4\pi} \int \sqrt{-\gamma(\Phi)} R(2) d^2\sigma, \]  \hspace{1cm} (123)

simply contributes a constant \( e^{-S_\Phi} = e^{-\Phi_0} = g_s^{-\chi} \), which suppresses worldsheets of high genus (as we add more handles, \( \chi \) grows more negative, so \( g_s^{-\chi} \) is a small number raised to a more positive power and thus smaller).

However, in the linear dilaton background \( \Phi(X) = V_\mu X^\mu \), it seems that strings become strongly coupled as we move in the direction of \( V_\mu \) (assuming the entries of \( V_\mu \) are positive) and weakly coupled as we move opposite \( V_\mu \). If so, we may worry that the technology of string perturbation theory breaks down here – we can perform a genus expansion only in regions of spacetime where \( \exp(\Phi(X^\mu)) \ll 1 \).

Perhaps surprisingly, the “pathological” contributes from the strongly coupled region in the linear dilaton background is actually suppressed by the tachyon field. One can show that the tachyon equation of motion is modified in the linear dilaton theory, which suggests that one should add a tachyon background term to the worldsheet action, where the tachyon profile is also of the form \( e^{\Phi(X)} \); this gives the so-called Liouville field theory.\(^4\)

\(^4\)See section 3.6 of Becker, Becker, and Schwarz’s string theory book for more on this!