RG Flows and a Theorem

Chih-Kai Chang

*University of Chicago*

*E-mail: ckchang@uchicago.edu*

**ABSTRACT:** In this note, I discuss some general aspects of RG flow in two and four dimension. RG flow is tightly related to the trace of energy momentum tensor, thus is constrained by the conformal symmetry. A CFT in the IR fixed point can be interpreted as RG flow of broken conformal symmetry. By matching the anomaly along RG flow, one is able to contract an effective action of the IR theory. I begin with a review on the trace anomaly in two dimension. A proof of c theorem is present in a way different from Zamolodchikov’s proposal, with more emphasis on the significance of dilaton background. Following the same idea, it can be shown that dilaton scattering leads to an anomaly in four dimension. Unitarity of S-matrix thus implies the monotonic property of a anomaly and establishes the a theorem.
1 Two Dimensional CFT

For a two dimensional theory, Zamolodchikov proved that there exists a quantity which monotonically decreases as the theory flows to low energy [1]. This result not only implies the irreversibility of RG flow, but also provides a measure of degree of freedom. As we integrate out the high frequency mode, we should expect that the degree of freedom decreases. The way we measure the degree of freedom is through calculating the central charge at the fixed point.

1.1 Central Charge

Consider a local unitary Euclidean two dimensional quantum field theory on $\mathbb{R}^2$. In such a theory, we can always find rank 2 energy momentum tensor $T_{\mu\nu}$, which is a local and conserved current of two dimension isometry. The conservation law puts a very strong constraint on the correlation function of energy momentum tensors. The two point function are forced to take the following form [2].

$$\langle T_{\mu\nu}(q)T_{\rho\sigma}(-q)\rangle = \frac{1}{2}(W_{\mu\rho}(q)W_{\nu\sigma}(q) + W_{\mu\sigma}(q)W_{\nu\rho}(q))f(q^2) + W_{\mu\nu}(q)W_{\rho\sigma}(q)g(q^2)$$  \hspace{1cm} (1.1)

The function $W(q)$ is given by $W_{\mu\nu}(q) = q_{\mu}q_{\nu} - q^2\eta_{\mu\nu}$. One can verify that this two point function satisfies the conservation law by checking the ward identity. Although energy momentum tensor has 4 components, the above expression is fixed by only two functions, $f(q^2)$ and $g(q^2)$, which are invariant under two dimension isometry. They can be determined by the detailed structure of the theory. In this note, I am specifically interested in scale invariant theory which allows us to further constraint the correlation function. Under scale
transformation, \( q \rightarrow \lambda^{-1} q \), energy momentum tensor acquires a factor \( \lambda^{-2} \). This allows us to fix the functions \( f(q^2) \) and \( g(q^2) \), up to constants.

\[
\begin{align*}
  f(q^2) &= \frac{b}{q^2}, \\
  g(q^2) &= \frac{d}{q^2}
\end{align*}
\]

In particular, we are mostly interested in the trace of energy momentum tensor, since it signals the conformal symmetry of the theory, which will be elaborated later.

\[
\langle T^\alpha_\alpha(q^2) \rangle = -(b + d) W_{\mu\nu}(q)
\]

Since this is just a polynomial in \( q_\mu \), we can transform it back to real space.

\[
\langle T^\mu_\mu(0) \rangle = -(b + d)(\partial_\alpha \partial_\beta - \partial^2 \eta_{\alpha\beta}) \delta^{(2)}(x)
\]

Ward identity of the trace vanishes at separate points, which is the condition of conformal symmetry, thus we achieve the conclusion that scale symmetry implies conformal symmetry in two dimension. So far we have restricted our discussion to a flat metric. The importance of the above trace formula can be further emphasized by coupling the theory to a slightly curved background metric. To linear order, the metric can be expanded as \( \eta_{\mu\nu} + h_{\mu\nu} \) and coupled to our theory by adding the following deformation to the original action, \( S_0 \).

\[
\delta S = \frac{1}{4\pi} \int d^2 x \ h_{\mu\nu} T^{\mu\nu}
\]

Let’s calculate the trace again but this time we work with the above deformation.

\[
\langle T^\mu_\mu(0) \rangle_{S_0} = \frac{1}{4\pi} \int d^2 x \ \langle T^\mu_\mu(0) T^{\alpha\beta}(x) \rangle h^{\alpha\beta}(x)
\]

\[
= -\frac{1}{4\pi} \int d^2 x \ (b + d)(\partial_\alpha \partial_\beta - \partial^2 \eta_{\alpha\beta}) \delta^{(2)}(x) h^{\alpha\beta}
\]

\[
= -\frac{b + d}{4\pi} (\partial_\alpha \partial_\beta - \partial^2 \eta_{\alpha\beta}) h^{\alpha\beta}(0)
\]

The trace doesn’t vanish, signaling the breakdown of conformal symmetry. This result is usually referred as trace anomaly or Weyl anomaly. The last line of the above equation can be identified as the Ricci scalar to linear order, \( (\partial_\alpha \partial_\beta - \partial^2 \eta_{\alpha\beta}) h^{\alpha\beta} \sim R \). In most literature, equation (1.5) is normalized to be

\[
\langle T^\mu_\mu \rangle = -\frac{c}{12} R
\]

The number \( c \) is called central charge. We use it to replace \( b + d \). Although equation (1.5) is derived at linear order of background field, one can show that equation (1.6) is correct to all order. The conformal symmetry is generically anomalous once coupled to a curved background.
1.2 Action Functional

Let’s examine the above result from a different angle. Under a local scale transformation, the metric transforms as $g_{\mu\nu}(x) \rightarrow e^{2\omega(x)} g_{\mu\nu}(x)$. This time we couple our scale invariant theory to a curved background which is conformal to a flat metric, that is to say, the metric looks like

$$g_{\mu\nu}(x) = e^{2\omega(x)} \eta_{\mu\nu}$$  \hspace{1cm} (1.7)

This metric is just a local transformation of the flat metric. The Ricci scalar and Laplacian becomes

$$R = -2 \nabla^2 \omega$$
$$\nabla^2 = e^{-2\omega} \partial^a \partial_a$$ \hspace{1cm} (1.8)

To further study equation 1.6, we can consider the following action functional.

$$Z[g] = \int [dX] \exp -S[X, g]$$ \hspace{1cm} (1.9)

The notation $X$ contains all field we need in this theory, including gauge fixing ghosts. The statement of trace anomaly implies that this functional is not invariant under scale transformation of $g \rightarrow e^{2\omega} g$, i.e. $Z[g'] \neq Z[g]$. Under an infinitesimal form of such transformation, the functional transforms as

$$\delta \omega Z[g] = - \frac{1}{2\pi} \int d^2 x \sqrt{g} \delta \omega(x) \langle T_{\mu\nu}(x) \rangle_g$$ \hspace{1cm} (1.10)

The trace appears naturally in this formalism. Let’s replace it by (1.6), (1.7), and (1.8).

$$\delta \omega Z[e^{2\omega} \eta] = - \frac{c}{12\pi} Z[e^{2\omega} \eta] \int d^2 x \delta \omega(x) \partial^a \partial_a \omega(x)$$ \hspace{1cm} (1.11)

This equation can be integrated to

$$Z[e^{2\omega} \eta] = Z[\eta] \exp \left( \frac{c}{24\pi} \int d^2 x \partial^a \partial_a \omega(x) \right)$$ \hspace{1cm} (1.12)

It is interesting that the metric dependence of the functional is uniquely determined by the trace anomaly. This result appears to be very useful when we consider dilaton action later. The conformal symmetry can be restored by promoting the function $\omega$ to be a background field, which is usually refered as dilaton.

1.3 Anomaly Match

Consider the following scenario. We have a CFT at UV which has moduli space of vacua. At certain point of moduli space, some operator acquires non-vanishing expectation value, denoted by $\nu$, thus breaks the conformal symmetry spontaneously. This results in a RG flow to a broken phase, which is assumed to be an IR theory. There exists a Nambu-Goldstone boson, denoted by $\tau$, as a consequence of broken conformal symmetry. It is required from Goldstone’s theorem that this boson couples linearly to the energy momentum tensor.

$$\langle 0 | T_{\alpha\beta} | \tau, q \rangle \sim \nu \eta_{\alpha\beta}$$
This boson is usually referred as dilaton. The statement of anomaly match is that the anomaly in the original CFT equals to which in the broken phase. In our case, we are interested in the trace anomaly. Anomaly match requires that we get the same anomaly from both UV and IR theories.

\[ \delta \omega S_{UV} = \delta \omega S_{IR} \]

To argue this match, we can consider calculating anomaly in both UV and IR theories\[4\]. In the IR theory, it depends only on the scale parameter \( \nu \) which characterizes the symmetry breaking. However, \( \delta \omega S \) is dimensionless, it can’t depends on scale \( \nu \). As a result, the anomaly is independent of vacua. As we move \( \nu \) to zero and the conformal symmetry is recovered, we should return to the anomaly in the original theory. Thus the anomaly is invariant as we move between the fixed points.

**1.4 Dilaton**

As mentioned before, a generic curved background breaks the conformal symmetry at the quantum level. In this section, let’s discuss the general feature of broken conformal symmetry and see what the broken symmetry can teach us about the RG flow. Consider a CFT in two dimension. One can break conformal symmetry by simply introducing a dimensionful parameter which serves as an energy scale. This violates the conformal symmetry at classical level, i.e. \( T_\mu^\mu \neq 0 \). Assume there exits a primary operator \( O \) with conformal dimension \( \Delta = 2 - \epsilon \). We introduce an energy scale \( \lambda \) with conformal dimension \( \epsilon \) by deforming the action with

\[ \delta S = \int d^2x \ \lambda O(x) \]  

(1.13)

This deformation moves the theory away from the fixed point. For simplicity, let’s restrict the discussion to the relevant deformation with positive \( \epsilon \). One can verify that the conformal symmetry is broken by checking the beta function of the dimensionless coupling \( g = \lambda \mu^{-\epsilon} \).

\[ \frac{dg}{d \log \mu} = -g\epsilon + \pi C g^2 + \cdots \]  

(1.14)

For a small \( \epsilon \), this deformation moves the theory toward an IR fixed point at \( g_{IR} = \epsilon / \pi C \), where the constant \( C \) is the coefficient of the \( O \) term appearing in the \( O \) OPE.

\[ O(x)O(0) \sim \frac{1}{x^{4-2\epsilon}} + C \frac{O(0)}{x^{2-\epsilon}} + \cdots \]  

(1.15)

However, it is possible to restore the conformal invariance by promoting the coupling constant \( g \) to a spacetime field. For each energy scale in the theory, \( \mu \) in this case, we replace it by \( e^{-\tau} \mu \), where \( \tau \) is some function of spacetime. The conformal symmetry is restored if \( \tau(x) \) transforms as \( \tau \rightarrow \tau + \omega \) under scale transformation. The RG flow can now be compensated by a scale transformation. This field is called dilaton. More specifically, we replace the perturbation by

\[ \delta S = \int d^2x \ g(e^{-\tau} \mu) \mu^2 O(x) \]  

(1.16)
To first order of $\tau$, the coupling looks like $\sim \int \tau \beta(g) \mu \epsilon O$. This is how we restore the conformal symmetry for an explicit violation. Now we can consider a special case where the symmetry is broken at the quantum level, so there is no explicit energy scale. That is to say, we have to set $\epsilon$ to zero. In a conformal theory, $\int T^{\mu}_{\mu} \sim \beta O$. As a result, we can interpret the coupling (1.16) as a coupling between dilaton and energy momentum tensor, $\int \tau T^{\mu}_{\mu}$. From the trace anomaly (1.6), we can replace the trace by Ricci scalar and expect the coupling to be $\sim c \int \tau R$.

1.5 c Theorem

Let’s consider the detailed action of the dilaton. From our discussion in section 1.2 and 1.4, its action must satisfy three conditions.

1. It must contain coupling $\int \tau T^{\mu}_{\mu}$.

2. Upon setting metric to be flat, it should give the action functional (1.12) with $\omega \rightarrow \tau$.

3. Under scale transformation $g^{\mu\nu} \rightarrow e^{-2 \omega} g^{\mu\nu}$, it must reproduce the trace anomaly.

There is an easy way to find the action. Define a scale invariant metric by $\hat{g}^{\mu\nu} = e^{-\tau} g^{\mu\nu}$. The action which follows the third condition must be $\sim \int \sqrt{\hat{g}} \tau \hat{R}$. Thus we find the action which satisfies the above three conditions.

$$S_{\tau} = \frac{c}{24 \pi} \int d^2 x \sqrt{g} (\tau R + (\partial \tau)^2)$$  \hspace{1cm} (1.17)

Consider a RG flow of two dimensional theory with at least two fixed points, one in UV while the other one in IR. As proved in section 1.1, scale invariance implies conformal invariance in two dimension, which shows that the theory is conformal at the fixed points, thus we can find the central charges, $c_{UV}$ and $c_{IR}$, at fixed points. Now we perform the same trick as before, we couple the theory with a background metric. At the UV fixed point, the theory is anomalous because we haven’t added the dilaton into our theory. The Weyl anomaly can be represented by

$$\delta_{\sigma} S = \frac{c_{UV}}{24 \pi} \int d^2 x \sqrt{g} \sigma R$$  \hspace{1cm} (1.18)

It follows immediately that we have the same anomaly at IR.

$$\delta_{\sigma} S = \frac{c_{IR}}{24 \pi} \int d^2 x \sqrt{g} \sigma R$$  \hspace{1cm} (1.19)

If $c_{UV} \neq c_{IR}$, the two anomaly don’t match each other. Since Weyl anomaly is a property of quantum theory, it must match regardless of the energy scale. The rest of the anomaly is provided by the dilaton in equation (1.17). To compensate the anomaly the coefficient in front the the dilaton action must be $(c_{UV} - c_{IR})/24 \pi$ such that the first term of equation (1.17) can support the rest of central charge. Upon setting the metric to be flat, the anomaly vanished since it is proportional to Ricci scalar. However the dilaton action doesn’t vanish entirely, it leaves a term

$$\frac{c_{UV} - c_{IR}}{24 \pi} \int d^2 x (\partial \tau)^2$$  \hspace{1cm} (1.20)
in the IR theory. From reflection positivity the coefficient of the kinetic term must be positive, thus we achieve the conclusion, $c_{UV} \geq c_{IR}$. This is the c theorem first proposed by Zamolodchikov [1] that the central charge is a monotonically decreasing function as we integrate out the high frequency modes. We can also evaluate the difference of central charge. The dilaton are coupled to the theory via $\int \tau T_{\mu}^\mu$. Consider its scattering, 

$$\langle e^{i \int d^2 x \tau T_{\mu}^\mu} \rangle = \cdots + \frac{1}{2} \int d^2 x d^2 y \tau(x) \tau(y) \langle T_{\mu}^\mu(x) T_{\nu}^\nu(y) \rangle + \cdots$$

$$= \cdots + \frac{1}{4} \int d^2 x d^2 y \tau(x) \partial_\alpha \partial_\beta \tau(x) (x - y)^\alpha (x - y)^\beta \langle T_{\mu}^\mu(x) T_{\nu}^\nu(y) \rangle + \cdots$$

$$= \cdots + \frac{1}{4} \int d^2 x \tau(x) \partial_\alpha \partial_\beta \tau(x) \frac{1}{2} \eta_{\alpha \beta} \int d^2 y \ y^2 \langle T_{\mu}^\mu(0) T_{\nu}^\nu(y) \rangle + \cdots$$

$$= \cdots + \frac{1}{8} \int d^2 \tau(x) \partial^2 \tau(x) \int d^2 y \ y^2 \langle T_{\mu}^\mu(y) T_{\nu}^\nu(0) \rangle + \cdots$$

We are specifically interested in the term with two dilaton since it can reproduce the equation (1.20). In the second line of the above calculation, we focus on the two derivative term. In the third line, we have used the translational invariance of the theory. From equation (1.20), we know the coefficient of $\partial^2 \tau$ is proportional to the difference of central charge. It follows that

$$c_{UV} - c_{IR} = 3\pi \int d^2 y \ y^2 \langle T_{\mu}^\mu(y) T_{\nu}^\nu(0) \rangle$$

(1.21)

This is exactly the result obtained by Zamolodchikov. Since $T_{\mu}^\mu(y) T_{\nu}^\nu(0)$ is positive definite, the difference of central charge is always positive or zero.

2 Four Dimensional CFT

It is natural to ask if we can find a similar monotonic quantity in dimension higher than two. In this section, we are most interested in four dimension. In fact, it is proved that there exists such a function in four dimension[3]. As in two dimension, it is also tightly related to the trace anomaly. Since the trace of energy momentum tensor is diff invariant, it can only equal to diff invariant quantity. In four dimension, there are three such quantities.

$$T_{\mu}^\mu = \alpha R^2 + \beta R_{\mu\nu}^2 + \gamma R_{\mu\nu\sigma\rho}^2$$

We can also use different basis.

$$T_{\mu}^\mu = c W_{\mu\nu\sigma\rho}^2 - a E_4 + b \partial^2 R$$

(2.1)

Here $W$ and $E_4$ are Weyl tensor and Euler density, $W_{\mu\nu\sigma\rho}^2 = R_{\mu\nu\sigma\rho}^2 - R_{\mu\nu\rho}^2 + R^2/3$, $E_4 = R_{\mu\nu\sigma\rho}^2 - 4 R_{\mu\nu}^2 + R^2$. It was first proposed by Cardy BUTTERFLY that the coefficient $a$ has monotonic property, $a_{UV} - a_{IR} > 0$. There are only limited examples where we can calculate $a$ explicitly. The situation is ameliorated when supersymmetry is involved, especially with $N = 1$ supersymmetry, where the a maximization[5] almost ensures the monotonic property. In this section, we consider a general proof of a theorem.
2.1 Dilaton and Anomaly Functional

We can follow the method we use in proving the $c$ theorem. We first couple the four dimensional CFT to a background metric as well as a dilaton field. The action functional of all those fields should reproduce the trace anomaly under a scale transformation $\tau \to \tau + \sigma$, that is to say

$$\delta_{\sigma} S = \int d^4x \sqrt{g} \sigma (cW_{\mu\nu\sigma\rho}^2 - aE_4 + b\partial^2 R)$$  \hspace{1cm} (2.2)

It follows immediately that the term proportional to $b$ is unnecessary, because it doesn’t involve any dilaton. This can be shown by

$$\delta_{\sigma} \int d^4x \sqrt{g} R^2 \sim \int d^4x \sqrt{g} \sigma \partial^2 R$$

As a result, this term can be produced by an action which is independent of dilaton. We are free to set $b = 0$. The anomaly action can be obtained by the same method as in section 1.5. It turns out to be

$$S = - 2a \int d^4x (\partial \tau)^4$$  \hspace{1cm} (2.3)

Here $G^{\mu\nu}$ is Einstein tensor. The dilaton interaction survives even in flat metric. The equation of motion of dilaton in flat metric is $\partial^2 \tau = (\partial \tau)^2$. Up to equation of motion, the action becomes

$$S_{\text{flat}} = 2a \int d^4x (\partial \tau)^4$$  \hspace{1cm} (2.4)

Because of anomaly match as discussed in section 1.3, the action in the IR should be

$$S_{\text{IR}}[g_{\mu\nu}] = CFT_{\text{IR}}[g_{\mu\nu}]$$

$$+ \frac{1}{6} f^2 \int d^4x \sqrt{\hat{g}} \hat{R} + \frac{\kappa}{36} \int d^4x \sqrt{\hat{g}} \hat{R}^2 + \kappa' \int d^4x \sqrt{\hat{g}} \hat{W}_{\mu\nu\sigma\rho}^2$$

$$- (a_{UV} - a_{IR}) \int d^4x \sqrt{\hat{g}} \left( \tau E_4 + 4G^{\mu\nu} \partial_{\mu} \tau \partial_{\nu} \tau - 4(\partial \tau)^2 \partial^2 \tau + 2(\partial \tau)^4 \right)$$

$$+ (c_{UV} - c_{IR}) \int d^4x \sqrt{\hat{g}} \tau \hat{W}_{\mu\nu\sigma\rho} + \cdots$$  \hspace{1cm} (2.5)

Here $\hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu}$. The term proportional to $a_{UV} - a_{IR}$ comes from anomaly match. As we take the flat limit, the result is

$$S_{\text{IR}}[\tau] = CFT_{\text{IR}}$$

$$+ \int d^4x \left( f^2 e^{-2\tau} (\partial \tau)^2 + \kappa (\partial^2 \tau - (\partial \tau)^2)^2 + (a_{UV} - a_{IR})(4(\partial \tau)^2 \partial^2 \tau - 2(\partial \tau)^4) \right)$$  \hspace{1cm} (2.6)

Let’s study this action term by term in details. For the first term, it contains two derivatives, so we can regarded it as a kinetic term. This becomes obvious if we redefine the field in terms of $\Psi = 1 - e^{-\tau}$. The first term now becomes

$$f^2 \int d^4x \Psi \partial^2 \Psi$$  \hspace{1cm} (2.7)
This is exactly the kinetic term of $\Psi$. Now we can move to the second term and consider the same field redefinition. Since it contains four derivatives, it The result is
\[ \frac{\kappa}{36} \int d^4x \frac{1}{(1 - \Psi^2)} (\partial^2 \Psi)^2 \] (2.8)
Finally, the last term, which is the anomaly contribution, gives
\[ 2(a_{UV} - a_{IR}) \int d^4x \left( \frac{2(\partial \Psi)^2 \partial^2 \Psi}{(1 - \Psi)^3} + \frac{(\partial \Psi)^4}{(1 - \Psi)^4} \right) \] (2.9)
It follows immediately that if the background field is null, i.e. $\partial^2 \Psi = 0$, we can get rid of the term in equation (2.8) as well as the first term in equation (2.9). The remaining term will show its importance in proving the a theorem in the following section.

2.2 a Theorem
We are especially interested in the following term since it contains $\Delta a$ as its coefficient.
\[ 2(a_{UV} - a_{IR}) \int d^4x \frac{(\partial \Psi)^4}{(1 - \Psi)^4} \] (2.10)
Consider an amplitude with four null $\Psi$ insertion. The above term appears to be the only possible interaction. Therefore, we see that $a_{UV} - a_{IR}$ is the amplitude of four null filed scattering. Consider summing all the diagrams of four null $\Psi$ interaction each with momentum $k_i$ satisfying $\sum_i k_i = 0$ and $k_i^2 = 0$. This interaction contains four derivatives, so the momentum dependence of the amplitude must take the form $s^2 + t^2 + u^2$, with $s = 2k_1 \cdot k_2$, $t = 2k_1 \cdot k_3$, $u = 2k_1 \cdot k_4$. The coefficient of $s^2 + t^2 + u^2$ is proportional to $\Delta a$.
\[ \mathcal{A}(s, t, u) = \frac{(a_{UV} - a_{IR})}{f^4} (s^2 + t^2 + u^2) + \mathcal{O}(s^4, t^4, u^4) \] (2.11)
Let’s study more closely the structure of this amplitude. Consider a forward kinematics with $t = 0$.
\[ k_1 = -k_3, \quad k_2 = -k_4 \]
The amplitude for this process is
\[ \mathcal{A}(s) = 2 \frac{(a_{UV} - a_{IR})}{f^4} s^2 + \mathcal{O}(s^4) \] (2.12)
Remember $s + t + u = 0$. We can now promote the momentum as complex number, thus $\Delta a$ becomes the residue of $\mathcal{A}/s^3$. There are branch cuts along the real axis, at both positive ans negative $s$. Negative cut correpnds to states in the $u$ channel and gives identical contribution as the positive cut because of the $s \leftrightarrow u$ symmetry. Therefore we find
\[ a_{UV} - a_{IR} = \frac{f^4}{\pi} \int_{s > 0} ds \frac{\text{Im} \mathcal{A}(s)}{s^3} \] (2.13)
From the optical theorem, the imaginary part of the amplitude is positivdef definite, thus we achieve the a theorem, $a_{UV} > a_{IR}$. 

– 8 –
2.3 Weakly Relevent Flows

Let’s consider an easy example which allows us to calculate $\Delta a$ explicitly. Assume there exists an operator $O$ with conformal dimension $4 - \epsilon$ in a four dimensional CFT. We can follow our discussion about RG flow in section 1.4, just replace two dimension by four dimension. Consider a relevent deformation with $\epsilon > 0$.

$$\delta S = \int d^4x \lambda O(x)$$

(2.14)

The beta function of dimensionless coupling $g = \lambda \mu^{-\epsilon}$ is

$$\frac{dg}{d\log \mu} = -g\epsilon + \frac{1}{2}\pi C\Omega_3 g^2 + \cdots$$

(2.15)

We have $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$. For a small $\epsilon$, this deformation moves the theory toward an IR fixed point at $g_* = 2\epsilon/C\Omega_3$, where the constant $C$ is the coefficient of the $O$ term appearing in the $OO$ OPE.

$$O(x)O(0) \sim \frac{1}{x^8-2\epsilon} + C\frac{O(0)}{x^{4-\epsilon}} + \cdots$$

(2.16)

Remember we can restore conformal symmetry by promoting the energy scale to a background field. We replace the deformation $\int g(\mu)\mu^\epsilon O$ by

$$\delta S = \int d^4x g(F(x)\mu)\mu^\epsilon O(x)$$

(2.17)

Here $F(x) = e^{\tau(x)}$. The background field $\tau$ also transforms under scale transformation, $\tau \rightarrow \tau + \sigma$. We are interested in the difference between the partition function of the deformed theory and the original CFT. To the third order, their difference is

$$\frac{\mu^\epsilon}{2} \int d^4xd^4y \frac{g(F(x)\mu)g(F(y)\mu)}{(x-y)^{8-2\epsilon}} - C\frac{\mu^{3\epsilon}}{6} \int d^4xd^4y d^4z \frac{g(F(x)\mu)g(F(y)\mu)g(F(z)\mu)}{(x-y)^{4-\epsilon}(x-z)^{4-\epsilon}(z-x)^{4-\epsilon}} + \cdots$$

(2.18)

Although we are able to expand to the third order, we only need this result up to second order, which is the first term in the above equation. Expanding the coordinate $y$ around $x$

$$\int d^4xd^4y \frac{g(F(x)\mu)g(F(y)\mu)}{(x-y)^{8-2\epsilon}} = \int d^4xd^4y \left( \frac{g(F(x)\mu)}{(x-y)^{8-\epsilon}} + \frac{1}{8} g(F(x)\mu) \frac{\partial^2 g(F(x)\mu)}{(x-y)^{6-2\epsilon}} + \frac{1}{192} g(F(x)\mu) \frac{\partial^4 g(F(x)\mu)}{(x-y)^{6-2\epsilon}} + \cdots \right)$$

(2.19)

We should concentrate on the four derivative term since we know the dilaton interaction also contains four derivative. Perform the $y$ integral over energy range $(\mu+d\mu)^{-1} < |x-y| < \mu^{-1}$ and expand the result in $\epsilon$. Although the equation looks tedious, the calculation is actually very simple. We arrive at the result

$$- \frac{\Omega_3}{384} \int d^4x g(F(x)\mu)\partial^4 g(F(x)\mu)d\log \mu = - \frac{\Omega_3}{384} \delta^2(\mu)d\log \mu \int d^4x (\partial^2 \epsilon)^2$$

(2.20)
For the above equality, I used the chain rule.

\[ \frac{\partial}{\partial x} g(F\mu) = \beta(F\mu) \frac{\partial}{\partial x} \log F = \beta \frac{\partial \tau}{\partial x} \]

Remember the coefficient of \( \int (\partial \tau)^4 \) is proportional to \( \Delta a \).

\[ \Delta a = -\frac{\Omega_3}{768} \int \beta^2(g(\mu))d\log \mu = -\frac{\Omega_3}{768} \int_0^{g*} \beta(g)dg = \frac{1}{1152} \frac{\epsilon^3}{C^2\Omega_3} \]

Here we integrate to the IR fixed point at \( g^* = 2\epsilon/C\Omega_3 \). We see that the result is positive.

References