Introduction to the Ryu-Takayanagi Formula

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1 Introduction

The idea of holography has played central roles in recent developments of string theory. Holography claims that the degrees of freedom in $(d+2)$-dimensional quantum gravity are comparable to those of quantum many body systems in $(d+1)$-dimensions [1]. This was essentially realized by remembering the Bekenstein-Hawking (BH) formula:

$$S_{BH} = \frac{\text{Area}(\Sigma)}{4G_N} \tag{1.1}$$

where S_{BH} is the black hole entropy, Σ is the event horizon, and G_N is the Newton constant. Owing to the the discovery of AdS/CFT correspondence [2], we know explicit examples of holographies where the quantum gravity on $(d+2)$-dimensional anti-de Sitter spacetime (AdS_{d+2}) is equivalent to a certain conformal field theory in $(d+1)$-dimensions (CFT_{d+1}).

It has been pointed out for more than 20 years that the black-hole entropy (1.1) shares similarities with an entanglement entropy $S_\mathcal{A}$ [3, 4]. Here, $\mathcal{A}$ is the space-like submanifold on a constant time slice Σ. Indeed, in d-dimensional free field theories, we can show that the leading divergent terms of $S_\mathcal{A}$ in the UV limit $\epsilon \to 0$ obey the area law:

$$S_\mathcal{A} = \gamma \frac{\text{Area}(\partial\mathcal{A})}{\epsilon^{d-1}} + \cdots \tag{1.2}$$

where γ is the coefficient that depends on the system, but not on $\mathcal{A}$, and $\partial\mathcal{A}$ is the boundary of $\mathcal{A}$ in the constant time slice Σ, i.e. $\partial\mathcal{A} = \Sigma \setminus \mathcal{A}$. In quantum field theory, the entanglement entropy is always divergent, so we needed to introduce a UV cutoff ϵ. The equation (1.2) means that the entanglement entropy is not an extensive quantity as opposed to the thermodynamic entropy. This is consistent with the intuition that the divergence would be proportional to the number of EPR pairs that straddle the entangling surface. Actually, the original motivation to study the entanglement entropy was its similarity to the BH entropy [3, 4].

Getting intuition from this area law (1.2), Ryu and Takayanagi generalized the Bekenstein-Hawking formula (1.1) in the context of CFT with the use of AdS/CFT correspondence:

$$S_\mathcal{A} = \frac{\text{Area}(\gamma_\mathcal{A})}{4G_N^{d+2}} \tag{1.3}$$

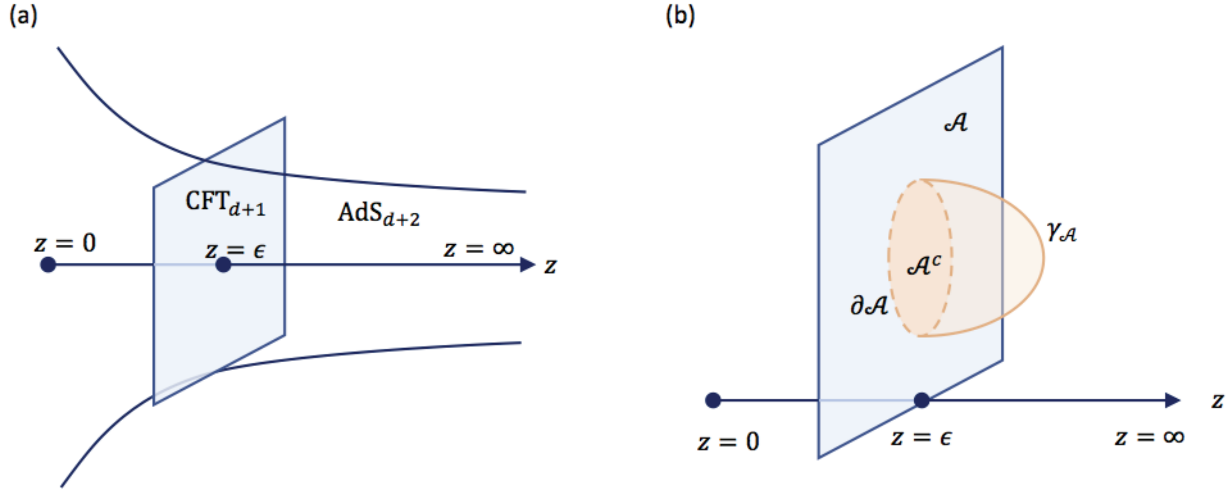

Figure 1: (a) Schematic of the AdS/CFT correspondence, where the Poincare metric with radius R is assumed: $ds^2 = R^2(dz^2 - dx_0^2 + \sum_{i=1}^{d} dx_i^2)/z^2$. Conformal field theory lives at $z = \epsilon$, where ϵ is the UV cutoff. (b) Schematic of the minimal surface $\gamma_{\mathcal{A}}$ in the AdS/CFT picture drawn in (a). Here, $\mathcal{A}^c$ is the compliment of $\mathcal{A}$.

where $\gamma_{\mathcal{A}}$ is the d-dimensional minimal surface whose boundary is given by $(d-1)$-dimensional manifold $\partial\gamma_{\mathcal{A}} = \partial\mathcal{A}$, and G_N^{d+2} is the Newton constant of the general gravity in AdS$_{d+2}$. The equation (1.3) is called the Ryu-Takayanagi (RT) formula [5]. Since the minimal surface tends to wrap the horizon in the presence of event horizon, RT formula (1.3) can be regarded as a generalization of the well-known BH formula (1.1).

In the original paper in 2006 [5], the authors gave a prescription for static time-independent situations. This prescription was subsequently generalized by Hubeny, Rangamani, and Takayanagi (HRT) in [6] to general states, including arbitrary time dependence. In this review, we will restrict ourselves to the discussion of time-independent situations. The generalization the the arbitrary time dependence can be achieved by nontrivial arguments considering Cauchy slice instead of constant time slice.

The Ryu-Takayanagi formula (1.3) provides an interesting insight into the AdS/CFT correspondence. The formula answers which region of AdS space is responsible to particular information in the dual CFT. Entanglement entropy is a useful universal viewpoint, since it does not depend on the details of theories such as specific operators or Wilson loops etc. In addition, the formula provides a useful tool for studying quantum many-body phases in condensed matter physics. For example, the fractional quantum Hall effect and quantum magnets on some geometrically frustrated lattices cannot be characterized by classical order parameters of some kind. These phases look featureless when one looks at correlation functions of local operators. Thus, the entanglement entropy is potentially useful to characterize these exotic phases.

When making use of the AdS/CFT correspondence, for example when calculating an entanglement entropy in condensed matter physics using a dual gravity theory, we need to pay attention that the gravity theory which is dual to the CFT is a general relativity plus quantum correction. But fortunately, in a certain limit, the quantum correction becomes negligible. The limit is expressed in terms of two parameters of CFT: $\lambda \to \infty, c_{\text{eff}} \to \infty$. Here, λ is the coupling constant and c_{eff} is the effective number of degrees of freedom. In the limit $c_{\text{eff}} \to \infty$, the number of degrees of freedom are scaled to be large, and the string interactions become weak. Then one can truncate to the tree level result. Furthermore, in the limit $\lambda \to \infty$, the classical string dynamics truncates to classical gravitational dynamics of the general relativistic form. In this limit, the massive string states in the dual description become heavy and decouple, leaving only the dynamics of semiclassical gravity. We will see that this holographic map simplifies dramatically the computation of entanglement entropy.

In the following, most of the work is based on the combination of two review articles [7, 8].

2 Entanglement Entropy

2.1 Entanglement Entropy in QFT

Let us define the entanglement entropy in a quantum field theory. If we start from a lattice model, where each lattice cites α have a finite-dimensional Hilbert space $\mathcal{H}_\alpha$, a pure quantum state of the system is an element of the tensor product Hilbert space:

$$|\Psi\rangle \in \otimes_\alpha \mathcal{H}_{tot} = \mathcal{H}_\alpha \tag{2.1}$$

Then the density matrix of a pure state is expressed as

$$\rho_{tot} = |\Psi\rangle \langle\Psi| \,. \tag{2.2}$$

The von Neumann entropy of the total system is clearly zero $S_{tot} = -\text{tr}\rho_{tot}\log\rho_{tot} = 0$. Now we imagine to divide the total system into two subsystems $\mathcal{A}$ and $\mathcal{A}^c$. Accordingly the total Hilbert space can be written as a direct product of two spaces $\mathcal{H}_{tot} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_{\mathcal{A}^c}$. The observer who is only accessible to the subsystem $\mathcal{A}$ will feel as if the total system is described by the reduced density matrix $\rho_\mathcal{A}$:

$$\rho_\mathcal{A} = \text{tr}_{\mathcal{A}^c}\rho_{tot} \tag{2.3}$$

where the trace is taken only over the Hillbert space $\mathcal{H}_{\mathcal{A}^c}$. Now we define the entanglement entropy of the subsystem $\mathcal{A}$ as the von Neumann entropy of the reduced density matrix $\rho_\mathcal{A}$:

$$S_\mathcal{A} = -\text{tr}_\mathcal{A}\rho_\mathcal{A}\log\rho_\mathcal{A} \tag{2.4}$$

This quantity provides us with a convenient way to measure how closely entangled a given wave function $|\Psi\rangle$ is. Although the separation of the constant time slice Σ into $\mathcal{A}$ and $\mathcal{A}^c$ is rather artificial here, we notice that the subsystem $\mathcal{A}^c$ is analogous to the inside of a black hole horizon for an observer sitting in $\mathcal{A}$, i.e., outside of the horizon.

It is also possible to define the entanglement entropy $S_\mathcal{A}(\beta)$ at finite temperature $T = \beta^{-1}$. This can be done just by replacing (2.2) with the thermal one

$$\rho_{thermal} = e^{-\beta\mathcal{H}} \tag{2.5}$$

where $\mathcal{H}$ is the total Hamiltonian.

It is also convenient to define another set of entropies called the Rényi entropies, which are simply defined in terms of the moments of the reduced density matrix:

$$S_{\mathcal{A}}^{(q)} = \frac{1}{1-q} \log \text{tr}_{\mathcal{A}}(\rho_{\mathcal{A}}^{q}) \tag{2.6}$$

The definition here requires $q \in \mathbb{Z}_{+}$, but we will see that oftentimes it is convenient to analytically continue the definition to $q \in \mathbb{R}_{+}$. This is especially useful when we use replica trick for computing entanglement entropy. The key point to note is the fact

$$S_{\mathcal{A}} = \lim_{q \to 1} S_{\mathcal{A}}^{(q)}. \tag{2.7}$$

2.2 Properties of the Entanglement Entropy

There are several useful properties that the entanglement entropy enjoys generally. Some of them are especially important in the context of Ryu-Takayanagi formula, so we will list them up in the following:

- Strong subadditivity relation:

$$S_{\mathcal{A}+\mathcal{B}+\mathcal{C}} + S_{\mathcal{B}} \leq S_{\mathcal{A}+\mathcal{B}} + S_{\mathcal{B}+\mathcal{C}} \tag{2.8}$$

$$S_{\mathcal{A}} + S_{\mathcal{C}} \leq S_{\mathcal{A}+\mathcal{B}} + S_{\mathcal{B}+\mathcal{C}} \tag{2.9}$$

  These are the most powerful inequalities obtained so far with respect to the entanglement entropy.

- If the density matrix ρ_{tot} is pure such as in the zero temperature system, we have

$$S_{\mathcal{A}} = S_{\mathcal{A}^c} \tag{2.10}$$

  This manifestly shows that the entanglement entropy is not an extensive quantity. This equality is violated at finite temperature.

The proofs are given, for example, in the textbook [9].

2.3 Path Integral and Replica Trick

In the following, we will first construct a path integral that computes the matrix elements of $\rho_{\mathcal{A}}$. Then we will see how to compute the Rényi entropies by considering a functional integral on a 'branched cover' geometry. By invoking an analytic continuum, we will finally get the entanglement entropy with the use of the equation (2.7).

We wish to define $\rho_{\mathcal{A}}$ on a constant-time slice Σ when there is no non-trivial time evolution. Since we are specifying a region $\mathcal{A}$, let's separate fields into two sets $\Phi(x) = \{\Phi_{\mathcal{A}}(x), \Phi_{\mathcal{A}^c}(x)\}$. Here we note that the reduced density matrix acts as an operator on $\mathcal{H}_{\mathcal{A}}$. Matrix elements of $\rho_{\mathcal{A}}$ can

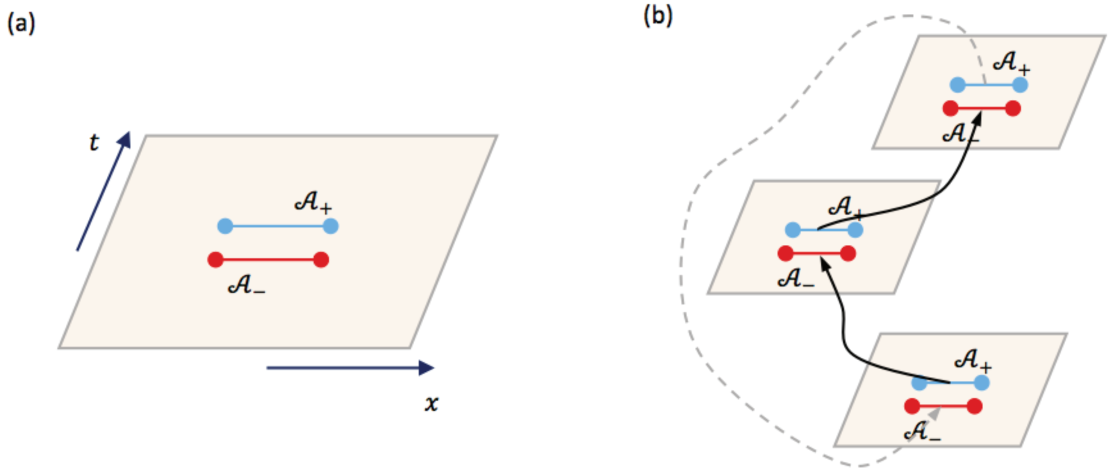

Figure 2: (a) Schematic of the Euclidean geometry for computing the matrix elements of the reduced density matrix $\rho_{\mathcal{A}}$. We have sketched the situation in two-dimensional Euclidean space as indicated. The two cuts at $\mathcal{A}$ have been separated in an exaggerated manner to indicate the boundary conditions we need to impose. (b) The n-sheeted Rieman surface $\mathcal{B}_n$ for the computation of the product of the reduced density matrices $(\rho_{\mathcal{A}})^q$ and the trace of it $\mathrm{tr}_{\mathcal{A}}\rho_{\mathcal{A}}$ (arrow plus the dashed arrow).

be defined by their action on fields supported in A. To see this, let us imagine regulating the path integral by imposing boundary conditions for fields in A as follows:

$$\Phi_{\mathcal{A}}|_{t=0-} = \Phi_-, \quad \Phi_{\mathcal{A}}|_{t=0+} = \Phi_+ \tag{2.11}$$

This is equivalent to cutting open the path integral in a restricted domain of space $\mathcal{A}$ at time $t = 0^{\pm}$, and projecting the result onto definite field values $\Phi_{\pm}$. [See figure 2(a)]. Thus, we can write down the reduced density matrix $\rho_{\mathcal{A}}$ using path integral:

$$\begin{aligned}(\rho_{\mathcal{A}})_{-+} &= \int [\mathcal{D}\Phi] e^{-S_{QFT}[\Phi]} \delta_E(\Phi_{\mp\mathcal{A}}) \\ \delta_E(\Phi_{\mp\mathcal{A}}) &\equiv \delta(\Phi_{\mathcal{A}}(t=0^-) - \Phi_-)\delta(\Phi_{\mathcal{A}}(t=0^+) - \Phi_+)\end{aligned} \tag{2.12}$$

We have constructed a functional integral to compute the matrix elements of $\rho_{\mathcal{A}}$. The multiplication of the reduced density matrices $(\rho_{\mathcal{A}})^q$ is achieved by simply integrating over the boundary conditions for the + component in the k^th density matrix with the $-$ component of the $(k+1)^{st}$ density matrix:

$$(\rho_{\mathcal{A}})^q_{-+} = \int \prod_{j=1}^{q-1} d\Phi_+^{(j)} \delta(\Phi_+^{(j)} - \Phi_-^{(j+1)}) \times \left[\int \prod_{k=1}^{q} [\mathcal{D}\Phi^{(k)}] \left\{ e^{-\sum_{k=1}^{q} S_{QFT}[\Phi^{(k)}]} \delta_E(\Phi^{(k)}_{\mp\mathcal{A}}) \right\} \right]. \tag{2.13}$$

Here, the outer integral performs the desired identification of the reduced density matrix elements, and the inner functional integral simply replicates the path integral which computes the individual

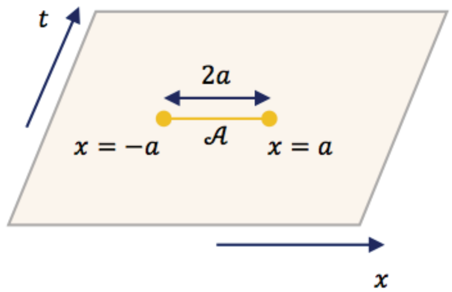

Figure 3: Schematic of the CFT_{1+1} on $\mathbb{R}^{1,1}$. We consider a static case. The specified region $\mathcal{A}$ is defined as $\mathcal{A} = \{x|x \in (-a, a)\}$ and the volume (length) of $\mathcal{A}$ is $2a$.

matrix elements. [See figure 2(b)]. Now, we should view each copy of $\rho_{\mathcal{A}}$ as being computed on a copy of the background spacetime $\mathcal{B}$. Then, we now can compute the path integral of the theory by integrating over all the fields living on the background $\mathcal{B}_q$, which is constructed by taking q-copies of these manifolds $\mathcal{B}$ and making identifications across them as prescribed by (2.13). We define $\mathcal{B}_q$ as being

$$\mathcal{Z}_q[\mathcal{A}] = \text{tr}_{\mathcal{A}}(\rho_{\mathcal{A}}^q) \equiv \mathcal{Z}[\mathcal{B}_q] \tag{2.14}$$

Then we can rewrite the equation (2.6) as

$$S_{\mathcal{A}}^{(q)} = \frac{1}{1-q} \log\left(\frac{\mathcal{Z}_q[\mathcal{A}]}{\mathcal{Z}_1[\mathcal{A}]^q}\right) = \frac{1}{1-q} \log\left(\frac{\mathcal{Z}[\mathcal{B}_q]}{\mathcal{Z}[\mathcal{B}]^q}\right). \tag{2.15}$$

We now have functional integrals that compute matrix elements of arbitrary integer powers of the density matrix. Taking the trace, which now simply involves identifying $\Phi_-^{(1)}$ with $\Phi_+^{(q)}$ for the Euclidean computation, we get the Rényi entropies defined in (2.6).

2.4 A single interval in CFT_{1+1} on $\mathbb{R}^{1,1}$

Let's try to compute the entanglement entropy in CFT_{1+1}. To start with we will consider simple static states for a CFT_{1+1} on $\mathbb{R}^{1,1}$. We will exploit the time independence to work in Euclidean signature, mapping the background geometry to the complex plane $\mathbb{C} = \mathbb{R}^2$. Consider the vacuum state $|0\rangle$ of the CFT_{1+1} on $\mathbb{C}$. We pick an instant of time $t = 0$ and set $\mathcal{A} = \{x|x \in (-a, a)\}$. What is clear for the complex plane is that the cyclic gluing of q copies of that plane does not change the topology, hence $\mathcal{B}_q$ is a genus-0 surface. we just have to deal with a function that is multi-branched.

As a first step, we are required to compute the partition function $\mathcal{Z}[\mathcal{B}_q]$. Since $\mathcal{B}_q$ is a genus-0 surface, we should be able to conformally map it back to the complex plane. We can start with fields $\phi(x, t)$, which live on a single copy of the complex plane, and upgrade them to $\phi_k(x, t)$ with $k = 1, 2, \cdots, q$ which live on the q-copies. The gluing conditions for constructing $\mathcal{B}_q$ can be mapped to boundary conditions for the fields:

$$\phi_k(x, o^+) = \phi_{k+1}(x, 0^-), \quad x \in \mathcal{A} = \{x|x \in (-a, a)\} \tag{2.16}$$

These boundary conditions can be equivalently implemented by passing from the basis of q-independent fields to a composite field $\varphi(x,t)$ living on $\mathcal{B}$ obeying twisted boundary conditions. The map one seeks should thus implement the twists by the cyclic $\mathbb{Z}_q$ replica symmetry. We are no longer working with the original CFT but rather with the cyclic product orbifold theory.

One introduces then, as in any orbifold theory, a set of twist fields which implement the twisted boundary conditions. The twists are by q^{th} roots of unity, and the main property we need for the twist operator $\mathcal{T}_q$ is that it induces a branch-cut of order q for the fields at its insertion point. Standard orbifold technology reveals that the scaling dimension of the twist operator is

$$h_q = \tilde{h}_q = \frac{c}{24}\left(q - \frac{1}{q}\right) \tag{2.17}$$

where c is the central charge. The main advantage of introducing these fields is that we can write down the partition function of our theory on $\mathcal{B}_q$ in terms of correlation functions of the twist fields:

$$\mathcal{Z}[\mathcal{B}_q] = \prod_{k=0}^{q-1} \langle \mathcal{T}_q(-a,0)\mathcal{T}_q(a,0)\rangle_{\mathcal{B}} \tag{2.18}$$

where we used the subscript $\mathcal{B}$ to indicate that the correlation function is meant to be computed on the original manifold. For our choice of $\mathcal{A}$ being a single connected interval, the above computation is very simple. Treating the twist fields as conformal primaries with scaling dimension given by (2.17), we learn that

$$\mathcal{Z}[\mathcal{B}_q] = \left(\frac{2a}{\epsilon}\right)^{-\frac{c}{6}\left(q-\frac{1}{q}\right)} \tag{2.19}$$

where we introduced a UV regulator ϵ to write down the correlation function. We now find the Rényi entropy with the use of equation (2.15) as

$$\begin{aligned} S_{\mathcal{A}}^{(q)} &= \frac{1}{1-q}\log\left(\frac{2a}{\epsilon}\right)^{-\frac{c}{6}\left(q-\frac{1}{q}\right)} \\ &= \frac{c}{6}\left(1+\frac{1}{q}\right)\log\left(\frac{2a}{\epsilon}\right) \end{aligned} \tag{2.20}$$

In this simple case it is trivial to analytically continue from $q \in \mathbb{Z}_+$ to $q \sim 1$. Thus, one clearly obtains the following:

$$S_{\mathcal{A}} = \frac{c}{3}\log\frac{2a}{\epsilon}. \tag{2.21}$$

3 Holographic Derivation of Entanglement Entropy

3.1 AdS dual of a single interval in CFT$_{1+1}$ on $\mathbb{R}^{1,1}$

We would like to derive the entanglement entropy (2.21) holographically. To compute the entanglement entropy in this situation using holographic formula (1.3), we need to find a geodesics between the two points $(x^1, z) = (-a, \epsilon)$ and $(x^1, z) = (a, \epsilon)$ in the Poincaré coordinate

$$ds^2 = R^2\frac{dz^2 - dx_0^2 + dx_1^2}{z^2}. \tag{3.1}$$

Here $dx_0^2 = 0$ because we are thinking of a static case and sitting on the constant-time slice. Therefore the geodesic action can be written as

$$S = R \int d\xi \frac{\sqrt{x'(\xi)^2 + z'(\xi)^2}}{z}. \tag{3.2}$$

Varying this action, one can check the resulting equations of motion are solved by half circle in the xz plane.

$$(x, z) = a(\cos\xi, \sin\xi), \quad (\epsilon/a \leq \xi \leq \pi - \epsilon/a) \tag{3.3}$$

The length of $\gamma_{\mathcal{A}}$ can be found as

$$\text{Length}(\gamma_{\mathcal{A}}) \quad = \quad 2R \int_{\epsilon/a}^{\pi/2} \frac{d\xi}{\sin\xi} = -2R\log(\epsilon/2a) = 2R\log\frac{2a}{\epsilon} \tag{3.4}$$

Finally the entanglement entropy can be obtained as follows

$$S_{\mathcal{A}} \quad = \quad \frac{\text{Length}(\gamma_{\mathcal{A}})}{4G_N^{(3)}} = \frac{R}{2G_N^{(3)}}\log\frac{2a}{\epsilon} = \frac{c}{3}\log\frac{2a}{\epsilon}. \tag{3.5}$$

Here in the third line, we used the relation given by the AdS/CFT correspondence

$$c = \frac{3R}{2G_N^{(3)}}. \tag{3.6}$$

In evaluating the integral, we converted the UV cut-off $z = \epsilon$ into a restriction on the domain of the affine parameter along the curve. The equation (3.5) is in consistent with the direct computation (2.21) in the CFT_{1+1} side. This is no coincidence! In both cases, the result is dictated purely by the conformal symmetry and we have indicated that the result is universally determined simply by the central charge.

3.2 Multiple Disjoint Intervals $\mathcal{A}$

The computation of entanglement entropy for multiple disjoint intervals in a CFT is a formidable task. However, the holographic answer however turns out to be very simple. Let us consider $\mathcal{A} = \cup_i \mathcal{A}_i$ with $\mathcal{A}_i = \{x \in \mathbb{R} | x \in (u_i, v_i)\}$. Then we can consider geodesics that connect the left endpoint of one-interval, say $\mathcal{A}_i$, with the right endpoint of any other $\mathcal{A}_j$ (including itself). The length of such geodesics are simply proportional to $2\log\frac{|u_i - v_j|}{\epsilon}$. Then holographic answer is then simplify

$$S_{\mathcal{A}} = \min\left(\frac{c}{3}\sum_{(i,j)}\log\frac{|u_i - v_j|}{\epsilon}\right) \tag{3.7}$$

with the sum running over all pairs of choices from which we pick the globally minimum result. For instance, for two intervals, we have

$$S_{\mathcal{A}} = \min\left(\frac{c}{3}\log\frac{|u_1 - v_1|}{\epsilon} + \frac{c}{3}\log\frac{|u_2 - v_2|}{\epsilon}, \frac{c}{3}\log\frac{|u_1 - v_2|}{\epsilon} + \frac{c}{3}\log\frac{|u_2 - v_1|}{\epsilon}\right) \tag{3.8}$$

This is illustrated in figure 4.

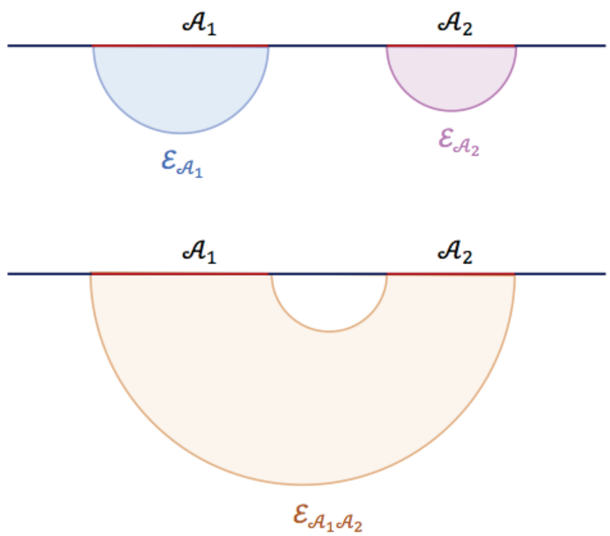

Figure 4: Sketch of the two potential extremal surfaces for a disjoint union of two regions $\mathcal{A}_1$ and $\mathcal{A}_2$. We either have the union of the two individual extremal surfaces $\mathcal{E}_{\mathcal{A}_1} \cup \mathcal{E}_{\mathcal{A}_2}$ or surface $\mathcal{E}_{\mathcal{A}_1\mathcal{A}_2}$ which connects the two regions.

3.3 Holographic proof of Strong Subadditivity

We have seen in section 2.2 strong subadditivity relation which entanglement entropy holds. In the context of AdS/CFT, we can prove these inequalities in a geometric manner.

Let us start with three regions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ on a time slice of a given CFT so that there are no overlaps between them. We extend this boundary setup toward the bulk AdS (see figure 5). Consider the entanglement entropy $S_{\mathcal{A}+\mathcal{B}}$ and $S_{\mathcal{B}+\mathcal{C}}$. In the holographic description 1.3, they are given by the areas of minimal area surfaces $\gamma_{\mathcal{A}+\mathcal{B}}$ and $\gamma_{\mathcal{B}+\mathcal{C}}$ which satisfy $\partial\gamma_{\mathcal{A}+\mathcal{B}} = \partial(\mathcal{A}+\mathcal{B})$ and $\partial\gamma_{\mathcal{B}+\mathcal{C}} = \partial(\mathcal{B}+\mathcal{C})$. Then it is easy to see that we can divide these two minimal surfaces into four pieces and recombine into (i) two surfaces $\mathcal{E}_{\mathcal{B}}$ and $\mathcal{E}_{\mathcal{A}+\mathcal{B}+\mathcal{C}}$, or (ii) two surfaces $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{C}}$, corresponding to two different ways of the recombination. Here we meant $\mathcal{E}_X$ is a surface which satisfies $\partial\mathcal{E}_X = \partial X$. Since in general $\mathcal{E}_X$'s are not minimal area surface, we have $\text{Area}(\mathcal{E}_X) \geq \text{Area}(\gamma_X)$. Therefore, we as we fan see from figure 5, this argument immediately leads to

$$\text{Area}(\gamma_{\mathcal{A}+\mathcal{B}}) + \text{Area}(\gamma_{\mathcal{A}+\mathcal{B}}) = \text{Area}(\mathcal{E}_{\mathcal{B}}) + \text{Area}(\mathcal{E}_{\mathcal{A}+\mathcal{B}+\mathcal{C}}) \geq \text{Area}(\gamma_{\mathcal{B}}) + \text{Area}(\gamma_{\mathcal{A}+\mathcal{B}+\mathcal{C}}) \tag{3.9}$$

$$\text{Area}(\gamma_{\mathcal{A}+\mathcal{B}}) + \text{Area}(\gamma_{\mathcal{A}+\mathcal{B}}) = \text{Area}(\mathcal{E}_{\mathcal{A}}) + \text{Area}(\mathcal{E}_{\mathcal{C}}) \geq \text{Area}(\gamma_{\mathcal{A}}) + \text{Area}(\gamma_{\mathcal{C}}) \tag{3.10}$$

4 Heuristic derivation of Holographic Formula

We have seen a simplest example of the Ryu-Takayanagi formula in the theory CFT_{1+1} on $\mathbb{R}^{1,1}$. So let's try to show the RT formula (1.3) now. In principle, we should be able to show the

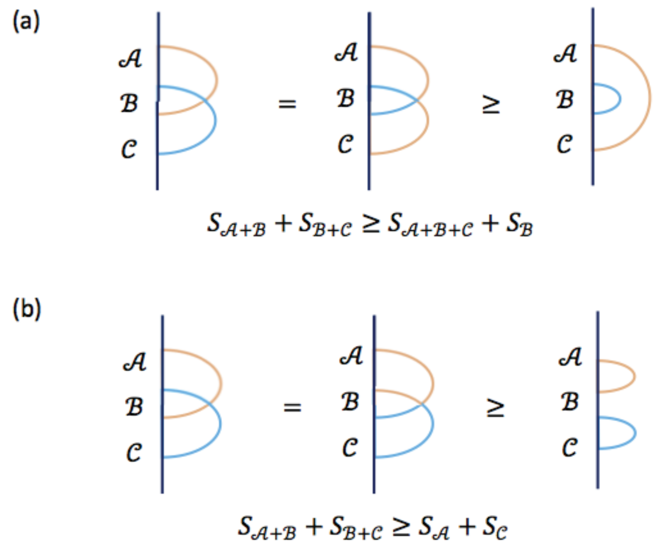

Figure 5: A holographic proof of the strong subadditivity of the entanglement entropy. To make the figures simple, we project the slice of a $(d+1)$-dimensional AdS space onto a two-dimensional plane. This simplification does not change our result.

holographic formula based on the first principle of the AdS/CFT correspondence known as the bulk to boundary relation:

$$Z_{CFT} = Z_{AdSGravity}. \tag{4.1}$$

In the CFT side, the entanglement entropy can be found if we can compute the partition function on the $(d+1)$-dimensional n-sheeted space (2.14) via the formula (2.15). This space$\mathcal{B}_q$ is characterized by the presence of the deficit angle $\delta = 2\pi(1-q)$ on the surface $\partial\mathcal{A}$. Then we need to find a $(d+2)$-dimensional back reacted geometry $\mathcal{S}_q$ in the dual AdS space by solving the Einstein equation with the negative cosmological constant such that its metric approaches to that of $\mathcal{B}_q$ at the boundary $z \to 0$. This is a technically complicated mathematical problem if we try to solve it directly.

To circumbent this situation, the following natural assumption is made [10]: the back reacted geometry $\mathcal{S}_q$ is given by a q-sheeted AdS_{d+2}, which is defined by puttingthe deficit angle δ localized on a codimension two surface $\gamma_{\mathcal{A}}$. Under this assumption, the Ricci scaler behaves like a delta function

$$R = 4\pi(1-q)\delta(\gamma_{\mathcal{A}}) + R^{(0)} \tag{4.2}$$

where $\delta(\gamma_{\mathcal{A}})$ is a delta function localized on $\gamma_{\mathcal{A}}$, and $R^{(0)}$ is the Ricci scaler of the pure AdS_{d+2}. Then we plug this in the supergravity action

$$\begin{aligned} \log\left(Z^{(q)}_{AdS}\right) &= -\frac{1}{16\pi G_N^{(d+2)}}\int_{\mathcal{M}} dx^{(d+2)}\sqrt{g}(R+\Lambda)+\cdots \\ &= -\frac{4\pi(1-q)\mathrm{Area}(\gamma_{\mathcal{A}})}{16\pi G_N^{(d+2)}} - \frac{1}{16\pi G_N^{(d+2)}}\int_{\mathcal{M}} dx^{(d+2)}\sqrt{g}(R^{(0)}+\Lambda)+\cdots \end{aligned} \tag{4.3}$$

where we only make explicit the bulk Einstein-Hilbert action. Now we make use of the bulk to boundary relation (4.1) to get

$$\log(\mathcal{Z}[\mathcal{B}_q]) = \log(Z^{(q)}_{AdS}) = \frac{(1-q)\text{Area}(\gamma_{\mathcal{A}})}{4G_N^{(d+2)}} + (q\text{-independent terms}) \tag{4.4}$$

With the use of equation (2.15), Rényi entropy becomes

$$S^{(q)}_{\mathcal{A}} = \frac{\text{Area}(\gamma_{\mathcal{A}})}{4G_N^{(d+2)}} \tag{4.5}$$

and therefore we get the desired holographic formula (1.3) with the use of the equation (2.7).

5 Other Examples of Holography

5.1 Holography in CFT_{1+1} on $\mathbf{S} \times \mathbb{R}$

We have carefully discussed the theory of CFT_{1+1} on $\mathbb{R}^{1,1}$. Now let's see the case on $\mathbf{S} \times \mathbb{R}$ instead of $\mathbb{R}^{1,1}$. This situation is a compactified circle at zero temperature. The CFT_{1+1} side result is given by [11, 5]

$$S_{\mathcal{A}} = \frac{c}{3} \cdot \log\left(\frac{L}{\pi\epsilon}\sin\left(\frac{\pi l}{L}\right)\right) \tag{5.1}$$

where l and L are the length of subsystem $\mathcal{A}$ and total system $\mathcal{A} \cup \mathcal{A}^c$, respectively. Corresponding AdS gravity dual has a metric

$$ds^2 = R^2(-\text{cosh}\rho^2 dt^2 + d\rho^2 + \text{sinh}\rho^2 d\theta^2) \tag{5.2}$$

In this coordinate, UV cut-off condition is $\rho \leq \rho_0$. Here we have an approximate relation

$$e^{\rho_0} \sim L/\epsilon. \tag{5.3}$$

The $(1+1)$-dimensional spacetime for the CFT_{1+1} is identified with the cylinder (t, θ) at the boundary $\rho = \rho_0$. The subsystem $\mathcal{A}$ is the region $0 \leq \theta \leq 2\pi l/L$. Then the minimal surface $\gamma_{\mathcal{A}}$ is identified with the static geodesic that connects the boundary points $\theta = 0$ and $\theta = 2\pi l/L$, with t fixed, traveling inside the cylinder. [See figure (6)].

With the cutoff introduced above, the geodesic distance $\text{Area}(\gamma_{\mathcal{A}})$ is given by

$$\cosh\left(\frac{\text{Area}(\gamma_{\mathcal{A}})}{R}\right) = 1 + 2\text{sinh}^2\rho_0 \text{sin}^2\frac{\pi l}{L} \tag{5.4}$$

Assuming the large UV cutoff $e^{\rho_0} \gg 1$, the equation (5.5) becomes

$$\text{Area}(\gamma_{\mathcal{A}}) = R \cdot \log\left(e^{2\rho_0}\text{sin}^2\frac{\pi l}{L}\right) \tag{5.5}$$

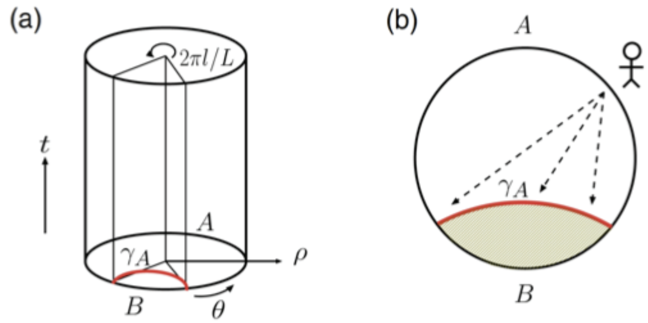

Figure 6: Sketch of the CFT_{1+1} on $\mathbf{S} \times \mathbb{R}$. The figures are cited from the original paper of Ryu and Takayanagi [5].

With the use of Ryu-Takayanagi formula 1.3, we get

$$S_{\mathcal{A}} = \frac{R}{4G_N^{(3)}} \log\left(e^{2\rho_0} \sin^2 \frac{\pi l}{L}\right) = \frac{c}{3} \log\left(e^{\rho_0} \sin \frac{\pi l}{L}\right). \tag{5.6}$$

Here, again we utilized the relation (3.6). Noticing the approximate relation (5.3), we realized that this holographic entanglement entropy is in consistent with the direct computation result given in (5.1).

5.2 Holography in infinite CFT_{1+1} at finite temperature

We consider CFT_{1+1} infinite system at finite temperature $T = \beta^{-1}$ from the viewpoint of AdS/CFT correspondence. It can be treated by applying the conformal map technique and analytic formulas have been obtained [11, 5]:

$$S_{\mathcal{A}} = \frac{c}{3} \cdot \log\left(\frac{\beta}{\pi\epsilon} \sinh\left(\frac{\pi l}{\beta}\right)\right). \tag{5.7}$$

We assume that the spacial length of the total system L is infinite, so we have $\beta/L \ll 1$. In such a high temperature region, the gravity dual of the conformal field theory is described by the Euclidean Banados-Teitelboim-Zanelli (BTZ) black hole [12]. Its metric looks

$$ds^2 = (r^2 - r_+^2)d\tau^2 + \frac{R^2}{r^2 - r_+^2} dr^2 + r^2 d\varphi^2. \tag{5.8}$$

The Euclidean time is compactified as $\tau \sim \tau + \frac{2\pi R}{r_+}$ to obtain a smooth geometry. We also impose the periodicity $\varphi \sim \varphi + 2\pi$. By taking the boundary limit $r \to \infty$, we find the relation between the boundary CFT and the geometry:

$$\frac{\beta}{L} = \frac{R}{r_+} \ll 1 \tag{5.9}$$

The subsystem $\mathcal{A}$ is the region $0 \le \varphi \le 2\pi l/L$ at the boundary. By extending the formula (1.3) into asymtoptically AdS spaces, the entropy can be computed from the length of the space-like

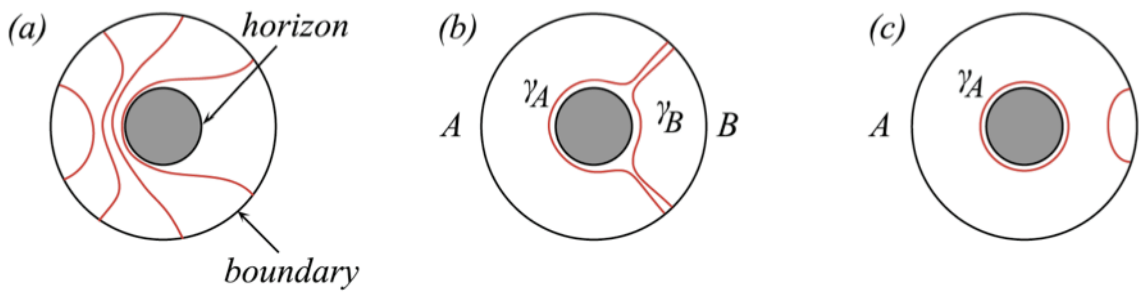

Figure 7: (a) Minimal surfaces $\gamma_{\mathcal{A}}$ in the BTZ black hole for various size of $\partial\mathcal{A}$. (b) $\gamma_{\mathcal{A}}$ and $\gamma_{\mathcal{B}}$ wrap the different parts of the horizon. (c) When $\partial\mathcal{A}$ gets larger, $\gamma_{\mathcal{A}}$ is separated into two parts: one is wrapped on the horizon and theother is localized near the boundary. All of the figure is cited from [7].

geodesic starting from $\varphi = 0$ and ending at $\varphi = 2\pi l/L$ at the boundary $r = r_0 \to \infty$ at fixed time. This geodesic distance can be found analytically as

$$\cosh\left(\frac{\text{Area}(\gamma_{\mathcal{A}})}{R}\right) = 1 + \frac{2r_0^2}{r_+^2}\sinh^2\left(\frac{\pi l}{\beta}\right) \tag{5.10}$$

The relation between the cut-off ϵ in CFT and the one r_0 of AdS is given by $\frac{r_0}{r_+} = \frac{\beta}{\epsilon}$. Then we can see that the CFT result (5.7) is achieved from the AdS part as well.

It is useful to understand the geometric meaning at finite temperatures. The geodesic line in the BTZ black hole takes the form shown in figure 7(a). When the size of $\mathcal{A}$ is small, it is almost the same as the one in the ordinary AdS$_3$. However, as the size becomes large, the turning point of the geodesic line approaches the horizon and eventually the geodesic line covers a part of horizon. This is the reason why we find a thermal extensive behavior of the entropy when $l/\beta \gg 1$ in equation (5.7). The thermal entropy in a conformal field theory is dual to the black hole entropy in its gravity description via the AdS/CFT correspondence. In the presence of a horizon, $S_{\mathcal{A}}$ is not equal to $S_{\mathcal{A}^c}$ since the corresponding geodesic lines wrap different parts of the horizon. [See figure 7(b)]. This is a typical property of the entanglement entropy at finite temperatures. As we discussed in section 2.2, at zero temperature, we have $S_{\mathcal{A}} = S_{\mathcal{A}^c}$, but at finite temperatures, genrerally $S_{\mathcal{A}} \neq S_{\mathcal{A}^c}$. Now we see this is due to the emergence of horizon in the dual AdS space!

We can also expect that when $\mathcal{A}$ becomes very large before it coincides with the total system, $\gamma_{\mathcal{A}}$ becomes separated into the horizon circle and a small half circle localized on the boundary like in the figure 7(c). In this situation, $S_{\mathcal{A}}$ will get large contribution from the event horizon.

6 Summary

We have reviewed simple three theories where Ryu-Takayanagi holographic formula can be easily checked: (i) CFT$_{1+1}$ on $\mathbb{R}^{1,1}$, (ii) CFT$_{1+1}$ on $\mathbf{S} \times \mathbb{R}$, and (iii) infinite CFT$_{1+1}$ at finite temperature. In addition, we reviewed geometric structure behind two of the important properties of the entanglement entropy: (a) strong subadditivity, (b) $S_{\mathcal{A}} \neq S_{\mathcal{A}^c}$ at finite temperature. There are several important, but more complicated theories that we could not cover, but those would be a natural extension beyond the topics covered in this paper.

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