Problem Set 1
Physics 445
Due April 8

Some abbreviations: P&S - Peskin & Schroeder

1. Take the quantum-mechanical action for a point particle with dynamical einbein $e(\tau)$ and $X(\tau)$ fields and a mass parameter $m$:

$$S = \frac{1}{2} \int d\tau \left( e^{-1} \dot{X}^{\mu} \dot{X}^{\nu} \eta_{\mu\nu} - em^{2} \right).$$

Find the equations of motion. Integrate out $e$ and find the action for a relativistic point particle.

2. Let’s turn to the example of a complex scalar field in four dimensions with action,

$$S = \int d^{4}x \left( -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi^{*} - V(|\phi|^{2}) \right), \quad (0.1)$$

where $V$ is the potential energy. This theory is Poincaré invariant where the Poincaré group is generated by translations, rotations and boosts. Derive the stress-energy tensor, $T_{\mu\nu}$, for this theory using Noether’s theorem for the space-time translational symmetries. Show that the stress-energy tensor is conserved. Is it a symmetric tensor in this example?

3. Let’s do one introductory exercise with differential forms. Take $\mathbb{R}^{3}$ as an example. The exterior derivative of a function $f(x, y, z)$, or a 0-form, is defined to be:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

We recognize the components of this resulting 1-form as the gradient of $f$. The wedge product of forms is antisymmetric, $dx \wedge dy = -dy \wedge dx$. More generally $\omega_{1} \wedge \omega_{2} = (-1)^{nm} \omega_{2} \wedge \omega_{1}$ where $\omega_{1}$ is an $n$-form and $\omega_{2}$ is an $m$-form. In this sense, odd forms are like fermions and anti-commute. We often omit the explicit “$\wedge$” symbol and write, for example, a 2-form as $dxdy$ rather than $dx \wedge dy$.

The exterior derivative, $d$, satisfies $d^{2} = 0$. Denote a general $p$-form by

$$\omega = \frac{1}{p!} \omega_{i_{1},\ldots,i_{p}}(x) dx^{i_{1}} \cdots dx^{i_{p}} \equiv \omega_{[i_{1},\ldots,i_{p}]}(dx^{i_{1}} \cdots dx^{i_{p}})$$

where vertical bars around a set of indices indicate they are summed only over $i_{1} < i_{2} \cdots < i_{p}$. The exterior derivative acts on $\omega$ by

$$d\omega = \frac{1}{p!} \frac{\partial \omega_{i_{1},\ldots,i_{p}}}{\partial x^{j}} dx^{j} dx^{i_{1}} \cdots dx^{i_{p}}.$$
to give a $p + 1$-form.

(i) For a general 1-form $\omega$ in $\mathbb{R}^3$, compute $d\omega$. Do you recognize the components of this 2-form?

(ii) For a general 2-form $\omega$ in $\mathbb{R}^3$, compute $d\omega$. Do you recognize the components of this 3-form?

Let's define the dual form, $\ast \omega$, for a $p$-form $\omega$ in $n$ dimensions. This dual form is an $n - p$ form with components

$$(\ast \omega)_{k_1, \ldots, k_{n-p}} = \omega_{i_1, \ldots, i_p} \epsilon^{i_1, \ldots, i_p}_{k_1, \ldots, k_{n-p}}.$$ 

Here $\epsilon_{i_1, \ldots, i_n}$ is the totally anti-symmetric Levi-Civita tensor. In QED, define a 1-form gauge potential $A = A_\mu dx^\mu$ as in lecture. Define the 2-form field strength $F = dA$.

(iii) Check that $F$ is indeed gauge-invariant. Also check the identification of the components of $F$ with the electric and magnetic fields.

(iv) Write Maxwell’s equations in terms of $d, F$, and $\ast F$.

(v) Consider the action $S = \alpha \int F \wedge F$ with constant $\alpha$. What equations of motion does it imply for the gauge-field $A$?
