1 Spin waves

1.1 Correlator of exponents

We write the correlation function as a path integral with sources

\[ \langle T e^{i\phi(x)} e^{-i\phi(0)} \rangle = \frac{1}{Z_0} \int D\phi \, e^{iS[\phi]+i\phi(x)-i\phi(0)} \]

\[ = \frac{1}{Z_0} \int D\phi \, e^{i \int d^4y (L[\phi]+J(y)\phi(y))} \]

\[ = \frac{Z[J]}{Z_0} \]

where \( J(y) = \delta(y-x) - \delta(y) \). The path integral is directly computable in the presence of a source

\[ Z[J] = Z_0 \exp \left( -\frac{1}{2} \int d^4zd^4y \, J(z)D(z-y)J(y) \right) \]

Plugging in the value of \( J \) and integrating over \( y \) and \( z \) gives the desired answer

\[ \langle T e^{i\phi(x)} e^{-i\phi(0)} \rangle = e^{D(x)-D(0)} \quad (1.1) \]

1.2 Effective Lagrangian

Since \( \phi \) is invariant under shifts, we can’t have any polynomials of \( \phi \) in the action. The Lagrangian must then be a function of derivatives of \( \phi \).

\( \phi \) has classical dimension \( d/2 - 1 \) and a single derivative has dimension 1. Therefore a term with \( m \) factors of \( \phi \) and \( n \) derivatives has dimension \( md/2 + n - m \). Furthermore, since each \( \phi \) has to have at least one derivative acting on it, we also need \( n \geq m \).

‘Renormalizability’ of the term now implies

\[ \frac{md}{2} \leq d, \quad \Rightarrow \quad m \leq 2 \]

The \( m = 1 \) term has to be a total derivative therefore \( m = 2 \). To make the term ‘renormalizable’, we need to add two derivatives, one for each \( \phi \). Since \( \phi \) is scalar, there is only one way to contract the indices of the derivatives, i.e., with each other. Throwing in an overall constant, this gives us the desired Lagrangian

\[ \mathcal{L} = \frac{1}{2} \rho (\nabla \phi)^2 \quad (1.2) \]

1.3 Spin correlator

The two point function of \( \phi \) has the Fourier transform

\[ D(p) = \frac{1}{\rho p^2} \]
In position space this is
\[ D(x) = \frac{\Gamma(d/2 - 1)}{4\pi^{d/2}} |x|^{2-d} = \alpha_d |x|^{2-d} \]
for \( d \neq 2 \). In 2 dimensions, we will find instead (with appropriate regulation) \( D(x) = \beta \log x \), where \( \beta \) is some constant.

This function is divergent in \( d \geq 2 \) at small distances. It hence needs to be regulated by a UV cutoff at the lattice spacing \( a \). More precisely, we need to substitute
\[
D(0) = \frac{\alpha_d}{a^{d-2}}, \quad d > 2 \\
D(0) = \beta \log a, \quad d = 2
\]

In one dimension, we are safe from the UV divergence. Using our result in the first part, the two point spin correlator is given by
\[
\langle s(x)s^*(0) \rangle = \begin{cases} 
A^2 e^{\beta_1 |x|}, & d = 1 \\
A^2 e^{\beta \log(x/a)} = A^2 \left( \frac{x}{a} \right) \beta, & d = 2 \\
A^2 e^{\alpha_d |x|}, & d = 2 > 1
\end{cases}
\]

The UV regulator can be ignored on the first two lines by appropriately defining \( A \) to absorb it.

Let’s try and interpret these results. For \( d > 2 \), the correlation function decays rapidly to 1 at large distances. If the system is in a state with nonzero average spin, i.e., spontaneously broken \( U(1) \) symmetry, any thermal fluctuation in the phase at a given point will not generate fluctuations significantly far away from it. Hence, the order of the state is protected.

On the other hand in two dimensions, the correlations between spin grows as a power law with the separation between them, and in one dimensions, exponentially. This means that any thermal fluctuation in an ordered state will propagate throughout the material and destroy the order. Hence, we can’t have a state with spontaneously broken symmetry in \( d \leq 2 \). This is the Mermin-Wagner theorem.

2 \( O(N) \) model with spontaneous symmetry breaking

2.1 Global symmetries

Thinking of \( \phi^i \) as a vector valued field, i.e., a function from Minkowski space to \( \mathbb{R}^N \), the Lagrangian
\[
\mathcal{L} = -\frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{2} m^2 \phi^i \phi^i - \frac{\lambda}{4} (\phi^i \phi^i)^2
\]
depends only on the norm of the \( N \)-dimensional vector vector \( \phi^i \) and it’s gradient vectors \( \partial_\mu \phi^i \). Therefore any \( O(N) \) rotation
\[
\phi^i \mapsto R^{ij} \phi^j
\]
where $R$ is an $N \times N$ matrix such that $R^T R = 1$, leaves the Lagrangian invariant. The global symmetry of the model is hence the rotation group $O(N)$. Hence, this model (with a general potential $V(\phi^i\phi^i)$) is called the $O(N)$ model.

The model also has spacetime symmetries $SO(d-1,1)$, and while those are important to consider in general\(^1\), we will ignore them for the time being because the minimum of the potential does not break them.

2.2 Vacuum structure

Note that the potential, when plotted as a function of the norm (or equivalently, the ‘radial’ coordinate in the target space $\mathbb{R}^N$) $\phi^i\phi^i$ has a local maximum at $\phi^i\phi^i = 0$ and a local minimum at a nonzero value. The value at which the potential is minimized is given by

$$\phi^i\phi^i = \frac{m^2}{\lambda}$$

This implies that the $O(N)$ vector $\phi^i$ picks up a vacuum expectation value (VEV)

$$\langle \phi^i \rangle = v^i , \quad \text{with} \quad v^i v^i = \frac{m^2}{\lambda}$$

Any $O(N)$ vector that satisfies the norm condition is a valid VEV for the field. Hence, the space of vacua (VEVs of the field) is an $(N-1)$-dimensional sphere $S^{N-1}$ in $\mathbb{R}^N$.

2.3 Small fluctuations

We can determine the number of Nambu Goldstone bosons without computing the Lagrangian governing small fluctuations. For a fixed VEV $v^i$, the subgroup of $O(N)$ that leaves the VEV invariant is $O(N-1)$, which corresponds to rotations in the $N-1$ dimensional subspace orthogonal to $v^i$. Goldstone bosons correspond to fluctuations that change the VEV to a different minimum, and therefore must correspond to elements of the coset $O(N)/O(N-1)^2$. The number of ‘broken’ generators is then

$$d_{O(N)} - d_{O(N-1)} = \frac{N(N-1) - (N-1)(N-2)}{2} = N - 1$$

Now let’s construct the Lagrangian for fluctuations. A generic field configuration $\phi^i(x)$ can be written as a local linear transformation $M_j^i(x)$ acting on the VEV

$$\phi^i(x) = M_j^i(x) v^j$$

The linear transformation $M_j^i(x)$ parametrizes a typical (not necessarily small) fluctuation of the fields. For simplicity let’s set the VEV to be nonzero only in the $N$th direction, i.e.,

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\(^1\)Theories with spontaneously broken spacetime symmetries tend to behave slightly differently from theories with spontaneously broken internal symmetries.

\(^2\)This is the modern approach to spontaneous symmetry breaking. One constructs a coset by quotienting the global symmetry (including spacetime symmetries) by the subgroup that isn’t broken. The Goldstone fields correspond to generators of this coset and one can write down a Lagrangian that’s invariant under the residual global transformations. For obvious reasons, this construction is called the coset construction.
\( v^i = (0, \ldots, 0, v = m/\sqrt{\lambda}) \). This reduces the linear transformation to an \( N \)-dimensional vector.

\[
\phi^i(x) = M_N^i(x)v
\]

Next, we consider small fluctuations. Expand \( M_N^i \) around identity

\[
\phi^i(x) = v \left( \delta_N^i + \pi^i \right)
\]

To linear order, the norm of the field is given by

\[
\phi \cdot \phi = v^2 (1 + 2\pi^N) + \mathcal{O}((\pi^i)^2)
\]

Therefore, the components \( \pi^a, a < N \) do not move the field away from the minimum of the potential, but the components \( \sigma \equiv \pi^N \) does. In terms of these fluctuations, the Lagrangian becomes

\[
\mathcal{L} = -\frac{1}{2} \partial_\mu \pi^a \partial^\mu \pi^a - m\sqrt{\lambda} \sigma (\pi^a \pi^a) - \frac{\lambda}{2} \sigma^2 (\pi^a \pi^a) - \frac{\lambda}{4} (\pi^a \pi^a)^2 \\
- \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - m^2 \sigma^2 - m\sqrt{\lambda} \sigma^3 - \frac{\lambda}{4} \sigma^4
\]

We see that the fluctuations in the transverse directions \( \pi^a \) are massless and are hence the \( N-1 \) Nambu-Goldstone bosons of the theory, while the fluctuation \( \sigma \) is massive with mass \( m_\sigma^2 = 2m^2 \).

### 3 Anrowitt-Fickler Gauge

We begin as usual by inserting an identity expanded as a path integral

\[
1 = \int D\alpha \; \delta(G(A^\alpha)) \det (\delta_\alpha G(A^\alpha))
\]

where \( \delta_\alpha \) is shorthand for \( \delta/\delta\alpha \), into the partition function\(^3\)

\[
Z = \int DA \; \exp \left[ i \int \left( -\frac{1}{4g^2} \text{tr} F^2 \right) \right]
\]

The Arnowitt-Fickler gauge \( A_3 = 0 \) is indeed linear in \( A \), and the gauge group volume hence factorizes

\[
Z = \text{Vol}_G \cdot \int DA \; e^{iS} \delta(G(A)) \det (\delta_\alpha G(A^\alpha))
\]

where \( \text{Vol}_G = \int D\alpha \) is the volume of the gauge group. Since the gauge transformation can be written succinctly as

\[
A^\alpha_\mu = A_\mu + D_\mu \alpha
\]

the variation of the gauge condition

\[
G(A) = A_3(x) - \omega(x)
\]

\(^3\)We work with the rescaled gauge field \( A_\mu \to gA_\mu \) in order to separate the coupling constant into a prefactor.
is given by
\[ \delta_a G(A^\alpha) = D_3 \]
The determinant is then given by the Grasmmanian path integral
\[ \det D_3 = \int Dc \bar{D}c \exp \left( -i \int \bar{c} D_3 c \right) \]
The gauge fixed action is then
\[ S = \int \left( -\frac{1}{4g^2} \text{tr} F^2 - \frac{1}{2\xi} \text{tr}(A_3)^2 - \bar{c} D_3 c \right) \]
Note in particular that the ghosts do not have a kinetic term (since there aren’t any time derivatives acting on them). Hence, the ghosts don’t propagate. The gauge field propagator can be obtained by inverting the quadratic term in the action as usual. Writing it slightly more covariantly by introducing a vector \( n^\mu = \delta^\mu_3 \), the propagator is given in Feynman gauge \( \xi = 1 \) by
\[ D^{\mu\nu}(p) = -\frac{i}{p^2} \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{(n \cdot p)^2} - \frac{p^\mu n^\nu + p^\nu n^\mu}{n \cdot p} \right) \]
Even though the ghosts don’t propagate, they still have a non-trivial vertex with \( A_3 \), which in this covariant form takes the value \( in^\mu \). However, the contraction of the vertex with the gauge field propagator is given by
\[ in_\mu D^{\mu\nu} = \frac{1}{p^2} \left( n^\nu - \frac{p^\nu}{n \cdot p} - n^\nu + \frac{p^\nu}{n \cdot p} \right) = 0 \]
Hence the vertex can’t contribute in internal loops either and the ghosts completely decouple from the theory (i.e., any diagram involving ghosts vanishes off-shell). Since the ghosts completely decouple, the argument for having two physical degrees of freedom for the gauge unfixed theory carries through for the gauge fixed one as well.

4 Proca action

4.1 Kinetic terms
Since we have two \( A \)'s and two \( \partial \)'s, each with a Lorentz index, the kinetic terms are simply obtained from the different ways of contracting the indices. We find two terms
\[ L = -\frac{c_1}{2} \partial_\mu A_\nu \partial^\mu A^\nu - \frac{c_2}{2} \partial_\mu A_\nu \partial^\sigma A^\mu - \frac{1}{2} m^2 A^\mu A_\mu \] (4.1)

4.2 Equations of motion
Let’s vary the action. We find
\[ \delta S = \int \delta A^\mu \left( c_1 \partial^2 A_\mu + c_2 \partial_\mu \partial \cdot A - m^2 A_\mu \right) \]
The equations of motion are then given by
\[ c_1 \partial^2 A_\mu + c_2 \partial_\mu (\partial \cdot A) - m^2 A_\mu = 0 \] (4.2)
4.3 Reducing degrees of freedom

The number of degrees of freedom is given by the number of fields whose equations of motion are dynamical, i.e., they evaluate time derivatives of the field in terms of the fields. In order to reduce the degrees of freedom from 4 to 3, we need to ensure that one of the fields, say $A_0$, does not show up with any time derivatives in the equations, and is hence completely determined at any instance in time in terms of the values as well as temporal and spatial derivatives of the other fields at that instance. Consider the $\mu = 0$ equation

$$c_1 \left(-\partial_0^2 + \nabla^2\right) A_0 - c_2 \partial_0^2 A_0 + c_2 \partial_0 \nabla \cdot \vec{A} - m^2 A_0 = 0$$

Evidently, if we set $c_2 = -c_1$, the terms with $\partial_0^2 A_0$ cancel and we are left with

$$(c_1 \nabla^2 - m^2) A_0 = c_1 \partial_0 (\nabla \cdot \vec{A})$$

This equation (supplemented with boundary conditions), entirely determines $A_0$ in terms of $A_i$ and eliminates it from the degrees of freedom, leaving us with 3. The kinetic term can now be written as

$$S_K = -\frac{c_1}{2} \int \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu = -\frac{c_1}{4} \int F_{\mu\nu} F^{\mu\nu}$$  \hspace{1cm} (4.3)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

4.4 Massless limit

In the massless limit, we recover the Maxwell action

$$S = -\frac{c_1}{4} \int F^2$$  \hspace{1cm} (4.4)$$