1 Finite gauge transformations

Let’s derive the general finite gauge transformation first and then specialize to $SU(2)$. Given a gauge group $G$, there exists a fundamental representation of a group element $h \in G$ given by a linear transformation $U_h(x)$ acting on a “vector”\(^1\) field $\psi(x)$ such that under the group transformation, we have

$$\psi(x) \mapsto U_h(x) \psi(x)$$

Note that $\psi(x)$ here isn’t necessarily a fermion field. It can be any abstract vector field on spacetime. The gauge field is an element of (the vector space of) the adjoint representation of $G$. More precisely, it is an element of the Lie algebra (which itself is a vector space) and can be expanded in a basis of generators

$$A_\mu(x) = \sum_a A^a_\mu(x) T^a$$

We also have the covariant derivative

$$[D_\mu \psi](x) = (\partial_\mu - igA_\mu(x)) \psi(x)$$

where $g$ is a coupling constant. The reason this is called a “covariant” derivative is because it transforms exactly like $\psi$ under a group transformation

$$[D_\mu \psi](x) \mapsto U_h(x)[D_\mu \psi](x)$$

Of course the transformed version can also be written as

$$(\partial_\mu - igA'_\mu)(U_h \psi)$$

where $A'_\mu$ is the transformed gauge field. Comparing the two, we find

$$\partial_\mu(U_h \psi) - igA'_\mu U_h \psi = U_h \partial_\mu \psi - ig U_h A_\mu \psi$$

$$\implies \left( \partial_\mu U_h - igA'_\mu U_h \right) \psi = \left( -igU_h A_\mu \right) \psi$$

Now this is true for any arbitrary vector $\psi$. Therefore, we linear operators acting on it on the left and right hand side must be equal

$$-igU_h A_\mu = \partial_\mu U_h - igA'_\mu U_h$$

\(^1\)A representation of a group is defined as a map from the group to the set of linear transformations on some abstract vector space $V$. Different choices of $V$ and different choices of the map define different representations. $\psi$ here is an element of the vector space $V$ and not necessarily a Lorentz vector.
Rearranging the terms a bit, we find the transformed gauge field

$$A'_\mu = U_h A_\mu U_h^{-1} - \frac{i}{g} (\partial_\mu U_h) U_h^{-1} \tag{1.1}$$

In terms of generators, we have

$$A_\mu = A^a_\mu T^a, \quad U_h = \exp \left( i \alpha^b T^b \right)$$

so that

$$(A')^a_\mu T_a = A^a_\mu \left( e^{i \alpha^b T^b} T^a e^{-i \alpha^c T^c} \right) + \frac{i}{g} \partial_\mu \alpha^a T^a \tag{1.2}$$

The first term can be further simplified using the identity

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2} [X, [X, Y]] + \ldots$$

to obtain

$$e^{i \alpha^b T^b} T^a e^{-i \alpha^c T^c} = T^a - f^{abc} \alpha^b T^c - \frac{i}{2} f^{def} f^{cba} \alpha^d \alpha^e + \ldots$$

For $SU(2)$, the structure constants are given by $f^{abc} = \epsilon^{abc}$ and we can use the Levi-Civita contraction formulas to simplify this even further. Equivalently, we can also calculate the final answer by instead using the identity

$$e^{i \alpha^a T^a} = 1 \cos \frac{|\alpha|}{2} + i (\hat{\alpha} \cdot \sigma) \sin \frac{|\alpha|}{2}$$

## 2 Nonabelian gauge theory in form notation

When dealing with Lie algebra valued forms, we have to generalize the definition of the wedge product, since there is no bilinear product in an abstract Lie algebra, unlike in $\mathbb{R}$. The wedge product of two Lie algebra valued forms now involves a commutator of their components instead of a product

$$\omega = \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \ldots dx^{\mu_p} = \omega_{\nu_1 \ldots \nu_q} dx^{\nu_1} \ldots dx^{\nu_q}$$

$$\eta = \eta_{\mu_1 \ldots \mu_p} dx^{\mu_1} \ldots dx^{\mu_p} = \eta_{\nu_1 \ldots \nu_q} dx^{\nu_1} \ldots dx^{\nu_q}$$

$$[\omega \wedge \eta] = [\omega_{\mu_1 \ldots \mu_p}, \eta_{\nu_1 \ldots \nu_q}] dx^{\mu_1} \ldots dx^{\mu_p} dx^{\nu_1} \ldots dx^{\nu_q} \tag{2.1}$$

Note in particular that this implies that $[\omega \wedge \omega] \neq 0$, unlike real valued differential forms.

In components, the field strength is given by

$$F^\mu_\nu = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu]$$

$$F^{a}_\mu = \partial_\mu A^a_\mu - \partial_\nu A^a_\nu + g f^{abc} A^b_\mu A^c_\nu \tag{2.2}$$
The first two terms are simply the components of the 2-form $dA$. The commutator can be put into form notation by contracting with $(1/2)dx^\mu \wedge dx^\nu$

$$\frac{1}{2} [A^\mu, A^\nu] dx^\mu \wedge dx^\nu = [A \wedge A]$$

The field strength can now be written as a two form\(^3\)

$$F = \frac{1}{2} F^a_{\mu\nu} dx^\mu dx^\nu T^a$$
$$F = dA - ig[A \wedge A]$$

(2.3)

Under a gauge transformation $\alpha = \alpha^a T^a$, we have

$$\delta A^\mu = \frac{1}{g} \partial^\mu \alpha + i[\alpha, A^\mu]$$
$$\delta F^\mu_{\nu} = i[\alpha, F^\mu_{\nu}]$$

Once again, since the Lie algebra commutator is included in the definition of the wedge product, we can write these as\(^4\)

$$\delta A = \frac{1}{g} d\alpha + i[\alpha \wedge A]$$
$$\delta F = i[\alpha \wedge F]$$

(2.4)

Finally, the (source-free) equations of motion in index notation are the pair

Yang-Mills equations: \quad $0 = \partial^\nu F^\mu_{\nu} - ig[A^\mu, F^\mu_{\nu}]$
Bianchi identity: \quad $0 = \epsilon^{\mu\nu\rho\sigma} (\partial^\nu F^\rho_{\sigma} - ig[A^\nu, F^\rho_{\sigma}])$

The form notation version of these is straightforwardly obtained by noting that index contraction is obtained by taking a generalized wedge product with the Hodge dual of the second form

$$d \star F - ig[A \wedge \star F] = 0$$
$$dF - ig[A \wedge F] = 0$$

(2.5)

2.1 Side note: Gauge covariant derivative of general Lie algebra valued forms

The gauge covariant derivative $D_\mu$ of a field $\psi(x)$ in the fundamental representation is defined as

$$D_\mu \psi \equiv \partial_\mu \psi - igA_\mu \psi$$

This can be generalized to arbitrary Lie-algebra valued forms by noting that the Lie algebra acts on itself via the adjoint representation, i.e. the Lie algebra commutator. For forms, the

\(^3\)The wedge product is often ignored in literature in which case we can write the second term as $A \wedge A = A^2$.

\(^4\)Recall that the wedge product of a 0-form with any p-form is simply the product of the two.
commutator is captured by the generalized wedge product. Therefore, the gauge covariant derivative of the Lie-algebra valued form $\omega$ is given by

$$D\omega = d\omega - ig[A \wedge \omega]$$

This allows us to write all of the above equations in a much more compact form

$$F = DA$$
$$D \ast F = 0$$
$$DF = 0$$

(2.6)

3 $su(3)$ algebra

3.1 Dimensionality of $su(3)$

The group $SU(3)$ consists of $3 \times 3$ unitary matrices with determinant 1. Out of a total 9 complex components, i.e., 18 real components, the unitarity equation $U^\dagger U = 1$ kills 9 real components. Furthermore, the determinant condition kills yet another, leaving us with 8 real numbers to characterize a given $SU(3)$ matrix. Therefore the group is 8-dimensional. Given that the Lie algebra and the Lie group have the same dimension, we find that the Lie algebra is also 8-dimensional. Hence, there are 8 matrices in the basis.

Another way of seeing this is the following: The Lie algebra is the set of logarithms of $SU(3)$ matrices. Unitarity implies that the Lie algebra matrices are Hermitian and hence have 9 real components (3 real diagonal components and 3 complex components in one of the off-diagonal triangles). The determinant condition translates to tracelessness of the Lie algebra matrices, which kills one of the independent components, leaving us with 8.

3.2 Structure constants

This calculation simply involves commuting matrices so I will only write the final answer for the structure constants. The only nonzero ones, up to appropriate permutations, are given by

$$f_{123} = 1$$
$$f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = 1/2$$
$$f_{458} = f_{678} = \sqrt{3}/2$$

3.3 Orthogonality

Once again, I’ll only write the final answer. We find that

$$C(r) = \frac{1}{2}$$
3.4 Quadratic Casimir

The quadratic Casimir for the fundamental representation is given by

\[ C_2(r) = \frac{4}{3} \]

We also have the dimensions of the fundamental representation and the adjoint representation respectively

\[ d(r) = 3, \quad d(G) = 8 \]

which implies that the difference

\[ d(r)C_2(r) - d(G)C(r) = 3 \cdot \frac{4}{3} - 8 \cdot \frac{1}{2} = 0 \]

4 More Casimirs

4.1 Decomposition into su(2) irreps

The fact the an irrep of \( g \) decomposes into irreps of \( \text{su}(2) \) means that the generators \( t^a_r \) can be written as direct sums of spin \( j_i \) irreps \( t^a_i \) of \( \text{su}(2) \), i.e.

\[ t^a_r = \bigoplus_i t^a_i \]

where the \( \bigoplus \) stands for direct sum of matrices. Think of this construction as follows: arrange the matrices \( t^a_i \) in some order, say, with increasing spin \( j_i \) Now consider a bigger matrix \( t^a_r \) which is block diagonal with the entries of the \( i \)th block being those of \( t^a_i \)

\[ t^a_r = \begin{pmatrix} t^a_1 & 0 & \ldots \\ 0 & t^a_2 & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \]

Fixing \( a \) to be some value, the factor \( C(r) \) for the representation \( r \) of \( G \) is given by

\[ C(r)\delta^{aa} = \text{tr}[t^a_rt^a_r] \]

\( \delta^{aa} = 1 \) and the right hand side can be simplified using the \( \text{su}(2) \) decomposition

\[ \text{tr} \left[ \bigoplus_i t^a_i \bigoplus_j t^a_j \right] = \text{tr} \left[ \bigoplus_i t^a_i t^a_i \right] \]

In order to see why this equality holds, note that taking the product of block diagonal matrices with blocks of identical sizes is equivalent to taking the products of the individual blocks. The direct sum can now be pull outside the trace (since the matrices are block diagonal) and we have

\[ C(r) = \sum_i \text{tr}[t^a_i t^a_i] \]

Now, note that each term in the sum on the right hand side is independent of the chosen index \( a \). Hence, we can choose different \( a \)’s in order to evaluate each term. In particular,
we chose the one that gives us the diagonal \( su(2) \) matrix in the spin \( j_i \) representation given by

\[
\begin{pmatrix}
    j_i \\
    j_i - 1 \\
    \vdots \\
    -j_i
\end{pmatrix}
\]

The trace of the square of this matrix is given by

\[
\text{tr}[t_i^2 t_i^2] = \sum_{-j_i}^{j_i} m^2 = \frac{1}{3} j_i (j_1 + 1)(2j_i + 1)
\]

Using this we find the required formula

\[
C(r) = \frac{1}{3} \sum_i j_i (j_1 + 1)(2j_i + 1)
\]  

\[ (4.1) \]

### 4.2 Fundamental and adjoint representations of \( su(N) \)

The fundamental representation of \( su(N) \) has the \( su(2) \) spins

\[
j_1 = 1/2, \quad j_i = 0, \quad i = 2, \ldots N - 1
\]

Another way of writing this is the following

\[
N = \frac{1}{2} \oplus 0 \oplus \ldots \oplus 0_{N-2}
\]

Using the above formula, we find that only the first spin contributes to the sum

\[
C(N) = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot 2 = \frac{1}{2}
\]

The adjoint representation is obtained from the tensor product

\[
N \otimes \bar{N} = \text{ad} \oplus 0
\]

The \( su(2) \) singlet on the right hand side is irrelevant since it doesn’t contribute to the normalization factor. For \( su(N) \), the antifundamental representation \( \bar{N} \) has the same decomposition as \( N \). Therefore we find

\[
\text{ad} \oplus 0 = \left[ \frac{1}{2} \oplus 0^\oplus_{N-2} \right] \otimes \left[ \frac{1}{2} \oplus 0^\oplus_{N-2} \right]
\]

\[
= \left( \frac{1}{2} \otimes \frac{1}{2} \right) \oplus \left( \frac{1}{2} \otimes 0 \right)^\oplus_{N-2} \oplus \left( 0 \otimes \frac{1}{2} \right)^\oplus_{N-2} \oplus 0^\oplus_{\ldots}
\]

\[
= 1 \oplus \frac{1}{2} \oplus 0^\oplus_{N-2} \oplus 0^\oplus_{\ldots}
\]

where the superscript \( \oplus (\ldots) \) stands for taking the direct sum of that object as many times as the number in parentheses. This completes the proof of the adjoint representation being
decomposable into one spin 1, 2(N − 2) spin 1/2’s, plus singlets. Now, using the formula for \( C(r) \), we find that

\[
C(\text{ad}) = \frac{1}{3} \cdot 1 \cdot 2 \cdot 3 + \frac{2(N - 2)}{3} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot 2
\]

\[
= 2 + N - 2
\]

\[
= N
\]

(4.2)

4.3 Symmetric and antisymmetric representations

In order to compute the quadratic Casimir for the symmetric and antisymmetric representations, we can use the formula

\[
C_2(r) = \frac{d(G)C(r)}{d(r)} = \frac{(N^2 - 1)C(r)}{d(r)}
\]

The vector spaces of the symmetric and antisymmetric representations are the symmetric and antisymmetric subspaces of the tensor product \( N \otimes N \), i.e., if \( v_i \in N \) is a vector of dimension \( N \), the vector space of the (anti)symmetric representation consists of (anti)symmetric matrices \( M_{ij} \). The corresponding dimensions are

\[
d(s) = \frac{N(N + 1)}{2}, \quad d(a) = \frac{N(N - 1)}{2}
\]

We now need only compute \( C(r) \) for both these representations, for which we can use the \( \mathfrak{su}(2) \) decomposition of the fundamental representation. This decomposition implies that a generic vector \( v \in N \) can be written as a stack of a 2-component column (in the spin 1/2 representation) and \( N - 2 \) single component columns (in the spin 0 representation)

\[
v = (t, s_2, \ldots s_N)
\]

Writing \( N = 1/2 \oplus 0_2 \oplus 0_3 \oplus \ldots \oplus 0_N \) with indices on the singlets in order to keep track of them, the tensor product \( N \otimes N \) can be decomposed as follows

\[
N \otimes N = \left[ \frac{1}{2} \otimes \frac{1}{2} \right] \oplus \left[ \sum_{i=2}^{N} \frac{1}{2} \otimes 0_i \right] \oplus \left[ \sum_{i=2}^{N} 0_i \otimes \frac{1}{2} \right] \oplus \left[ \sum_{i,j=2}^{N} 0_i \otimes 0_j \right]
\]

Each of the terms in the square brackets decomposes into \( \mathfrak{su}(2) \) irreps and these irreps distribute themselves into symmetric and antisymmetric combinations. Let’s start with the first. We already know from the \( \mathfrak{su}(2) \) decomposition of the tensor product from quantum mechanics

\[
\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0
\]

that the triplet 1 is symmetric and the singlet 0 is antisymmetric.

Next, the direct sum

\[
\left[ \frac{1}{2} \otimes 0_i \right] \oplus \left[ 0_i \otimes \frac{1}{2} \right]
\]

(4.3)
also splits evenly into symmetric and antisymmetric subspaces. Therefore, we get $N - 2$ symmetric 1/2 representations and $N - 2$ antisymmetric 1/2 representations from this piece.

Finally, the last term is a double sum which consists of two types of tensor products

$$\sum_{i=2}^{N} 0_i \otimes 0_i, \quad \sum_{i \neq j} 0_i \otimes 0_j$$

The first type is identically symmetric, while the second one splits up evenly into symmetric and antisymmetric representations. This gives us

$$\frac{(N - 2)(N - 3)}{2}$$

antisymmetric 0 representations and

$$\frac{(N - 2)(N - 3)}{2} + N - 2 = \frac{(N - 2)(N - 1)}{2}$$

symmetric 0 representations. Putting all of these together, we find the decomposition

$$s = 1 \oplus \frac{1}{2} \oplus 0^{(N-2)} \oplus 0^{(N-1)(N-2)/2}$$

$$a = 0 \oplus \frac{1}{2} \oplus 0^{(N-2)} \oplus 0^{(N-2)(N-3)/2}$$

$$= \frac{1}{2} \oplus 0^{(N-2)} \oplus 0^{(N^2-5N+8)/2}$$

We can now compute the normalization constants for both of these

$$C(s) = \frac{1}{3} \cdot 6 + (N - 2) \cdot \frac{1}{3} \cdot \frac{3}{2} + 0$$

$$= \frac{N + 2}{2}$$

$$C(a) = 0 + (N - 2) \cdot \frac{1}{3} \cdot \frac{3}{2} + 0$$

$$= \frac{N - 2}{2}$$

Finally, the quadratic Casimirs for both irreps are given by

$$C_2(s) = \frac{(N - 1)(N + 2)}{N}$$

$$C_2(a) = \frac{(N - 2)(N + 1)}{N}$$

The identity for the decomposition of product representations

$$N \otimes N = s \oplus a$$

is given by

$$[C_2(N) + C_2(N)] d(N)^2 = C_2(s) d(s) + C_2(a) d(a)$$
The quadratic Casimir for the fundamental representation is simply
\[ C_2(N) = \frac{N^2 - 1}{N} C(N) = \frac{N^2 - 1}{2N} \]
so that the left hand side reduces to
\[ N(N^2 - 1) \]
The right hand side is then
\[ \frac{(N-1)(N+2)}{2} \frac{N(N+1)}{N} + \frac{(N-2)(N+1)}{N} \frac{N(N-1)}{2} \]
\[ = \frac{(N^2 - 1)(N+2)}{2} + \frac{(N^2 - 1)(N-2)}{2} \]
which is indeed equal to the left hand side.

5 Coulomb potential

5.1 Wilson loop

The Wilson loop operator is given by
\[ W_P = \exp \left( -ie \oint_P dx^\mu A_\mu \right) = \exp \left( -ie \int d\tau \dot{x}^\mu A_\mu \right) \]
where \( \tau \in [0, 2\pi) \) parametrizes the loop \( P \) and the dot superscript stands for derivative with respect to \( \tau \). The exponent can be written as an integral over all of space time by defining a delta function supported on \( P \), denoted by \( \delta(P) \), and also defining a four vector \( j^\mu \) whose value is the unit tangent vector to \( P \) times \( \delta(P) \) and which vanishes everywhere off the curve.
\[ W_P = \exp \left( -ie \int dx^4 j^\mu A_\mu \right) \]
It’s expectation value is simply the normalized path integral
\[ \langle W_P \rangle = \frac{1}{N} \int DA \exp \left( -i \int F^{\mu\nu} F_{\mu\nu} + \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + ej^\mu A_\mu \right) \]
where
\[ N = \int DA e^{-\frac{i}{2} \int F_{\mu\nu} F_{\mu\nu} + \frac{1}{2\xi} (\partial_\mu A^\mu)^2} \]
and we have added a gauge fixing term. In terms of the gauge field, the exponent is simply
\[ \langle W_P \rangle = \frac{1}{N} \int DA \exp \left( \frac{1}{2} \int A_\mu (G^{-1})^{\mu\nu} A_\nu - ie j^\mu A_\mu \right) \]
where \( (G^{-1})^{\mu\nu} \) is the inverse propagator. This is simply a Gaussian integral whose result is given by
\[ \langle W_P \rangle = \exp \left( \frac{ie^2}{2} \int d^4x d^4y j^\mu(x) G_{\mu\nu}(x-y) j^\nu(y) \right) \]
Finally, using the fact that the source $j^\mu$ is localized on the loop $P$, we find the expression

$$\langle W_P \rangle = \exp \left( \frac{i e^2}{2} \oint_P dx^\mu \oint_P dy^\nu \, G_{\mu\nu}(x-y) \right)$$

The (Feynman) propagator in position space is the Fourier transform

$$G_{\mu\nu}(x-y) = \int \frac{d^4 q}{(2\pi)^4} e^{-iq(x-y)} \frac{i}{q^2 + i \epsilon} \left[ \eta_{\mu\nu} - \left( 1 - \frac{1}{\xi} \right) q_\mu q_\nu \right]$$

We can choose Feynman gauge $\xi = 1$ and work out the integral to obtain the final answer\footnote{The sign mismatch is due to the difference in signature - we’re working in mostly positive signature here.}

$$\langle W_P \rangle = \exp \left( -\frac{e^2}{2} \oint \oint \langle AA \rangle \right) \quad (5.1)$$

Another way of deriving this result is to expand the exponent into a power series. Schematically, we have

$$W_P = 1 - ie \oint j \cdot A + \frac{(ie)^2}{2} \oint \oint j^\mu j^\nu A_\mu A_\nu + \ldots$$

Since Maxwell theory is free, Wick’s theorem applies here (i.e., higher point functions factorize into two point functions and correlators of an odd number of fields vanish) and taking the expectation value gives us a power series of the schematic form

$$\langle W_P \rangle = 1 - \frac{e^2}{2} \oint \oint \langle A A \rangle + \ldots$$

Resumming the series results in the exponent of the double integral of the propagator

$$\langle W_P \rangle = \exp \left( -\frac{e^2}{2} \oint \oint G \right)$$

which then gives us (5.1).

5.2 Rectangular path

Since the length $T$ of the temporal segments of the rectangular loop is much larger than the length $R$ of the spatial segments, and the integrand above is non-oscillatory on the temporal segments, we can safely ignore the contribution of the spatial segments. Label the forward and backward directed temporal segments by the index $f$ and $b$ respectively, so that coordinates on them are labelled by $x^\mu_{f,b}$ and $y^\mu_{f,b}$ respectively. The expectation value of the Wilson loop becomes

$$\langle W_P \rangle = \exp \left( -\frac{e^2}{2} \sum_{i,j=f,b} \int dx^0_i dy^0_j \frac{\eta_{00}}{8\pi^2 (x^\mu_i - y^\mu_j)^2} \right)$$
where the limits on the integral are determined based on the direction in which that segment points. There are two distinct types of contributions to the sum - the first type is where the integrals run over the same segment, which is divergent. The second type is where the two integrals run over different segments, which is regulated by the spatial separation $R$.

Consider the first type. There are two terms in this part of the sum

$$\int_0^T dx_0^f \int_0^T dy_0^f \frac{1}{8\pi^2 (x_0^f - y_0^f)^2} + \int_0^T dx_0^b \int_0^T dy_0^b \frac{1}{8\pi^2 (x_0^b - y_0^b)^2}$$

Both terms are equal to each other and add up to the divergent integral

$$\frac{1}{4\pi^2} \int_0^T du \int_0^T dv \frac{1}{(u - v)^2} \quad (5.2)$$

This integral needs to be regulated appropriately by splitting the Wilson loop into a band. However, it’s contribution to the expectation value of the loop is an overall prefactor which does not depend on the spatial geometry of the loop - in particular the separation $R$. Therefore we can safely ignore it and focus on the other contribution instead. The relevant integrals are

$$- \frac{e^2}{8\pi^2} \int_0^T dx_0^f \int_T^0 dy_0^f \frac{1}{(x_0^f - y_0^f)^2} - \frac{e^2}{8\pi^2} \int_0^T dx_0^b \int_0^T dy_0^b \frac{1}{(x_0^b - y_0^b)^2} - R^2$$

$$= \frac{e^2}{4\pi^2} \int_0^T du \int_0^T dv \frac{1}{(u - v)^2 - R^2 - i\epsilon}$$

where the $i\epsilon$ is a regulator that avoids the divergence obtained when $(u - v)^2 = R^2$. The $u$ integral can be done using residue theorem and we will only write down the final answer in the limit $T \gg R$, which is given by

$$\langle W_P \rangle = \exp \left[ -i \left( -\frac{e^2}{4\pi R} \right) \right] T = \exp \left[ -iE(R)T \right] \quad (5.3)$$

The interpretation of this fact is that the energy of this configuration is given by the (attractive) Coulomb energy

$$E(R) = -\frac{e^2}{4\pi R} \quad (5.4)$$

5.3 Aside: Wilson lines in pure gauge theory

In order to make sense of the result and interpretation of the above problem, it helps to understand the physical intuition behind Wilson lines and Wilson loops in pure gauge theory. These operators, when acting on the vacuum, create “defects”. Defects are states in the Hilbert space of the gauge theory that aren’t eigenstates of the Hamiltonian. The expectation value of the Hamiltonian in these states gives the energy, which turns out to be infinite. Defects are also non-dynamical - they are external objects that can be inserted into the theory and change the structure of the theory.

A single future-directed timelike Wilson line represents an infinitely massive charged particle. Because it is charged, it acts as a source for gauge fields. However, since it is infinitely massive, it does not backreact to any other excitation (e.g. a photon).
Since Wilson lines with fixed end points aren’t gauge invariant, we need to consider infinitely long Wilson lines for this interpretation to be valid - another way of thinking about this is that a charged particle can’t be created from nothing. The energy of a single Wilson line is divergent - corresponding to the self-energy of a charged particle.

Now let’s understand the previous problem in the context of this particle interpretation. Calculating the interaction energy of two oppositely charged particles separated by a distance $R$ corresponds to calculating the expectation value of two oppositely directed Wilson lines separated by a distance $R$. Since we want to get a non-divergent answer for this, we need these lines to have finite length in time. But the resulting configuration is not gauge invariant. Therefore, we attach two spatial Wilson lines between the end points, but ignore their contribution to the expectation values. Finally, the expectation value that we calculate will have two separate types of contributions: one from the self energy of each particle, which is infinite, and another from the interaction energy. The self energy terms are the ones that we ignored in the solution, in order to obtain the Coulomb potential.

5.4 Nonabelian Wilson loop

The nonabelian Wilson loop is defined by the expression

$$W_P = \text{tr}_r \mathcal{P} \exp \left( -ig \oint_P \, dx^\mu A^a_\mu T^a_r \right)$$  (5.5)

where the symbol $\mathcal{P}$ stands for path ordering and $\text{tr}_r$ is the trace in the representation $r$. $T^a_r$ are the generators of the Lie algebra in the $r$ representation. Since we only need to compute its expectation value up to order $g^2$, we can expand the exponential

$$W_P = \text{tr}_r \mathcal{P} \left( 1_r - ig T^a_r \oint_P \, dx^\mu A^a_\mu(x) - g^2 T^a_r T^b_r \oint_P \, dx^\mu \oint_P \, dy^\nu A^a_\mu(x) A^b_\nu(y) \right) + \mathcal{O}(g^3)$$

where $1_r$ is the identity in the $r$ representation. The path ordering is trivial for the first two terms. Upon taking the expectation value, the term linear in $g$ vanishes since no operator in the theory can have a nonvanishing one-point function. The trace distributes over the three terms. Owing to the identity

$$\text{tr}(T^a_r T^b_r) = C(r) \delta^{ab}$$

the path ordering is irrelevant for the second term as well. The trace of the identity operator can be factorized from the expression

$$\langle W_P \rangle = \text{tr}_r \, 1_r - g^2 C(r) \oint_P \, dx^\mu \oint_P \, dy^\nu \langle A^a_\mu(x) A^b_\nu(y) \rangle \delta^{ab} + \mathcal{O}(g^3)$$

The calculation from here on is identical to the abelian case, in particular because

$$\langle A^a_\mu(x) A^b_\nu(y) \rangle \delta^{ab} = \delta^{ab} \delta^{ab} C_{\mu\nu}^{\text{abelian}}(x - y) = d(G) C_{\mu\nu}^{\text{abelian}}(x - y)$$

Finally, using the fact that

$$d(G) C(r) = d(r) C_2(r) = C_2(r) \text{tr}_r \, 1_r$$
we can factorize the trace of the identity operator to find

\[ \langle W_P \rangle = \text{tr}_r \mathbf{1}_r \left( 1 - g^2 C_2(r) \oint_P dx^\mu \oint_P dy^\nu \langle A_\mu^a(x) A_\nu^b(y) \rangle \delta^{ab} \right) + \mathcal{O}(g^3) \]  

(5.6)

and the expression above is identical to that of the abelian theory expanded to quadratic order in \( e \), with the replacement

\[ e^2 \to g^2 C_2(r) \]

and the overall factor \( \text{tr}_r \mathbf{1}_r \). Therefore, the calculation of the potential follows through identically and we find

\[ V_r(R) = -\frac{g^2 C_2(r)}{4\pi R} + \mathcal{O}(g^3) \]  

(5.7)

The overall factor \( \text{tr}_r \mathbf{1}_r \) corresponds to the summing over the various “colours” of particles running through the loop and hence is dropped when evaluating the Coulomb potential between two particles, for which we need to restrict to a single colour. More precisely, for the nonabelian rectangular Wilson loop, we should compare the expectation value to the object

\[ \langle W_P \rangle = \text{tr}_r \mathbf{1}_r \ e^{-iE(R)T} \]