Introduction

Instantons are defined as the finite-action solutions to the equations of motion of field theories on a Euclidean metric. Instantons are important to quantum theories because they appear in the path integral formulation as the leading quantum corrections to the classical system behavior. Thus, instantons can be used to study quantum effects such as the tunneling behavior of quantum particles. Before diving directly into the role instantons play in quantum field theory, we will first work through the mathematical formalism. We will start by describing solitons and monopoles. Then, we will discuss instantons in SU(2) gauge theory, wrapping up the discussion with examples of instantons in various theories. Further reading on the topic can be found in Rajaraman (1982) [1] and Weinberg Vol II (1996) [2], which were helpful books to read for this report and go much deeper than what is covered here.

1 Solitons

Solitons are solutions to the equations of motion that maintain their shape. In other words, the energy density of a soliton is localized in space and does not dissipate as time passes. These pseudoparticles are useful to the description of instantons because the math required to derive solutions is similar. The reason for this will become clear later but, to put it succinctly, instantons are a special class of solitons that are described by “kink” solutions. We will derive a simple example of a soliton for a double well potential scalar field theory below.

Take a scalar field $\phi$ with a lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi)$$

where

$$V(\phi) = \frac{1}{8} \lambda (\phi^2 - v^2)^2$$

This potential is familiar. It yields two ground states at $\phi(x) = \pm v$ that, when we shift the field by its VEV, induce a particle mass $m = \frac{\lambda}{2} v$. This scenario is useful to illustrate soliton properties when we look at it in 1+1 dimensional spacetime.

In this case, where there is one spatial dimension, the boundaries of space exist at the two points $x = \pm \infty$. The topology of the spatial boundary is therefore similar to the space of vacuum field configurations. However, looking at the identity map, which maps $x = -\infty$ to $\phi = -v$ and $x = \infty$ to $\phi = v$, we can see that this mapping does not correspond to a vacuum and, therefore, the field must be smooth in between $\phi = -v$ and $\phi = v$. This statement requires energy and the question that will be answered below is: Can this be done with finite energy?

We will look for a time independent solution to the classical field equations with boundary conditions

$$\lim_{x \to \pm \infty} \phi(x) = \pm v$$

The energy is defined as

$$E = \int_{-\infty}^{\infty} -\mathcal{L} dx$$

Using our boundary conditions in (3) and the condition for time independence, the equation in (4) reduces as

$$E = \int_{-\infty}^{\infty} \left[ \frac{1}{2} \phi'^2 + V(\phi) \right] dx$$
\[ E = \int_{-\infty}^{\infty} \frac{1}{2} (\phi' - \sqrt{2V(\phi)})^2 + \sqrt{2V(\phi)} \phi' \, dx \] (6)
\[ E = \int_{-\infty}^{\infty} dx \frac{1}{2} (\phi' - \sqrt{2V(\phi)})^2 + \int_{-v}^{v} d\phi \sqrt{2V(\phi)} \] (7)
\[ E = \int_{-\infty}^{\infty} dx \frac{1}{2} (\phi' - \sqrt{2V(\phi)})^2 + \frac{2}{3} (m^2/\lambda)m \] (8)

From equation (8), we can now gather an understanding of the soliton solution. The second term in equation (8) is directly related to the coupling and therefore will not be affected by our solution to the field equations. Therefore, to minimize the energy, the first term \((\phi' - \sqrt{2V(\phi)})^2\) must go to zero, which implies that
\[ \phi' = \sqrt{2V(\phi)} \] (9)

We can integrate this result and plug in equation (2) to get
\[ \phi(x) = v \tanh \left( \frac{1}{2} m(x - x_0) \right) \] (10)

where \(x_0\) is a constant of integration.

This solution is a soliton as it meets the requirements. It is obviously time independent and the energy density does not dissipate. Additionally, the energy density is localized and goes to zero exponentially for \(|x - x_0| > 1/m\). The soliton behaves much like a particle; the rest energy and momentum could be derived and the soliton could be lorentz boosted, to find other solutions.

In this case, we saw the derivation of a soliton solution for a 1+1 dimensional classical field theory. There are soliton solutions for higher dimensional field theories and complex mathematical tricks that can be used to solve for them, which we will discuss later. But, a lesson can be gleaned without diving into more complex scenarios. The important takeaway from this demonstration is the relationship between the topology of the boundary space and the topology of the vacua set. The existence of the soliton is related to the non-trivial map from the boundary space to the vacua set. This relationship will help motivate the later discussion of instantons and how the study of a vacuum structure of a quantum field theory may lead to the study of instantons.

## 2 Monopoles

The derivation in section 1, while helpful, does not reflect the 3+1 dimensional structure of our world. However, the generalization of our the understanding of solitons from section 1 to higher dimensions is a non-trivial process. Previously, we chose to work in a 1+1 dimensional frame, with our solutions constrained to be static in one coordinate frame. This allowed us to treat our field equations as ordinary differential equations. As we move to theories with two or more spatial dimensions, we are now working with partial differential equations, which are much more difficult to solve directly. While there are exact solutions for certain, special systems, which will be discussed here, it is also important to discuss general features that can be derived without exact solutions, such as homotopy classification and topological charges.

### 2.1 Winding Number

Before looking into soliton solutions, it will be important to discuss homotopy classification and the winding number, which are important topics for understanding the higher dimensional solutions.

We will examine the case of \(S_2\). All non-singular mappings of a spherical surface \(S_2\) into a different \(S_2\) can be classified into homotopy sectors, meaning that mappings from continuous sectors can be within one sector can be continuously deformed into one another, while mappings from two different sectors cannot. The homotopy classes form a group with is isomorphic to the integers
\[ \pi_2(S_2) = \mathbb{Z}_2 \] (11)
From this relationship, we can derive an understanding of what the winding number represents. The winding number quantifies the number of times $S_2$ is traversed during the mapping $S_2 \to S_2$. The winding can generate an infinite number of homotopy classes, but it remains an integer value. This is what equation 11 succinctly represents.

This idea can be connected to our study of field theories by examining the non-linear O(3) model, which consists of three real scalar fields $\phi(x,t) \equiv \{\phi_a; a = 1,2,3\}$ that satisfy $\phi \cdot \phi = 1$. The topology of these scalar fields can be thought of as the vacuum three-sphere. The lagrangian can be written out as

$$L = \frac{1}{2} (\partial_{\mu} \phi) \cdot (\partial^{\mu} \phi)$$

In this model, the finite energy static configurations in two spatial dimensions $\phi(x)$ can be classified into homotopy sectors, using the formalism above, characterized by a label $Q$. We can write out $Q$ as an integral over the field function by

$$Q = \frac{1}{8\pi} \int \epsilon_{\mu\nu} \phi \cdot (\partial_\mu \phi \times \partial_\nu \phi) d^2 x$$

where $\mu, \nu = 1,2$ refer to spatial indices. We can show that this expression provides the winding number below by relating the surface element to cartesian and spherical variables as

$$dS_a = d^2 \zeta \left( \frac{1}{2} \epsilon_{rs} \epsilon_{abc} \frac{\partial \phi_b}{\partial \zeta_r} \frac{\partial \phi_c}{\partial \zeta_s} \right)$$

where we describe the sphere $S_2$ in internal space by two polar variables $\zeta_1, \zeta_2$. We can rewrite equation 13 in spherical variables as

$$Q = \frac{1}{8\pi} \int \epsilon_{rs} \epsilon_{abc} \phi_a \frac{\partial \phi_b}{\partial \zeta_r} \frac{\partial \phi_c}{\partial \zeta_s} d^2 \zeta$$

and input equation 14 into our formula for $Q$ to get

$$Q = \frac{1}{4\pi} \int dS_a \cdot \phi_a = \frac{1}{4\pi} \int dS$$

which shows that $Q$ gives the number of times the sphere is traversed and that $Q$ is indeed the winding number $n$.

### 2.2 Derrick’s Theorem

There is one more topic to touch on before working out soliton solutions in higher dimensions. In Section 1, we were able to work with only scalar fields in 1+1 dimensions. In studying higher dimensional solutions, Derrick (1964) [3] showed that there are no finite-energy, time-independent solutions that are localized in more than one dimension if we consider scalar fields only. Formally, this means that there is no stable, localized solution to the equation

$$\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = \frac{1}{2} f'(\phi), \quad \phi(x,t) \in \mathbb{R}, \quad x \in \mathbb{R}^3$$

Conceptually this can be understood as the gradient energy of the solution always diverging at large distances from the soliton core. This presents an obstacle in interpreting soliton solutions, but higher dimensional soliton solutions can still be found when with introduce gauge or fermion fields. In the next section we will show this process using gauge fields.

### 2.3 The ’t Hooft-Polyakov Monopole

Finally, we can discuss soliton solutions for the familiar case of 3+1 dimensions. The discussion in section 2.2 requires us to broaden our theory beyond spin-zero fields. Our lagrangian in equation 1 has a global
U(1) symmetry; therefore we can gauge it. We can take \( \phi^a \) to be in the adjoint representation of an SU(2) gauge group, producing a lagrangian with three spatial dimensions that takes the form

\[
\mathcal{L} = -\frac{1}{2} (D^\mu \phi)^a (D_\mu \phi)^a - V(\phi) - F^{a\mu\nu} F^a_{\mu\nu}
\]

(18)

where

\[
(D_\mu \phi)^a = \partial_\mu \phi^a + e \epsilon^{abc} A_\mu^b \phi^c
\]

(19)

\[
F^{a\mu\nu} = \partial_\mu A^a_\nu + \epsilon^{abc} A^b_\mu A^c_\nu
\]

(20)

\[
V(\phi) = \frac{1}{8} \lambda (\phi^a \phi^a - v^2)^2
\]

(21)

where we have denoted the gauge coupling as \( e \).

The gauge symmetry of this theory is spontaneously broken to U(1) and we can take the vacuum field configuration to be \( \phi^a = v \delta^a_3 \) so that the \( A^a_\mu \) field remains massless. This results in two bosons \( W^+_\mu = (A^3_\mu \mp iA^1_\mu)/\sqrt{2} \) with induced mass \( m = ev \) and charge \( \pm e \). This theory is known as the Georgi-Glashow model, which was once an alternative to the Standard Model, but is now ruled out because it does not have a \( Z^0 \) boson. But, this model has a soliton solution and is therefore a good model to illustrate soliton solution behavior in higher dimensions.

We will define the value \( \hat{\phi}^a = \phi^a/|\phi| \) and we can write down a gauge-invariant expression for \( F_{\mu\nu} \) when we take \( \hat{\phi}^a \) according to the vacuum conditions described above resulting in

\[
F_{\mu\nu} = \hat{\phi}^a F^{a\mu\nu} - e^{-1} \epsilon^{abc} \hat{\phi}^a (D_\mu \hat{\phi})^b (D_\nu \hat{\phi})^c
\]

(22)

This can be used as the electromagnetic field strength, or \( A^a_\mu \), at any point where \( |\phi| \neq 0 \), which is just all the points where the SU(2) symmetry is broken. We can input equations 19 and 20 to the above expression and use the identity \( \epsilon^{abc} \epsilon^{ade} = \delta^{bd} \delta^{ce} - \delta^{be} \delta^{cd} \) to rewrite equation 22 as

\[
F_{\mu\nu} = \partial_\mu (\hat{\phi}^a A^a_\nu) - \partial_\nu (\hat{\phi}^a A^a_\mu) - e^{-1} \epsilon^{abc} \hat{\phi}^a \partial_\mu \hat{\phi}^b \partial_\nu \hat{\phi}^c
\]

(23)

and we can then write the magnetic field as

\[
B^i = \frac{1}{2} \epsilon^{ijk} F_{jk} = \epsilon^{ijk} \partial_j (\hat{\phi}^a A^a_k) - (2e)^{-1} \epsilon^{ijk} \epsilon^{abc} \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c
\]

(24)

Let us now look at the magnetic flux through a sphere at \( r \to \infty \), given by \( \Phi = \int dS \cdot B \). The first term in equation 24 is just the curl and it has no divergence so the surface integral is zero. This reduces the flux equation to

\[
\Phi = -\frac{1}{2e} \int d^2 \theta e^{abc} \epsilon^{ijk} \hat{\phi}^a \partial_i \hat{\phi}^b \partial_j \hat{\phi}^c
\]

(25)

where we have inserted \( dS_k = r^2 \sin \theta \, d\theta \, d\phi \, \delta_k \). Additonally, \( \delta_k = x/r \) and \( d^2 \theta = d\theta d\phi \).

This equation looks awfully familiar to equation 15 derived in section 2.1. The reappearance of our equation for the winding number makes sense when we think about what \( \hat{\phi}^a \) represents. When we defined \( \hat{\phi}^a \), we mapped the original field onto a unit vector. We can define a mapping with a winding number \( n \) by taking \( \hat{\phi}^a \) as \( \theta \) and \( n \phi \) as \( \varphi \). We have defined a mapping from the vacuum sphere onto \( S_2 \), which means equation 25 is just a function of the winding number and the total flux through the sphere can be written as

\[
\Phi = -\frac{4\pi n}{e}
\]

(26)

implying that any soliton solution in this theory with a non-zero winding number is a magnetic monopole. This is called the ’t Hooft-Polyakov monopole. There are an infinite number of potential soliton solutions due to an infinite number of winding numbers. We will focus on the simplest case and, as in section 1, look at the identity map \( (n=1) \). We will now find the soliton solution for this monopole.

As in section 1, we will use the identity to map the boundary onto the vacuum structure resulting in the boundary condition for the scalar field

\[
\lim_{r \to \infty} \phi^a(x) = v x^a/r
\]

(27)

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and the boundary condition for the gauge field can be derived by requiring \((D_\mu \phi)^a = 0\) in the limit \(r \gg 0\), yielding
\[
\partial_i (x^a / r) + e e^{abc} A^b_i x^c / r = 0
\]
(28)
and, skipping some algebra, we can use these two conditions to derive our ansatz
\[
\phi^a(x) = v f(r) a^a / r
\]
(29)
\[
A^a_i(x) = a(r) e^{aoj} x_j / er^2
\]
(30)
where \(f(\infty) = a(\infty) = 1\) and \(f(0) = a(0) = 0\), which results in \(A^a_i\) and \(\phi^a\) having the desired asymptotic limits and being well defined at \(r = 0\).

We can plug the above three equations into equation 31 and use the variational principle to get a set of second order differential equations to solve for \(a\) and \(f\).

We can now construct the soliton energy, which is just the mass of the monopole. The mass is given by
\[
M = \int d^3x \left[ \frac{1}{2} B^a_i B^a_i + \frac{1}{2} (D_i \phi)^a (D_i \phi)^a + V(\phi) \right]
\]
(31)
where, based on our ansatz
\[
B^a_i = -\frac{1}{e} \left[ a' \left( \delta_{ai} - \hat{x}_a \hat{x}_i \right) + 2a - a^2 \right] \hat{x}_a \hat{x}_i
\]
(32)
\[
(D_i \phi)^a = v \left[ \frac{(1-a)}{r} \left( \delta_{ai} - \hat{x}_a \hat{x}_i \right) + f' \hat{x}_a \hat{x}_i \right]
\]
(33)
\[
V(\phi) = \frac{1}{8} \lambda \nu^4 (f^2 - 1)^2
\]
(34)
We can plug the above three equations into equation 31 and use the variational principle to get a set of second order differential equations to solve for \(f(r)\) and \(a(r)\).

To find the lower bound on \(M\) we use a technique similar to section 1 by rewriting
\[
\frac{1}{2} B^a_i B^a_i + \frac{1}{2} (D_i \phi)^a (D_i \phi)^a = \frac{1}{2} [B^a_i + (D_i \phi)^a]^a - B^a_i (D_i \phi)^a
\]
(35)
which we can apply to distribution rule to so that the last term is reformed as
\[
B^a_i (D_i \phi)^a = \partial_i (B^a_i \phi^a) - (D_i B_i)^a \phi^a
\]
(36)
The Bianchi identity implies \((D_i B_i)^a = 0\), which means the above equation reduces to \(B^a_i (D_i \phi)^a = \partial_i (B^a_i \phi^a)\), a total divergence. Gauss’s theorem shows that when we integrate over the total divergence it is equal to integrating \(B^a_i \phi^a\) over the surface of a sphere at spatial infinity where \(\phi^a = v\hat{x}^a\)
\[
\int d^3x \partial_i (B^a_i \phi^a) = \int dS_i B^a_i \phi^a = \frac{4\pi |n| v}{e}
\]
(37)
and our monopole mass can now be written as
\[
M = \frac{4\pi |n| v}{e} + \int d^3x \left[ \frac{1}{2} (B^a_i + (\text{sign } n)(D_i \phi)^a)^2 + V(\phi) \right]
\]
(38)
which we can now use to construct the Bogomolny bound on the monopole mass. In the case \(\lambda > 0\), the monopole with winding number \(n\) is unstable and will break up into \(|n|\) monopoles with winding number \(n = \pm 1\), depending on sign.

We can take \(m_W = ev\) and \(\alpha = e^2 / 4\pi\) the Bogomolny bound is
\[
M \geq \frac{m_W}{\alpha} |n|
\]
(39)
which is much heavier than the W boson. When we consider the limit \(\lambda \to 0\) so that \(V(\phi)\) vanishes, we can extract a pair of first-order differential equations for \(a\) and \(f\). These are
\[
a' = (1-a)f
\]
(40)
\[ f' = \frac{(2a - a^2)}{(m_W)^2} \]  
(41)

which have the closed-form solution

\[ a(m_W r) = 1 - \frac{m_W}{\sinh(m_W r)} \]  
(42)

\[ f(m_W r) = \coth(m_W r) - (m_W r)^{-1} \]  
(43)

This is the Bogomolny-Prasad-Sommerfeld solution, a soliton solution that satisfies the Bogomolny bound.

The Georgi-Glashow model has monopole solutions, but it is not in accord with observables in nature. The Standard Model, which is in accord, does not have monopole solutions. In the Standard Model, the electric charge is a linear combination of an SU(2) generator and U(1) hypercharge generator. One can introduce an SU(2) singlet field with an arbitrarily small hypercharge, and therefore an arbitrarily small electric charge so that the Dirac charge quantization condition would prevent any magnetic monopoles. But, the Georgi-Glashow model is a good theory to examine how soliton solutions can be found in higher spatial dimensions, and now we will finally be able to turn to instantons.

3 Instantons

Instantons are the localised finite-action solutions to the classical Euclidean field equations of a theory. The Euclidean field equation are directly related to the Minkowskian metric by a Wick rotation. We can take real time \( t = (x_0)_{\text{Mink}} \) of the Minkowski metric and analytically continue it to the imaginary values so that \( (x_4)_{\text{Euc}} = i t \) can serve as a real fourth component. We can analytically continue the Minkowski action in the same manner using a factor of \(-i\) in order to derive field equations that exist in Euclidean space. The special solutions described above to these field equations represent instantons. Therefore, the preceding discussion on solitons will be of great importance. The derivation of instantons follows much of the same procedure with the added constraint that we are working in a wick rotation of Minkowski space. Below we will derive and discuss instantons in SU(2) gauge theory, the solutions of which describe the transitions between vacua.

3.1 Instantons in SU(2) Gauge Theory

Consider SU(2) gauge theory with gauge fields only. This results in the Yang-Mills lagrangian

\[ \mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \]  
(44)

with \( F^a_{\mu\nu} \) described the usual way for non-abelian gauge theory. We will restrict our discussion to gauge transformations \( U \) that are time independent. This fixes temporal gauge, \( A_0 = 0 \) and we will also impose the condition that \( U(x) \to \text{const} \) as \( |x| \to \infty \). The space then has the topology of a three-dimensional sphere \( S^3 \).

If every \( U \) can be smoothly deformed into every other \( U \), then all the field configurations are gauge equivalent and there is only one vacuum state. If they cannot be smoothly deformed into the other, there must be more than one vacuum state. To see this, supposed \( U \) and \( \tilde{U} \) cannot be deformed. The vector potentials, \( A_u = \frac{i}{g} U \partial_\mu U^\dagger \) and \( \tilde{A}_u = \frac{i}{g} \tilde{U} \partial_\mu \tilde{U}^\dagger \), are gauge transformations of zero so \( F_{\mu\nu} \) and \( \tilde{F}_{\mu\nu} \) vanish. But, if we smoothly deform \( A_u \) into \( \tilde{A}_u \), we will pass through vector potentials that are not gauge transformations of zero, which therefore have non-zero field strengths, and therefore have non-zero energy. So, there is an energy barrier between the two vector potentials and they must each represent two different minima in the classical field space, also known as vacuum states in quantum theory!

In our scenario, every \( U \) cannot be smoothly deformed into every other \( U \) and we can classify the field configurations by a winding number. This can be shown by taking a 4-vector \( a_\mu = (\vec{a}, a_4) \), which we can use to describe a \( 2 \times 2 \) special unitary matrix \( U \) as

\[ U = a_4 + i\vec{a} \cdot \vec{\sigma} \]  
(45)

where \( \vec{\sigma} \) is just a vector notation of the usual pauli matrices and the length of \( a \) is equal to one. As can be seen, \( a \) can be thought of as a point on a three-sphere, which we will call the vacuum three-sphere. Since
our spatial boundary conditions give the topology of a three-sphere as well, $U$ is a map from the spatial three-sphere to the vacuum three-sphere. Following the procedure of section 2.3, we can define a winding number $n$ for this map, which describes the number of times the vacuum three-sphere covers the spatial three-sphere. The winding number can be written out as

$$n = \frac{-1}{24\pi^2} \int d^3 x \epsilon^{ijk} Tr[(U \partial_i U^\dagger)(U \partial_j U^\dagger)(U \partial_k U^\dagger)]$$  \hspace{1cm} (46)$$

Summarizing the previous few paragraphs: We have shown that SU(2) gauge theory has an infinite number of zero energy classical field configurations characterized by a winding number $n$ and separated by energy barriers. We can visualize the potential of this system by relating it to a scalar field theory with potential

$$V(\phi) = \lambda \nu^4 [1 - \cos(2\pi \phi/\nu)]$$  \hspace{1cm} (47)$$

The potential has minima when $\phi = n \nu$ and we can let $|n\rangle$ be the quantum states corresponding to these minima.

The tunneling amplitude between two vacua takes the form $\langle n' | H | n \rangle$. In scalar field theory, this amplitude scales by exponential decay with the volume of space and thus vanishes in the infinite volume limit, keeping the minima degenerate. In SU(2) gauge theory, the story is not the same. There is a classical solution to the euclidean field equations that describes the mediation between states with different winding numbers. For the case $n' = n + 1$, this solution is the instanton, which is accompanied by the case $n' = n - 1$, the anti-instanton. Other cases where $|n' - n\rangle$ are examples of a dilute gas of instantons (or anti-instantons). We will explicitly construct the solutions below.

At euclidean times $-t$, $+t$ we can set $A_{\mu \pm} = \frac{i}{g} U_{\pm \mu} U_{\pm}^\dagger$ where $U_{\pm}$ has winding number $n_{\pm}$. We set the boundary condition $A_\mu = 0$ at $|x| = r$. Thus, we have set up $U$ on the cylindrical boundary of 4 dimensional euclidean spacetime. The boundary is, as should be familiar by now, topologically a three-sphere. Since we will take $r$, $\pm t$ to $\infty$, the shape of the boundary does not matter and we can consider it to be a three sphere where we have a map $U(\hat{x})$, where $\hat{x}_\mu = x_\mu/\rho$ and the winding number $n = n_+ - n_-.$

We will now follow our intuition gained in section 2.3 and construct a Bogomolny bound on the euclidean action

$$S = \frac{1}{2} \int d^4 x \ Tr[F_{\mu \nu} F_{\mu \nu}]$$  \hspace{1cm} (48)$$

for a field with boundary condition

$$\lim_{\rho \to \infty} A_\mu(x) = \frac{i}{g} U(\hat{x}) \partial_\mu U^\dagger(\hat{x})$$  \hspace{1cm} (49)$$

Equation 46 can be rewritten in terms of polar angles $\theta$, $\psi$ and the azimuthal angle $\varphi$ and reformulated as a surface integral over a surface at infinity in four dimensional euclidean space. In this form, equation 46 resembles

$$n = \frac{1}{24\pi^2} \int dS_\mu \epsilon^{\mu \nu \sigma \tau} \ Tr[(U \partial_\nu U^\dagger)(U \partial_\sigma U^\dagger)(U \partial_\tau U^\dagger)]$$  \hspace{1cm} (50)$$

and we can use the boundary condition in equation 49 to write the above equation in terms of the vector potential,

$$n = \frac{ig^3}{24\pi^2} \int dS_\mu \epsilon^{\mu \nu \sigma \tau} \ Tr[A_\nu A_\sigma A_\tau]$$  \hspace{1cm} (51)$$

We will rewrite this surface integral as a volume integral using the Chern-Simons current

$$J^\mu = 2 \epsilon^{\mu \nu \sigma \tau} \ Tr[A_\nu F_{\sigma \tau} + \frac{2}{3} ig A_\nu A_\sigma A_\tau]$$  \hspace{1cm} (52)$$

which is not gauge invariant, but does have a divergence that is gauge invariant,

$$\partial_\mu J^\mu = \epsilon^{\mu \nu \sigma \tau} \ Tr[F_{\mu \nu} F_{\sigma \tau}] = 2 \ Tr[\tilde{F}_{\mu \nu} F_{\mu \nu}]$$  \hspace{1cm} (53)$$

where $\tilde{F}_{\mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \sigma \tau} F_{\sigma \tau}$ represents the dual field strength.
Remembering that on the surface at infinity the vector potential is a gauge transformation of zero, and thus the field strength vanishes, we can use equation 52 to rewrite equation 51 to be

\[ n = \frac{g^2}{32\pi^2} \int dS_\mu J^\mu = \frac{g^2}{32\pi^2} \int d^4x \partial_\mu J^\mu \] (54)

where we have used Gauss’s theorem to produce the rightmost formula. Our last step in expressing the winding number as an integral of a gauge invariant expression is to use equation 53 to get

\[ n = \frac{g^2}{16\pi^2} \int d^4x \frac{\rho}{\rho} \text{Tr} \left[ \tilde{F}_{\mu\nu} F_{\mu\nu} \right] \] (55)

From this result we can construct a Bogomolny bound. Using \( \tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu} = F_{\mu\nu} F_{\mu\nu} \), we can construct the identity

\[ \frac{1}{2} \text{Tr} \left[ \tilde{F}_{\mu\nu} \pm F_{\mu\nu} \right]^2 = \text{Tr} \left[ F_{\mu\nu} F_{\mu\nu} \right] \pm \text{Tr} \left[ \tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu} \right] \] (56)

The lefthand side of the equation is obviously non-negative, so we can construct the inequality

\[ \int d^4x \text{Tr} \left[ F_{\mu\nu} F_{\mu\nu} \right] \geq \left| \int d^4x \frac{\rho}{\rho} \text{Tr} \left[ \tilde{F}_{\mu\nu} F_{\mu\nu} \right] \right| \] (57)

Picking apart equation 57, we can see that the lefthand side is just twice the euclidean action and the righthand side is, according to equation 55, just the winding number multiplied by a constant. We can therefore rewrite the inequality as

\[ S \geq \frac{8\pi^2 |n|}{g^2} \] (58)

Here, we have constructed a Bogomolny bound for the minimum value of the euclidean action for a classical solution that describes the mediation between vacuum configurations \( n_- \) at \( x_4 = -\infty \) and \( n_+ = n_- + n \) at \( x_4 = +\infty \).

From equation 56, we note that we can only saturate the bound in our inequality if

\[ \tilde{F}_{\mu\nu} = (\text{sign } n) F_{\mu\nu} \] (59)

We can find an explicit solution to this equation in the case of \( n = 1 \), which is our instanton solution.

Once again, since we are working in \( n = 1 \), we will take the identity map,

\[ U(\hat{x}) = \frac{x_4 + i \vec{\hat{x}} \cdot \vec{\sigma}}{\rho} \] (60)

which recovers the ansatz

\[ A_\mu(x) = i \int f(\rho) U(\hat{x}) \partial_\mu U^\dagger(\hat{x}) \] (61)

where at the boundaries \( f(\infty) = 1 \) and \( f(0) = 0 \). From this we can derive our field strength,

\[ F_{\mu\nu} = \frac{i}{g} \left[ (\partial_\mu f) U \partial_\nu U^\dagger + f(1 - f) \partial_\mu U \partial_\nu U^\dagger - (\rho \leftrightarrow \nu) \right] \] (62)

where we have used the identity \( (\partial_\mu U^\dagger) U = -U^\dagger \partial_\mu U \) to simplify the equation.

We will now write out the field strength in spherical coordinates, where \( \theta, \psi \) represent polar angles and \( \varphi \) represents the azimuthal angle. Written this way, we get

\[ F_{\rho\varphi} = \frac{i}{g} f^\prime \rho^{-1} U \partial_\theta U^\dagger \] (63)

\[ F_{\psi\varphi} = \frac{\partial_\psi U \partial_\varphi U^\dagger - \partial_\varphi U \partial_\psi U^\dagger}{\frac{i}{g} f(1 - f)(\rho^2 \sin^2 \theta \sin \psi)} \] (64)
The final step to produce our differential equations is to note that our definition of $\mathbf{F}_{\mu\nu}$ implies $\mathbf{F}_{\rho\theta} = -\mathbf{F}_{\psi\varphi}$ and since, in our case of the instanton solution, $\mathbf{F}_{\mu\nu} = F_{\mu\nu}$, equation 63 is equal to minus equation 64. Using this fact, and our normal separation of variables technique, we can derive the equations

$$\rho f' = cf(1 - f) \quad (65)$$

$$U\partial_\theta U^\dagger = \frac{\partial_\psi U\partial_\phi U^\dagger - \partial_\phi U\partial_\psi U}{c(\sin^2 \theta \sin \psi)} \quad (66)$$

where $c$, in this instance, is just the separation constant. Plugging our mapping, $U(\hat{x})$, from equation 60 into equation 65, we can integrate to get

$$f(\rho) = \frac{\rho^2}{\rho^2 + a^2} \quad (67)$$

where $a$ is a constant of integration and represents the size of the instanton. This solution could also be parametrized by its origin. We used the spacetime origin for our calculations but, as noted back in section 1, we could lorentz boost into a new frame. The solution we have derived here for Yang-Mills theory is called the BPST instanton.

### 3.2 The Role of Instantons in Quantum Field Theories

Having gone through the explicit calculation of the BPST instanton in the previous section, we will now zoom out to discuss the conceptual role instantons play in quantum field theories. Sadly, the previously described BPST instanton, which we worked hard to derive, is not an exact solution to the field equations when the Higgs mechanism is introduced. To understand why, we need only to look back to the definition of Derrick’s theorem in section 2.2, massive theories do not have exact instanton solutions. However, instantons are still a robust topic to study and have interesting uses in various theories— we will briefly describe some instances of instantons below.

In QCD instantons have been theorized to play a role in the vacuum structure. The instanton liquid model, a method of describing the ensemble of instantons as a liquid rather than a dilute gas of individual models has been proposed to analyze the phenomenology of QCD. The study of instantons has also been applied to the confinement principle in Yang-Mills theory, without great success. A comprehensive review of instantons in QCD can be found in Schäfer & Shuryak (1998) [4].

In electroweak theory, instantons are still being studied although they would introduce a baryon number violation. The Higgs mechanism results in instantons not being exact solutions but, one can use approximations or the constrained instanton formulation. Additionally, instantons are non-perturbative so their effects are heavily suppressed. One idea is that the presence of gauge boson mass suppresses large instantons. As we saw deriving instanton solutions, the mass of the solution is much larger than the gauge boson mass.

In conclusion, it is clear that instantons are integral in understanding the vacuum structure of many theories. While there are no exact solutions in the Standard Model, instantons have potential uses both as approximations in the standard model and beyond.

### References


