I. Abstract group theory

A. Definition of group

Let $G$ be a group.

1. $a, b, c \in G$

   i) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity)

   ii) $e \in G \implies \forall a \in G \ a \cdot e = e \cdot a = a$

   iii) $\forall a \in G \ a^{-1} \in G \ a \cdot a^{-1} = a^{-1} \cdot a = e$

2. $G = \mathbb{Z}, \mathbb{R}, \mathbb{C}$ with addition

   i) $(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$

   n-tuples

3. $G = \mathbb{R}^* = \mathbb{R} - \{0\}$ with multiplication

   Def. $H \subseteq G$ also a group is a subgroup

   i) $H \subseteq \mathbb{R}^+ (\mathbb{C} +)

   ii) $H = \mathbb{R}^*$ subgroup?

   iii) $1 \in \mathbb{R}^+$ positive real number. Subgroup?
Order of $G$, $|G|$ number of elements in $G$.

Finite group, $|G| < \infty$ otherwise $\infty$.

Let $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$

$n = r + Nl$ where $r$: integer $\pmod{N}$

$\mathbb{Z}/N\mathbb{Z} = \mathbb{Z}_N$

$a \cdot b = b \cdot a$ for all $a, b \Rightarrow$ abelian group.

Let $k = \begin{cases} \mathbb{R}, & \text{non-abelian} \\ \mathbb{F}, & \text{abelian} \end{cases}$

$GL(n, k) = \{ A | A = n \times n \text{ invertible over } k \}$

(i) infinite order. Non-abelian, $n > 1$

(ii) matrix group is a subgroup of $GL(n, k)$:

$SL(n, k) = \{ A \in GL(n, k) : \det A = 1 \}$

$O(n, k) = \{ A \in GL(n, \mathbb{F}) : AA^T = 1 \}$

$SO(n) = \{ A \in O(n, \mathbb{F}) : \det A = 1 \}$

$SO(2, \mathbb{R}) = \{ R(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} : 0 \leq \phi \leq 2\pi \}$

$SO(3, \mathbb{R})$ rotations.
iii) Function defines $G$ group by set $\mathcal{F} \left[ x \rightarrow G \right] = \{ f : f \text{ is a function } x \rightarrow G \}
$
\text{claim: } \mathcal{F} \text{ is a group} 
(f_1, f_2) (x) = f_1(x) f_2(x)
$
If $x \in X$ has $\alpha$ rel $y$ points then
\text{this is an infinite order group.}

If $X$ is a manifold, $G$ Lie group
\Rightarrow \alpha \text{-dim. space.}

B. Sym. group

Permutation is a 1-1 invertible $\phi : x \rightarrow x$
$\phi_1 \circ \phi_2$ is a perm.

$X = \{ 1, \ldots, n \}$ symmetric group on $n$ elements
is group of perm.

$S_n = \{ \text{perms of } X = \{ 1, \ldots, n \} \}$

$\phi_1 \circ \phi_2 = \phi_2 \circ \phi_1 \phi_1 \circ \phi_2 = \phi_1 \circ \phi_2$

\text{our convention}

$\phi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \phi_1 & \phi_2 & \cdots & \phi_n \end{pmatrix} \quad \phi(1) = \phi_1 \quad \cdots$

$\phi = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$
\[ \phi_1 = (v_1 \ldots v_n) \quad \phi_v = (t \ldots t) \]
\[ \phi_{L \cdot v + t} = (v_1 \ldots v_n) \]

Cyclic group of length \( l \)
\[ a_1 \to a_2 \to \cdots \to a_l \to a_1 \]
and other element fixed.
\[ \phi = (a_1, a_2, \ldots, a_l) \]
\[ S_2 = \{ 1, (12) \} \]
\[ S_3 = \{ 1, (12), (13), (23), (123), (132) \} \]

S_2 abelian, \( S_3 \) non-abelian \( (12)(13) \)
\[ (123) = (12)(13) \]

Transposition \( (ij) \) perm.

1. Every element of \( S_n \) is a product of transpositions.

2. \( 1 \leq i < j \leq k \)
\[ (ij)(jk) = (ik)(ij)(jk) \]
\[ (ij)^2 = 1 \]
\[ (ij)(kl) = (kl)(ij) \quad i,j \neq k,l \]
Pictorial illustration of

Perms. can be written as products of transps. in many ways but number is always same mod 2. Distinguish even/odd perms.

Modify × group of braids finite or infinite

Present a group by generators and relations. A set \( S' \subseteq G \) generates a group \( G \) if every element of \( G \) is a 'word' made from the letters of \( S' \).

Relations are equalities between different words. Presenting a group by gen. and relation means giving a finite set of relations from which all other follow.
C. Cosets & Conjugacy

1. Equivalence relations

Def: Let \( X \) be any set. A binary relation \( \sim \) is an equivalence relation if

\[ a, b, c \in X \]

i) \( a \sim a \)
ii) \( a \sim b \Rightarrow b \sim a \)
iii) \( a \sim b \) and \( b \sim c \Rightarrow a \sim c \)

In the above examples:

i) \( [a] = \{a\} \)
ii) \( [1] = \{n: n \text{ is odd}\} \)
iii) \( [4] = \{n: n \text{ is even}\} \)

The disjoint equivalence classes of an equivalence relation on \( X \) decompose \( X \) into a union of mutually disjoint subsets. Conversely, given a disjoint decomposition \( X = \bigsqcup X_i \), we can define an equivalence relation by saying \( a \sim b \) if \( a, b \in X_i \).
2. Lagrange's Theorem.

(i) \( H \leq G \) subgroup \( gHg^{-1} \subset H \) is called a left coset of \( H \).

\[
\begin{align*}
G &= \{ e, 1, 12 \} \\
H &= \{ e, 12 \} \\
(12) H &= \{ (12), (122) \} \\
(122) H &= \{ (23), (12) \} \\
(122) H &= \{ (23), (12) \}
\end{align*}
\]

(ii) \( G = S_3, H = \{ e, 2 \} \) are the 2 cosets.

Claim: 2 left cosets are either identical or disjoint. Every \( g \in G \) lies in some coset. \( G \) acts on \( G \) by \( g \cdot H = gH \). 

pf.: Suppose \( g, g' H \cap g'H = \emptyset \)

 implies \( g' = g h \) \( \in H \)

\[ g' = g h \Rightarrow g' h^{-1} = h \Rightarrow H = g' h^{-1} \Rightarrow g' H = g H \]

Theorem: \( H \) subgroup of \( G \) (finite) then order of \( H \) divides order of \( G \).

\[ |G| / |H| \geq 2 \]

pf.: If \( G \) is finite \( G = \{1\} \) gH for some g; order of any coset is order of \( H \)

\[ |gH| = |H| \]

\[ \Rightarrow 161 / |H| = 2 \]
with $\mathbb{Z}_p$ prime has no subgroups.

$\mathbb{Z}_2$ has no subgroups.

Def: set of left cosets is denoted $G/H$. The order of this set is the index of $H$ in $G$ and denoted \( [G : H] \).

\( G = S_3 \), \( H = S_5 \). \( G/H = \{ H, (12), H, (12) \cdot H \} \).
\( [G : H] = 3 \).

3. Conjugacy

The notion generalizing the idea of similarity of matrices.

Def: i) group element $h$ is conjugate to $k$ if $h = g h g^{-1}$.

ii) conjugacy class of $h$ is \( \mathcal{C}_h = \{ g h g^{-1} : g \in G \} = \{ k : k \sim h \} \).

iii) $H \leq G$, $K \leq G$ be two subgroups. We say $H$ is conjugate to $K$ if $g \in G \implies K = g H g^{-1}$.

By $G = GL(n, k)$ be a matrix group then conjugacy is same notion as similarity of matrices. Conjugacy class of a matrix $A$ is set of conjugate matrices with same eigenvalues as $A$. 
Groups which are self-conjugate are very special.

Def: A subgroup \( N \vartriangleleft G \) is called a normal subgroup (or invariant subgroup) if \( gN = N \forall g \in G \).

Now have a nice theorem. In general, set of left cosets of \( H \in G \), \( G/H \), is not a group.

If \( H \) is normal,

Then: \( N \vartriangleleft G \), normal then set of left cosets \( G/N \equiv \{gN\} \) is itself a group with group mult. \((g, N) \cdot (g', N) \in (gg', N)\).

So: well-defined? \( gN = g', N \) some answer.

Check 3 axioms for a group.

w) i) All subgroups of abelian groups are normal. \( \text{and } N \triangleleft Z \) is normal.

and the quotient is \( Z/N \triangleleft Z \).

ii) \( A_3 = \{1, (123), (132)\} \subseteq S_3 \) is normal. What is \( S_3/A_3 \)
D. Homomorphism and Isomorphism

Def:

1. A homomorphism \( \mu : G \rightarrow G' \) is a mapping that preserves group law

\[
\mu(\langle g_1g_2 \rangle) = \mu(g_1) \cdot \mu(g_2)
\]

in \( G \)

in \( G' \)

2. If \( \mu \) is 1-1 and onto, it is called an isomorphism

3. One often uses the term automorphism of \( G \) when \( G = G' \).

Slogan: isom. goes as the same.

We will show that any finite group is isom. to a subgroup of perm. group.

let \( S_3 = \{1, 2, 3\} \) perm on 3 letters

define \( G = \langle 12 \rangle \) and consider group

\[
S_3 = \{ (1) \text{, } (12) \text{, } (13) \text{, } (23) \text{, } (123) \text{, } (132) \text{, } (123) \text{, } (132) \}
\]

matrix form

\[
\mu : S_3 \rightarrow \mathbb{R}^3 \text{, } \mu(\langle 1 \rangle) = (1)
\]

\[
\mu(\langle 2 \rangle) = (1)
\]

Is it home? \( \mu(1 \cdot 2) = \mu(1) \cdot \mu(2) \) yes!

Isomorphism and example of matrix rep.

ii) \( Z_n \) and \( \{1, 2, \ldots, n-1\} \) multiplication.
iii) Sign homomorphism

\[ \sigma : S_n \to \mathbb{Z}_2 = \{ \pm 1, \text{mult.} \} \]

\[ \sigma : \sigma \mapsto +1 \text{ if } \sigma \text{ is even } \]

\[ \sigma : \sigma \mapsto -1 \text{ if } \sigma \text{ is odd } \]

2. Kernel and Image

\[ \mu : G \to G' \text{ homo then there is autom. } \]

a) "Bad-game" subgroup of both \( G \) and \( G' \).

Def: i) Ker of \( \mu \)

\[ K = \{ g \in G \mid \mu(g) = 1 \} \]

ii) Im of \( \mu \)

\[ \mu(g) \in G' \]

Thm: Let \( K \subseteq G \) be kernel then \( K \) is a normal subgroup.

\[ \Rightarrow G/K \text{ is a group.} \]

These 2 groups are closely related

\[ \mu(g) = \mu(g') \iff gK = g'K \]

Thm: \( \mu(g) \sim G/K \)

PF: we associate each \( gK \) to element

\[ \mu(g) \text{ in } G' \]
\[ \gamma : gK \to \mu(g) \]

Claim: \( \gamma \) is an isomorphism

i) \( \gamma \) well-defined:

\[ gK = g'K \Rightarrow \exists k \in K, g' = gk \]

\[ \Rightarrow \mu(g') = \mu(gk) = \mu(g)\mu(k) = \mu \]

ii) \( \gamma \) is 1-1 i.e.

\[ \mu(g) = \mu(g') \Rightarrow gK = g'K \]

**Exact Sequence**

\[ k \to G \to \mu(g) \]

**Im of one arrow defines kernel of next to immediate right.**

Then mean

\[ 1 \to k \to G \to \mu(g) \to \]

E. Comments on the general structure theory of groups

Useful mathematical objects have associated problems, namely classification.

Groups:
- Finite abelian
- Infinite nonabelian

Look for 'good criterion'
one "good criterion" is that of finitely generated abelian groups. Need 2 notions:

1. **Def:** subset $S \subseteq G$ is a generating set for a group if every element $g \in G$ can be written as a "word" or product of elements in $S$: $g = g_1 \cdot \ldots \cdot g_k$ for $g_i \in S$.

Finitely generated means that $S$ is finite, so all elements can be obtained by taking products of $S$.

- Sym.: $G$ is finitely generated by i) **transpositions**.
- ii) $G_0$ by any $g \neq e$ of order $N$.

**Def:** Given $G_1$, $G_2$, the **product group** $G_1 \times G_2$ is the Cartesian product as a set with component-wise multiplication.

Any finitely generated abelian group is isomorphic to $\mathbb{Z}^r \times G_T$

Integers $r$ is rank of the group. $G_T$ is a finite abelian group called the **torsion subgroup** (finite order elements).

What can we say about finite abelian groups?
2. Finite abelian groups

rig rel. prime integers

$\mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$

more generally $\mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{\gcd(p,q)} \times \mathbb{Z}_{\frac{pq}{\gcd(p,q)}}$

used implicitly in physics whenever we have 2 deg. of freedom with different but commensurable frequencies.

Suppose you do high every other day then you do high on Monday every other week because 1 and 7 are rig. prime.

more generally $\mathbb{Z}_p \times \mathbb{Z}_q$, multiplication group.

At time problem we set $T = 1$ period is $T = 1$

2 oscillations of periods $p,q$, then config

same is $\mathbb{Z}_p \times \mathbb{Z}_q$. Basic period is long (e.g., 7). Using this,

The Kramers Structure Theorem:

Any finite abelian group is isomorphism to $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$

where $n_1 | n_2 | n_3 | \ldots | n_k$

See S. Lang
3. Finite simple groups (classification)

Generalization to non-abelian groups is very hard. (Good criterion is that a group be simple.)
A group is simple if it has no non-trivial normal subgroups

\[ G \] has no non-trivial homomorphic images.

\[ \mathfrak{p} \mid \mathfrak{p} \text{ prime.} \]

(20th Century)

Complete classification of finite simple groups. Key is Feit-Thompson theorem says a finite simple non-abelian group has even order. Classification completed in 1981. Takes ~ 10,4 journal pages!

\{ An normal subgroup of } 

\{ A_n \text{ for } n \geq 5 \}

\{ Simple lie groups over finite fields \}

\{ 26 sporadic oddballs \}

(iii) has a few more infinite sequences like (i) and (ii). For example An(s) of n x n matrices of det 1 with entries in 2 \mid 1 \text{ and } 26 \mid \text{ prime power.}
26 odd balls. = 2 X largest one. It is the monster group with order

\[ |\text{monster}| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 76 \cdot (12 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71) \]

Has only 194 conjugacy classes.

Best construction from string theory!

First shown to exist in 1980 by Griess.

\[ o(\mathcal{G}) \leq 8 \]

4. Group of Schrödinger Eqn.

Symmetry: \[ \hat{\psi} \rightarrow x \hat{\psi} \rightarrow x^* \hat{\psi} \]

\[ S H S^{-1} = H \]

Simultaneously diagonalize. Good quantum numb

\[ \langle \psi | H | \phi \rangle = \text{matrix element if quantum number different.} \]
Insect $S'$ and commute. 

If $S'$ commutes, how about $S$? 

Indeed $S$ commutes. 

In a basis of eigenvectors, we can regard $S$ by a matrix. Basis of representation theory. 

What happens if $S$ and $S'$ commute with $H$ but $[S, S'] \neq 0$? We shall see later. 

A group of coordinate transforms that leave $H$ invariant forms a group. 

"A group of Schrödinger eigen."

"Invariance group of $H". 

$H = p_x^2 + p_y^2 + p_z^2 + V(x)$

Group of pure rotations of $H$ 

is invariance group. 

Discuss $D_4$? 

A group of rotations that leave invariant a crystal lattice is a "crystallographic point group."
point = origin (1 point) is left unmoved.

In such groups:
Abelian rotations by angles \( \frac{2\pi}{n} \) for \( n = 2, 3, 4, 6 \)

Call rotations about \( oj \) axis by \( \frac{2\pi}{n} \) \( \text{(o)} \).

\( C_2 \) x rotation by \( \pi \) about \( x-\text{axis} \).

\( D_4 \) in \{ \( \text{e}, C_4, C_{2v}, C_{2}, C_{4v}, C_{4}, C_{2h}, C_{2d} \) \}

\( D_4 = \{ \end{array} \}

\[ D_x = -2 \]

\[ \text{a} \]

\[ \text{c} \]

\[ -\pi \]

\[ +\pi \]

\[ \text{1-d example and application: Potential well} \]

\[ V(x) \]

\[ \infty \]

\[ x \]

\[ H = \frac{p^2}{2m} + V(x) \]

\[ E_n = \left( \frac{n\pi}{L} \right)^2 \frac{\hbar^2}{2m} \]

\[ T_n(x) = \delta \left( x - \frac{n\pi}{L} \right) \]

For even: \[ \mathcal{H} T(x) = T(-x) \]

\[ T T^{-1} = T \]

\[ \mathcal{H} H^{-1} = H \]

Even did not degenerate.
\[ H(\mathbf{p}_n) = \mathbf{T} H \mathbf{p}_n = \mathbf{e}_n \mathbf{T} \mathbf{p}_n \]

\[ = \begin{cases} \mathbf{e}_n & \text{if } \mathbf{p}_n \text{ is degenerate} \\
\end{cases} \]

\[ \mathbf{I}^2 = 1 \quad \text{c.v. } \pm 1 \]

\[ (\mathbf{i} \text{ is Hamilton } \mathbf{T} \mathbf{x}) = + \delta(x + x') \]

\[ \Rightarrow \mathbf{I}_x \mathbf{x}^2 = + \delta(x + x') \]

\[ = \mathbf{I}_x \mathbf{x} + \]

Given any e.f. \( \mathbf{p}_n \), \( \mathbf{p}_n \pm \mathbf{I} \mathbf{p}_n \) with definite c.v. under \( \mathbf{I} \).

All e.f. are even/odd. Selection rule.

\[ \frac{\text{even (odd)}}{x \text{ even/odd}} \]

When is this non-zero?

\[ \text{case } + \quad \mathbf{I} \mathbf{I} = \mathbf{I} \]

\[ \langle \pm, 1 \times (\pm) \rangle = \langle (\pm), 1 \times (\pm) \rangle = \]

\[ - \langle (\pm), 1 \times (\pm) \rangle = \]

\[ - (\pm)(\pm) \leq 1 \times 1 \Rightarrow \]

\[ \Rightarrow - (\pm)(\pm) = + \text{ or vanishes.} \]
II

Groups & Symmetry

1. Theory of groups is the theory of symmetry. Often problems in physics are intractable. All you can say is based on symmetry. In physics, group theory enters in many ways:
   - Special relativity: Lorentz group
   - Atomic physics, condensed matter: Regs.
   - Of rotation groups, crystallographic groups
   - And classification of lattices.
   - Nuclear physics: reps. of $c$pt and non-
     $c$pt groups.
   - Particle physics: standard model:
     Compact Lie groups and their reps.
   - Gauge theory, gen. rel., string theory:
     Discrete groups, arithmetic groups, non-$c$pt
     groups, $L^2$-dim. groups.

2. Groups & Symmetry

2.1 Permutation groups

Connection to symmetry is provided by transformation
groups. Let $X$ be any set (possibly $\emptyset$)

Def. 2.1.1. A perm of $X$ is a 1-1 onto mapping

$X \to X$. The set $S_X$ of all perms forms
a group under composition.
Def 2.1.2a: A transform group on $X$ is a subgroup of $S_X$.

Important. So let's put it another way:

Def 2.1.2b: $G$ acts on $X$ as a transform group if $\forall g \in G \exists \phi : 1-1 \text{ map } X \to X$

$x \mapsto g \cdot x \Rightarrow g_1 \cdot (g_2 \cdot x) = (g_1g_2) \cdot x$

$x = \{1\} \Rightarrow S_X = S_1$ as before

Ex 2.1.2: Group actions on the plane

$X = \mathbb{R}^2$, $G = \text{SO}(2, \mathbb{R})$

$R_\theta : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

defines a group of $1-1$ mappings of $\mathbb{R}^2$ hence a transformation group.

2.1. Cayley's Theorem.

Note: $G$ always acts on itself as a transform group.

To each $a \in G$ associate a mapping $L(a)$ where

$L(a) : G \to G$ by $L(a) : g \mapsto \phi(g, a)$

"left translation" by $a$.

Mapping is $1-1$ and onto $\Rightarrow L(a) \in S_G$.

Satisfy $L(ab) = L(a)L(b)$

Can also check $L(e) = 1 \iff a = 1$
So $L : a \rightarrow L(c)$ is a homomorphism $L : G \rightarrow S_n$.

Also since $L = \{T(1)\}$, so the image of $L$ is isomorphic to $S_n$. We have proved

**Theorem 2.1.1** (Cauchy's Theorem)

Any group $G$ is isomorphic to a subgroup of the full perm. group $S_n$. If $n = |G| < \infty$ then $G$ is isomorphic to a subgroup of $S_n$.

**Caution:** $L(a) : b \rightarrow b$ is not a homomorphism!

**3. Isometry Groups**

In any group $G$, a group of transformations preserving some notion of distance (e.g., $G = \text{translations preserving length and angles}$), we take $\mathbb{R}^2$ with Euclidean metric

$$12 \times 11^2 = (2)^2$$

**Def.** An isometry of the Euclidean metric is a map $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which is norm-preserving:

$$1 \leq \|T(x) - T(y)\| = \|x - y\|$$

The group of all isometries is denoted $E(3)$.

\[
\begin{align*}
&1)\text{ translations } T_x : x \rightarrow x + t \\
&T_x \circ T_y = T_{x+y} \\
&\text{abelian subgroup } T_x \subset E(3), \ T \sim \mathbb{R}^3 \\
&2)\text{ rotations } x \rightarrow (R_\theta x)^t = R_\theta \text{ if } x \in \mathbb{R}^3
\end{align*}
\]
The isometry group of $\mathbb{R}^3$ is,

$$E(3) = \text{Group of all isometries of 3-space}$$

$$\cong T \times O(3)$$

(Translations)

A semi-direct product appears in a problem set.

4. Symmetries of regular objects

Let $X$ be the set of points of the regular object and $\mathfrak{g}$ be for isometries preserving $X$. Subgroup of $O(2)$ or $O(3)$ for 2D-3D objects.

4.1 Subgroups of $O(2)$

$C_n$ = group of rotations $k\pi$ about origin

$\mathfrak{g}$ in $\mathbb{R}^2$ by $\theta = \frac{n}{2}k$,

$k = 0, 1, ..., n$

Dihedral group. Consider the group generated by rotations in $C_n$ together with a 2nd group.

Transformation, $P = \text{reflection in } x$-axis

ops. of 124 and $P$ on 4 pts.
These generalize \( D_n = \langle R_k, \sigma \rangle \) order in

4.2. Symmetries of polygons in the plane

\[ \text{Symm. of a regular } n\text{-gon centered at the origin of the plane are } D_n. \]

Square \( a \times b \) rotations and reflectons permute \( \{1, 2, 3, 4\} \) subgroup of \( S_4 \).

Dihedral group \( D_4 \)

4.3. \[ \text{Symm. of } n\text{-dim. latttices} \]

Let \( g \in \text{SO}(n) \) be a symm. of a \( n\text{-dim. lattice} \).\n
Generates discrete subgroup \( G \subset \text{SO}(n) \) which must be \( \cong \mathbb{Z}^n \otimes \mathbb{K} \) / \( \mathbb{Z}^n \).

In an integral basis for the lattice, the matrix \( g \in \text{SO}(n) \) must be an integral matrix.

\[ g : \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z} \]

In a canonical basis, \( g \) must be rep. by a rotation \( R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \).
true is independent of basis and must be integral.

\[ 2 \cos \theta \in \mathbb{Z} \]

only possible if \( \cos \theta = 0, \pm \frac{1}{2}, \pm 1 \)

\[ \theta = \frac{2\pi}{n}, \quad n = 1, 2, 3, 4 \text{ or } 6. \]

5-fold symm. of 2-dim. lattice is forbidden (quasi-crystals !!)

4.4 Discrete subgroups of \( \text{SO}(3) \)

Generalization non-trivial

Discrete subgroups of \( \text{SO}(3) \) are symm. of regular solids in 3-space.

Subgroups of \( \text{SO}(3) \) can be classified

i) \( C_n \quad n \geq 1 \)

ii) \( D_n \quad n \geq 1 \)

iii) symm. of tetrahedron

iv) symm. of cube = symm. of octahedron

v) symm. of dodecahedron = symm. of icosahedron

Similar to list of finite simple groups

\begin{align*}
\text{tetra} & \quad \text{cube} & \quad \text{octahedron} \\
(4 \text{ faces}) & \quad (6 \text{ faces}) & \quad (8 \text{ faces}) \\
\text{dodecahedron} & \quad \text{icosahedron} & \quad (20 \text{ faces}) \\
(12) & &
\end{align*}
5. Orbits

5.1 Def: Suppose \( x, y \in X \) and \( G \) is a trans. of \( X \)
we say "\( x \) is \( G \)-equivalent to \( y \)," \( x \sim y \), if
\( \exists \ g \in G \) \( g \cdot x = y \). This is an equivalence
corollary.

Def 5.2 The equiv. classes of \( x \) under \( x \sim y \)
are the \( G \)-orbits of \( X \):
\[
\text{Orb}_G(x) = \{ y \mid y \sim x \} = \{ g \cdot x \mid g \in G \}
\]

Lw 5.1 Rotations \( S_0(1,1) \) produce orbits on \( 1R^2 \)
point

orbit is a circle

2 different (qualitatively)
orbits of \( S_0(1,1) \)

Lw 5.2 2-2 Lorentz group defined by:
\[
S_0(1,1) ; 1R = \{ B(\theta) = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} \mid -\infty < \theta < \infty \}
\]
(notation explained later). Group is nice because
\[
B(\theta_1) B(\theta_2) = B(\theta_1 + \theta_2)
\]
so \( S_0(1,1) \cong 1R \) as groups

3 distinct kinds of
orbits
So, \((1,1)\) acts on \(\mathbb{R}^4\) as Lorentz boosts. Choose \((t, x)\) on \(\mathbb{R}^4\) thought of as \(1+1\) spacetime and define
\[
 t \rightarrow e^{\omega} t - \sinh \omega x
 x \rightarrow \sinh \omega t + e\omega x
\]
(light-cone \(x^+ = \pm x^0 x^1 \\text{ and } x^- = \pm x^2 x^3\) is invariant. Orbit are hyperbolae unless \(x^+ = 0\) or \(x^- = 0\). Problem?

i) hyperbolae in forward / backward light-cone

ii) 4 disjoint light-rays

iii) the origin.

By a change \(t \rightarrow E \ x \rightarrow \phi\), interpret some orbit as the mass-shell orbits for

i) massive particles: \(m^2 > 0\) in forward and backward light-cone, \(m^2 < 0\) i.e. tachyons on the left / right.

ii) massless particles: right or left-moving

iii) massless "particle" of zero energy / mom.

5.1 Transitive group actions

Def: A \(G\)-action on \(X\) is transitive if there is only one \(G\) orbit.

\(A, x, y \in X \rightarrow y = g \cdot y\)

Any \(x, y\) are \(G(x, y)\) related.
let $\mathbb{S}^2$ consider $\text{SO}(n,\mathbb{R})$ subgroup of $\text{SO}(3,\mathbb{R})$
defined by rot. about $z$-axis. Acts on sphere but not transitively.

Now we relate algebraic notion of subgroups
and orbits to geometric notions of orbits and fixed points.

Proposition 5.1: $G$ acts on $X$ as a transitive group. For\[ \forall x \in X \]
define $G^x \equiv \{ g \in G : g \cdot x = x \}$

$G^x$ is the set of group elements for which
$x$ is a fixed point and is called the isotropy group at $x$.

Problem: i) $G^x$ is a subgroup
   ii) Some example
      \[ (ii) \ x, y \ in \ some \ G\text{-orbit} \ y \in G^x, \ (y \ not \ unique) \]

Theorem 5.1: Each left coset of $G^x$ in $G$ is in 1-1 correspondence with the orbits in the $G$-orbit of $x$:

\[ \varphi : \text{Orb}_G(x) \to G/G^x \]

For a 1-1 map \( \varphi \)

\[ \text{Suppose } y \ in \ G\text{-orbit } \varphi(x) \ then \exists \ g \in G \]

\[ g \cdot x = y \ . \ Define \ \varphi(y) = g \cdot G^x \]
Check $+$ is well-defined

$y = g' \cdot x \Rightarrow \exists h \in G^x \quad g' = g \cdot h \Rightarrow$

$g' \cdot x = g \cdot h \cdot x = g \cdot (g' \cdot x)$

Conversely, given coset $g \cdot G^x$ we may define

$\varphi^{-1}(g \cdot G^x) \equiv g \cdot x$

must check that this is well-defined. Since it

inverts $\varphi$, $\varphi^{-1}$ is 1-1.

Cor: If $G$ acts transitively on $X$ with isotropy

group $H$ then there is a 1-1 correspondence between $X$

and the set of cosets of $H$ in $G$. At least:

$X \cong G/H$

ex 5.5

$SO(1, IR)$ acts on $S^1$ transitively. Its isotropy

group at any point is $\cong SO(2, IR)$.

As a set $SO(3, IR)/SO(2, IR) \cong S^2$

ex 5.6a

$G = Z$ acts on $1R$ via $n \cdot x = x + n$. The

orbits are in 1-1 correspondence with $[0, 1]/0 \cong 1$. Not

that therefore the set of orbits is in 1-1 correspondence

with $1R/2\delta$ and can be identified with $x \in S^1$.

ex 5.6b

This generalizes: let $\Lambda^2$ be a 2-d lattice then

$\Lambda^2/\Lambda$ is isomorphic to a doughnut or

torus.
Def of a lattice: Let $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ be lin. indep. vectors. Defined

$$\Lambda \equiv \{ \sum_{i=1}^{n} \lambda_i \mathbf{e}_i \mid \lambda_i \in \mathbb{Z} \}.$$ 

Going under vector addition.

Lattice in $\mathbb{R}^2$

Atoms in a crystal form a lattice

Quadratic forms:

Def: A quad. form on a vect. $V$ is a bilinear form

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F} \subset \mathbb{R}, \mathbb{C}$$

Also defines form on other obj. objects, like lattices

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

To any lattice, associate a quad. form from the matrix of inner-products of the vectors $G_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle$ called the Gram-Matrix. Real quadratic form
Classification of lattices is connected to classification of equivalence classes,
\[ G \rightarrow S \text{ GL}(n, \mathbb{R}) \]
where \( S \) is an invertible integral matrix:
\[ S \in \text{ GL}(n, \mathbb{Z}) \]

\[ (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \rightarrow (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) \text{ not equivalent over } \mathbb{Z}! \]
\[ (\text{are equivalent over } \mathbb{R}) \]
\[ \text{ det } S = \pm 1 \text{ is even or odd} \]
\[ \text{ det } S = \text{ even} \]

Back to our example.
\[ \mathbb{Z}^n \subset \mathbb{R}^n \text{ for n-dim. lattice is an n-dim. forms.} \]

The Brillouin Zone is solid straight is a 3-d forms.

Some group element may be written \( \{ R \} + t \)
\[ x' = R x + t \]
\[ \{ R + t \}^{-1} = \{ R \}^{-1} - R^{-1} t \]
\[ t = 0 \text{ R form a group -- point group of the crystal} \]
\[ \{ E \} + t \text{ pure translations form a subgroup. (labels} \]
\[ \{ \mathbf{e}_n = n \mathbf{a}_1 + \cdots + n \mathbf{a}_n \} \]

3 primitive translations define periodic of the lattice.

For finite crystals, use periodic b.c.

Bloch's Thm: consider just translations (suppose no more symmetry) Come back after reg. theory...
6. Group actions on fn. spaces

General principle: Suppose $G$ acts on $X$ as a trans. group. Then $G$ also acts on any fn. space $F: \{ f: X \to Y \}$ where $f \to g \cdot f$ and $g \cdot f$ is defined by $(g \cdot f)(x) := f(g^{-1}x) \quad \forall x \in X.$

Often how group actions appear in general functions being examined.

$\mathcal{G} = \text{Sym}(\mathbb{R})$ acts on $\mathbb{R}^n$.

$R^+(\mathbb{R})$ acts on $\mathbb{R}^n$.

By our basic principle $G = \text{SO}(2, \mathbb{R})$ also acts on

$F := \{ f: \mathbb{R}^n \to \mathbb{R} \}_{f \in F}(R \cdot f)$

where $(R \cdot f)$ is defined by the values

$(R \cdot f)(x_1, x_n) \equiv f(Rx_1, x_n)$ and the

association $f \to R \cdot f$ is an action of $G$ on $F$.

$G$-orbits in space $f$ for are collections of

for which can be rotated into one-another.

$G$-orbits of one $f$ are invariant for

$f(x_1, x_n) = f(x)$
6.1 Symmetric Tors.

Example 1. Going a long on the square is the
theory of symmetric tors.

Form, going a long on \( \mathbb{R}^n \)

\[
6: (x_1, \ldots, x_n) = (x_{(1)}, \ldots, x_{(n)})
\]

\( \therefore \) \( S_n \) acts on \( \mathbb{R}^n \). Show for more

Symm. for:

Elementary Symm. Join:

\[
E(t) = \sum_{i=0}^{n} a_r t^r = \frac{n!}{i!} \left(1 + x_i + \cdots \right)
\]

\( a_0 = 1 \)

\( a_1 = x_1 + \cdots + x_n \)

\( a_r = x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n \)

\( a_r = x_1 \cdots x_r \)

ii. \( i = 1 \) to \( r \)

Theorem 1: Any Symm. poly. in \( n \) var. can be
uniquely written as a poly. in the first \( a_r(x) \).

Another way is via power sum for \( P_r(x) = \frac{x^r}{r!} \).

Theorem 2: Any Symm. poly. in \( n \) var. can be
uniquely written as a poly. in \( P_r(x) \).

Write \( P_r \) in term. of \( a_r \) and vice versa.
Newton's identities:

\[
\sum_{r=0}^{n} (-1)^{n-r} P_r a_{n-r}
\]
\[ a_1 = e_1 \]
\[ a_2 = \frac{e_1}{2} (e_2 - e_1) \]
\[ a_3 = \frac{1}{3!} e_3 - \frac{1}{5!} e_1 e_3 + \frac{1}{7!} e_3 \]


- 7 examples of symmetric functions in physics
  1) wavefn. of bosonic particles
  2) inv. fn. of matrices
  3) total wavefn in integrable systems
  Laughlin wavefn of QH etc.
Matrix Groups

7. Introduction

Symmetry of 2 flavors \( \rightarrow \) discrete

Survey groups with parameters which appear frequently in physical problems, and describe some of their global properties.

\[ \mathbb{Z}/N\mathbb{Z} = \{ a \in \mathbb{Z} \mid k = 0, \ldots, N-1 \} \]

\[ \text{centre} \quad SO(2,\mathbb{R}) \approx \{ e^0 \mid \theta \in \mathbb{R} \} \quad \cong \mathbb{R} \]

Discrete set of points

Circle also a manifold

Groups which are also manifolds are called Lie groups (cont. parameters). \( SO(3) \), \( GL(n) \), \( SL(n) \) are examples of Lie groups.

2. Definition

Definition 2.1: A Lie group \( G \) is a group which is also a manifold, \( \ast \) group multiplication \& inversion are differentiable maps.

\[ \mu : G \times G \to G \quad \mu(g_1, g_2) = g_1 g_2 \]
\[ \iota : G \to G \quad \iota(g) = g^{-1} \]

In local coordinates, we choose \( (x^1, \ldots, x^n) \in \mathbb{R}^n \) and parameterize \( g(x^1, \ldots, x^n) \)
\( g(x_1, \ldots, x^n) \cdot g(x'_1, \ldots, x'_n) = g(z_1, \ldots, z^n) \)

\( z^i = z^i(x, x', z) \) — differentiable functions.

Similarly for the inverse.

\[ w(1) \text{ coordinate } \theta \]
\[ g(\theta) \cdot g(\theta') = g(\theta'') \]
\[ a''(\theta, \theta') = a + o' \text{ analytic!} \]

3. General Linear Groups

\[ \text{GL}(n, \mathbb{R}) = \{ A : \det A \neq 0 \} \]
\[ \text{M}_n(\mathbb{R}) = \{ n \times n \text{ matrices } \rightarrow \mathbb{R}^n \text{ coordinate } \]
\[ \text{matrix elements } a_{ij} \]

\[ \text{GL}(n, \mathbb{R}) \text{ is a submanifold which is the complement of } \det A = 0 \text{ so we can use } a_{ij} \text{ coordinates.}\]
\[ a'_{ij}(u, v) = \sum_{k=1}^n a_{ik}(u) k_{jk} \text{ analytic!} \]
\[ (u^{-1})_{ij} = \frac{1}{(\det u)^{ij}} \text{ also analytic.} \]

Same for \( \text{GL}(n, \mathbb{C}) = \{ A : \det A \neq 0 \} \)

What about global structure?

Use the fact \( \det : \text{M}_n(\mathbb{R}) \rightarrow \mathbb{R} \) is continuous
So \( \det^{-1}(0) \) is a closed submanifold of \( \mathbb{R}^n \).
Thm: \( GL(n, \mathbb{R}) \) has 2 components. \( GL(n, \mathbb{C}) \) has one component.

The set of the \( \det \): studies isomorphism of the subelev
\( \det A = 0 \) in \( GL(n, \mathbb{R}) \) sent dimension 1
\( \det > 0 \) or \( \det < 0 \). Can't go from one
to another smoothly.

In \( GL(n, \mathbb{C}) \) \( \det = 0 \) is codim. 2 (real cod.)
so you can deform.

3.1 Special linear groups
\[ \mathfrak{sl}(n, \mathbb{R}) = \{ A \in GL(n, \mathbb{R}) \mid \det A = 1 \} \]
\( \dim \mathfrak{sl}(n, \mathbb{R}) = n^2 - 1 \) locally \( \mathbb{R}^{n^2-1} \)

Global properties:
- one component (connected!)
- non-connected \( \mathfrak{gl}(n, \mathbb{R}) \)

\( \exp \) means finite volume.

4. Orthogonal groups
\[ O(n, \mathbb{R}) = \{ A \in GL(n, \mathbb{R}) \mid A A^T = 1 \} \]
\[ SO(n, \mathbb{R}) = \{ A \in O(n, \mathbb{R}) \mid \det A = 1 \} \]
in terms of matrix elements, définires \( A_{ij} \) and
\[ \sum_{j} A_{ij} A_{kj} = \delta_{ik} \quad 1 \leq i, k \leq n \]
are quadratic définires defining submanifold of \( \mathbb{R}^{n^2} \).
Example of a compact manifold. Sum \( (x) \)
on \( i = 2 \) to get
\[
(i,j) \quad (a_{ij})^2 = n
\]
shows that \( o(n) \) is a submanifold of the sphere
of radius \( \sqrt{n} \) in \( \mathbb{R}^n \). \( o(n) \) is closed set.
A convergent set of orthogonal matrices converges to an
orthogonal matrix. Hence \( o(n, \mathbb{R}) \) is cpt.

4.1

Lorentz group and Poincaré group

Lorentz metric in \( D+1 \) dim. \( (x^0, \ldots, x^D) \) is
defined by
\[
x \cdot x = -(x^0)^2 + (x^1)^2 + \cdots + (x^D)^2
\]
\[
= x^\mu \eta_{\mu \nu} x^\nu
\]
\[
x \cdot y = -(x^0)y_0 + x^1y_1 + \cdots (x^D)y_D
\]
\[
= x^\mu \eta_{\mu \nu} y^\nu
\]
\[
\eta = \begin{pmatrix}
 -1 & 0 & 0 & \cdots \\
 0 & 1 & \cdots & 0 \\
 \vdots & \ddots & \ddots & \vdots \\
 0 & \cdots & 1 & -1
\end{pmatrix}
\]
The Lorentz group \( o(1, D) \) is the set of lin.
transform preserving \( x \cdot x \). Equivalently, it is the
set of \( (D+1) \times (D+1) \) real matrices \( A \in \mathbb{R}^{D+1} \)
\[
A \eta = \eta A
\]
The Lorentz group in \( D+1 \) dim. is the group \( o(1, 1) \)
described above.

As in Euclidean space, one can show that
\( \Theta(1, D) \equiv \text{Isometries of Lorentz metric} \) is just
the group \( T \times o(1, D) \).
The group of isom. of the Lorentz metric is called
the Poincaré group.
4.1.1. Pseudo-orthogonal groups

Another way to define $O(n, \mathbb{R})$ is that it is the set of linear transformations $\mathbb{R}^n \rightarrow \mathbb{R}^n \quad x \rightarrow x' = A \cdot x$ preserving the quadratic form

$$x_1^2 + \ldots + x_n^2$$

More generally, we can consider the real quadratic form $q + 1$ of signature $(p, q)$ on $\mathbb{R}^n$

$$(x, x)_{p, q} = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$$

where $p + q = n$. The set of linear transformations $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ preserving $(x, x)_{p, q}$ defines the group $O(p, q, \mathbb{R})$.

As a matrix group, we have $A \cdot \gamma = \gamma$

$$\gamma = \{ (-1)^p, (-1)^q \}$$

(classical mechanics)

4.2. Connected components

Orthogonal groups have an important property: non-trivial components.

As a set $O(2) = SO(2) \cup \{ -I \}$

As a group $O(2) = Z \times \text{SO}(2)$
How about \( o(1,1) \)? 4 connected components.

\[
o(1,1) = S_0(1,1) \cup P \cdot S_0(1,1) \cup T \cdot S_0(1,1) \cup P \cdot T \cdot S_0(1,1)
\]

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad P \cdot T = -1
\]

As a manifold, a disjoint union of 4 copies of \( \mathbb{R} \).

As a group:

\[
o(1,1) = (2\pi \times 2\pi) \times S_0(1,1)
\]

Component containing identity

Comments:

Similar for all orthog. groups:

\[
o(n) = S_0(n) \cup P \cdot S_0(n)
\]

\[
o(3,1) = S_0(3,1) \cup P \cdot S_0(3,1) \cup T \cdot S_0(3,1) \cup P \cdot T \cdot S_0(3,1)
\]

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2\pi \times 2\pi) \times S_0(3,1)
\]

Physical interpretation:

- \( P \), \( T \) correspond to spatial parity inversion and time inversion, respectively.

\[
P: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad P \cdot T = -1
\]

Fact that components are disconnected when not to have physical theories which use \( S_0(1,1) \) or \( S_0(3,1) \) invariant but not \( P \) or \( T \) invariant.

Work interactions which give \( \beta \)-decay are not parity invariant. Interactions of \( K \)-mesons are not even \( T \)-invariant!
5. Unitary Groups

\[ U(n) := \{ A \in GL(n, \mathbb{C}) \mid AA^* = I \} \]

Special unitary groups \( SU(n) := \{ A \in U(n) \mid \det A = 1 \} \)

Global properties: \( U(n) \), \( SU(n) \) are connected

Just like the orthogonal case, there is a group \( U(p, q) \) preserving Hamiltonian forms

\[ U(p, q) := \{ \eta \mid \eta \eta^* = \eta \} \]

\[ \eta = \{ (-1)^p, (+1)^q \} \]

6. Symplectic Groups

Appears in canonical quantization of Hamiltonian systems (electric/magnetic dualities)

Deeply connected with quaternions.

\( Sp(2n, \mathbb{H}) \) (sometimes called \( Sp(n) \) — rank)

Matrix version \( \mathbb{F}_1 \)

\[ J = \begin{pmatrix} 0 & I_{2n} \\ -I_{2n} & 0 \end{pmatrix} \]

\[ J^2 = -1 \text{ for } n \in \mathbb{Z} \]

Def: A symplectic matrix is a matrix \( A \in \mathbb{F}_2 \)

\[ A^T J A = J \]
\[ S_{0}(2n, \mathbb{R}) = \left\{ A \in GL(2n, \mathbb{R}) \mid A^{T}JA = J \right\} \]

\[ S_{0}(2n, \mathbb{C}) = \left\{ A \in GL(2n, \mathbb{C}) \mid A^{T}JA = J \right\} \]

\[ \dim S_{0}(2n, \mathbb{R}) = n(2n+1) \]

\[ \dim S_{0}(2n, \mathbb{C}) = 2(2n^2 + n) \]

\[ \text{Quaternions ?} \quad \bar{q} = q_1 - q_2 i - q_3 j - q_4 k \]

\[ q_0 = |q|^2 \]

\[ uS_{0}(2n, \mathbb{R}) \] is group of unit \( n \times n \) matrices with quaternion entries.

\[ uS_{0}(2n, \mathbb{R}) \cong \{ \bar{q} \mid \bar{q} \in U(n) \} \]

natural definition.

\[ S_{0}(2n, \mathbb{R}) \] is non-cpt but the unitary

Complex form group \( U(n) \) \( \subseteq S(2n, \mathbb{C}) \)

Group is cpt of real dimension \( n(2n+1) \).

7. Classification of compact Lie groups

A Lie group is simple if there are no non-trivial normal Lie subgroups.

Classification of simple Lie groups (Killing-Cartan)
(1) \( A_n = SU(n+1) \), \( n \geq 1 \)
(2) \( B_n = SO(2n+1) \), \( n \geq 2 \)
(3) \( C_n = SU(2n) \), \( n \geq 3 \)
(4) \( D_n = SO(2n) \), \( n \geq 4 \)
(5) \( G_2 \) dim = 14
(6) \( F_4 \) dim = 52
(7) \( E_6 \) dim = 78
(8) \( E_7 \) dim = 133
(9) \( E_8 \) dim = 248

Every single Lie group can be associated with a unique compact real form.

\[ O(6, -6) \to O(n) \text{ "with rotation"} \]

Basic study single unit: \( e^{i\theta} = e^{i\phi} \), \( \phi \) and \( \theta \) are continuously analytically continued, \( \mathbb{R}^8 \) x real \( \mathbb{R}^2 \), \( e^{i\theta} \) is non copt.

Classification gives sequence of regular cases and some "exceptional cases."

(First current basic, \( U(n, \mathbb{H}) \))

8. \( SU(2) \) (most important in physics)

\[ g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = g \rightarrow |e^{i\theta}| = 1 \]

8.1

\[ |x|^2 + |y|^2 = 1 \]

8.2. 4 vector \( \mathbf{F} = (i\beta, i\alpha, i\beta, 0) \)

\[ \mathbf{v} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} \alpha \rightarrow \beta \\ \beta \rightarrow \alpha \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]
\[ g = x^m \theta_p \quad x^n x^m = 1 \]

5.5

is therefore \( S^3 \) is a manifold.

8.3

Furc angles

\[ \begin{align*}
\alpha &= \frac{1}{2} (\phi + \Psi) \\
\beta &= \frac{1}{2} (\phi - \Psi) \\
\epsilon &= \frac{1}{2} \sin \theta \\
\end{align*} \]

Equivalently

\[ g = \begin{pmatrix}
\cos \theta & i \sin \theta & 0 \\
-i \sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\cos \frac{\phi}{2} & i \sin \frac{\phi}{2} & 0 \\
-i \sin \frac{\phi}{2} & \cos \frac{\phi}{2} & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\cos \theta & i \sin \theta & 0 \\
-i \sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix} \]

Need \( \Psi = 2 \beta \) becomes \( (\phi, \Psi) \rightarrow (\phi, \Psi + \pi) \)

\( i \) gives different phases for \( \alpha, \beta \).

\( ii \)

Many Coord. Diagrams, Different convention

\[ \phi \rightarrow \phi + \pi \quad \Psi \rightarrow \Psi - \pi \]

\[ g = \begin{pmatrix}
\cos \theta & i \sin \theta & 0 \\
-i \sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\cos \frac{\phi}{2} & i \sin \frac{\phi}{2} & 0 \\
-i \sin \frac{\phi}{2} & \cos \frac{\phi}{2} & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\cos \theta & i \sin \theta & 0 \\
-i \sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix} \]

8.4

\( SL(2) \) and \( SL(2, \mathbb{C}) \)

\( i \)

\( SL(2, \mathbb{C}) \) matrix

\[ A = \begin{pmatrix}
\cos \theta & i \sin \theta \\
-i \sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix} \]

\( \lambda \geq 0 \quad \eta \in \mathbb{R} \)

ii) \( A \in SL(2, \mathbb{C}) \) matrix

\[ A = \begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix} \begin{pmatrix}
1 & x \\
x & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \]

\( a \in SU(2) \quad a \geq 0 \quad \mu \in \mathbb{C} \)
9. Rotation and Rovindas

Space and Spacetime symmetries in 3 and 4 dimensions

Lorentz transformations described by 2x2 matrix groups. Depends on signature.

\[
\begin{align*}
1R^2_{1,0} & : \text{SO}(2) = SU(2)/Z_2 \\
1R^2_{1,1} & : \text{SO}(1,2) = SL(2,\mathbb{R})/Z_2 \\
1R^4_{1,0} & : \text{SO}(4) = (SU(2) \times SU(2))/Z_2 \\
1R^3_{1,1} & : \text{SO}(1,3) = SL(2,\mathbb{C})/Z_2 \\
1R^2_{1,0} & : \text{SO}(1,2) = (SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/Z_2
\end{align*}
\]

9.1 \text{su}(2) and rotations in 2-space

\text{su}_2 \text{ is the Lie algebra of Hamilton 2x2 matrices}

\text{su}_2 \text{ is the Lie algebra of traceless matrices}.

Claim: \( R^3 \cong \mathbb{R}^2 \) as vector spaces

\[
\begin{align*}
\mathbf{x} & \mapsto \mathbf{m} = \begin{pmatrix} x_1 & -ix_2 \\ ix_2 & x_1 \end{pmatrix} \\
& = \begin{pmatrix} \cos \theta & -i \sin \theta \\ i \sin \theta & \cos \theta \end{pmatrix}
\end{align*}
\]

\[
\mathbf{m} \mathbf{x} \mathbf{m}^T = \begin{pmatrix} \cos^2 \theta + \sin^2 \theta & -i \sin \theta \cos \theta + i \sin \theta \cos \theta \\ i \sin \theta \cos \theta + i \sin \theta \cos \theta & \cos^2 \theta + \sin^2 \theta \end{pmatrix}
\]

\text{su}(2) acts on \( \mathbb{R}^2 \) as the transformation group

\[
\mathbf{m} \to A \mathbf{m} A^T \quad A \in \text{su}(2)
\]

preserves subgroups \( \mathbb{R}^2 \).

Check: \( (A \mathbf{m} A^T)^T = A \mathbf{m} A^T \) preserves \( \mathbb{R}^2 \)

\[
Tr(A \mathbf{m} A^T) = Tr(\mathbf{m}) = 0 \quad \text{preserves } \mathbb{R}^2
\]

Action in \( \mathbb{R}^2 \): \( \mathbf{m} \mathbf{A} \mathbf{m}^T \mathbf{A}^T = \lambda \mathbf{m} \mathbf{x} \mathbf{A}^T + \rho \mathbf{m} \mathbf{y} \mathbf{A}^T \)
\[
\Rightarrow \quad c \mapsto A m a^+ \quad \text{defines a lin. transf. of } \mathbb{R}^2
\]
\[
A \hat{z} = \hat{z}' \cdot \hat{z}
\]
\[
\hat{z}' = R(A) \hat{z}
\]
Note \( \det M \hat{z} = -\hat{z}' = \det A^+ M \hat{z} \cdot A \)
\[
= \det M \hat{z}' = -(\hat{z}')^2
\]
so \( R(A) \in O(2) \)

Check \( \Phi : \text{su}(3) \rightarrow R(A) \in O(2) \) is a homomorphism.

Define \( R : \text{su}(3) \rightarrow O(3) \)
\[
\begin{cases}
R(AB) \hat{z} = AB \hat{z} \cdot \hat{z} = (AB)^+ \\
R(\hat{z}) \hat{z} = \hat{z} = (R(\hat{z}) \hat{z}) A^+ \\
- (R(A) R(\hat{z}) \hat{z}) \hat{z}
\end{cases}
\]
i) \( \ker R = \{1, -1\} \approx 2\pi \)
ii) \( \text{Im } R = S_0(3) \)

9.2
\( \text{sl}(2, \mathbb{C}) \) \( \Delta \) Lorentz Transf. in \( 3+1 \)

3+1 space \( \text{in } \mathbb{R}^4 \) \( \text{as } 2 \times 2 \) matrix \( x \)
\( x \in \mathbb{R}^4 \Rightarrow M_x = \begin{pmatrix}
x^0 + x^3 \\
x^1 + i x^2
\end{pmatrix}
\]
\( A \in \text{sl}(2, \mathbb{C}) \rightarrow \quad A M_x A^+ \equiv M_{x'} \)
\( \det M_x = (\hat{z})^2 - (\hat{z}')^2 = (\hat{z}^2)^2 - (\hat{z}'^2)^2 \)
\( \Lambda(A) : x \mapsto x' \) is a Lorentz transf.

Homomorphism \( \Lambda : \text{sl}(2, \mathbb{C}) \rightarrow O(1, 3) \)
\( \ker \Lambda = \mathbb{Z}_2 \quad \text{dim } (\Lambda) = 5 \text{ at } (1, 3) \)
dim. \( \mathfrak{su}(2) \oplus \mathfrak{su}(2) = 6 \) (real Lie group) same as \( O(1,3) \) 
3 rotations \( + 3 \) boosts.

1.3 \( \mathfrak{su}(2) \times \mathfrak{su}(2) \) and rotations in 4 Euclid. dim.
\[
\begin{align*}
\mathfrak{su}(2) & = \begin{pmatrix} x_1 + ix_2 & ix_3 + x_4 \\ ix_3 - x_4 & x_1 - ix_2 \end{pmatrix} = x^m \tau_m \\
& = x^4 \tau_1 + x^5 \tau_2 \\
\text{untorsion} & = \mathbb{H} \\
\mathbb{H} & = \{ x^m \tau_m : x^m \in \mathbb{R} \} = \{ a \in M_2(\mathbb{C}) : a^* = a \} \\
\end{align*}
\]

Next Lie algebras!

Next problem set.

i) center of group

ii) Poisson bracket & symplectic group

iii) magnetic trans. group

iv) \( O(4) \sim \mathbb{R}^4 \oplus \mathbb{R}^4 \) alg. level