1. Let’s return to an old friend the dihedral group, in particular, $D_4$.

(i) How many conjugacy classes are there for $D_4$? How many irreducible representations? Is your result consistent with the order of the group?

(ii) Construct the character table for $D_4$ and the irreducible representations.

(iii) Suppose we have a Hamiltonian $H$ describing an electron moving in some crystal such that the symmetry group of the Hamiltonian is $D_4$. We perturb the crystal field by a perturbation $H'$ that is not invariant under $D_4$ but transforms in the largest irrep that you found above, call the irrep $\Gamma_{\text{big}}$. In general, we can always decompose the perturbation into pieces that transform according to different irreps. Let’s consider 2 component wavefunctions for the electron $\psi_k^i$ transforming in the $i^{th}$ irrep of $D_4$. When is the matrix element,

$$\langle \psi_{k'}^i | H' | \psi_k^i \rangle,$$

necessarily zero? Check that this is the case explicitly for at least 2 different choices of $(i, i')$.

(iv) The preceding observation is the basis for the use of group theory to determine selection rules. Let’s try to make a more general statement. Suppose we have a symmetry group $G$ under which $H$ is invariant. Perturb $H$ by some operator $H'$ which transforms in the $\Gamma_p$ irrep. We have 2 wavefunctions $\psi^i$ and $\psi^{i'}$ transforming in some representation $\Gamma_i$ and some other representation $\Gamma_{i'}$ of $G$ (not necessarily irreps). Under what conditions on $\Gamma_p, \Gamma_i$ and $\Gamma_{i'}$ is the matrix element,

$$\langle \psi^{i'} | H' | \psi^i \rangle,$$

possibly non-zero?

2. To get familiar with projection operators, let’s do an example. In lecture, we constructed the projection operator for the 2-dimensional irrep of $S_3$ for the 3-dimensional reducible representation. There are 2 other irreps which we called $1_+$ and $1_-$. Construct the projection operators for these 2 irreps and check that the image of the projection operator acting on

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

has the right dimension.

Actually, in lecture, we took a short cut and computed $P_2$ but not each $P^{ij}_2$. Fill in the missing steps and compute each projection operator $P^{ij}_2$ for the 2-dimensional irrep. Check that

$$\sum_i P^{ii}_2$$
gives the matrix we found in class.

**3.** Characters are enormously useful for quickly determining which representations appear in a product of wavefunctions etc. Let us define fusion coefficients in the following way. Consider,

$$T^\mu \otimes T^\nu = \oplus_\lambda N^\lambda_{\mu\nu} T^\lambda,$$

where each $T$ is an irrep. The $N^\lambda_{\mu\nu}$ tell us how many times $T^\lambda$ appears in the tensor product. These are the fusion coefficients, and they show up in a myriad of places including addition of angular momentum.

(i) Start by showing that $\chi_{T_1 \otimes T_2}(g) = \chi_{T_1}(g) \chi_{T_2}(g)$ so characters behave nicely under tensor products.

(ii) Using this observation, show that $N^\lambda_{\mu\nu} = \langle \chi^\lambda, \chi^\mu \chi^\nu \rangle$. Here is $\langle, \rangle$ is the inner product for characters.

(iii) We have used characters for reps of finite groups but we can also use characters for reps of Lie groups. Consider the spin $j$ representation of $SU(2)$. Parametrize the conjugacy classes of $SU(2)$ conveniently and compute the character $\chi_j$ as a function of this parameter in the spin $j$ representation (hint: use the fact that a unitary matrix can be diagonalized using unitary matrices).

(iv) Using the expression for the characters, $\chi_j$, show that

$$\chi_j \chi_{j'} = \sum_{j'' = |j - j'|}^{j+j'} \chi_{j''}.$$

(v) Lastly, show that this implies the usual rule for addition of angular momentum,

$$D^j \otimes D^{j'} = \oplus_{j'' = |j - j'|}^{j+j'} D^{j''}.$$

Check that the dimensions work correctly.