1. Prove part (iii) of Schur’s lemma. Namely, if $T(g)A = AT(g)$ for all $g$ then $A$ is proportional to the identity matrix.

2. Let $M$ be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Find the eigenvalues of $Ad(M)$ acting on $n \times n$ complex matrices.

3. Consider the Poincaré group in one dimension with coordinate $x$. Let’s also consider rescalings of the coordinate $x$ so we consider transformations,

$$f(x) = ax + b,$$

where $a, b$ are real numbers and $a$ is non-zero.

(i) Show that a matrix representation of this group is given by,

$$
\begin{pmatrix}
  a & b \\
  0 & 1
\end{pmatrix}.
$$

(ii) Show that this matrix rep is reducible but indecomposable. Is it unitary?

4. Coherent state representation: take the vector space $V^{2j}$ of polynomials $f(z)$ in $z$ of degree $\leq 2j$. Define an action of $SU(2)$ on this space by,

$$[T(g)f](z) = (-\beta^*z + \alpha^*)^n f\left(\frac{\alpha z + \beta}{-\beta^*z + \alpha^*}\right),$$

for

$$g^{-1} = \begin{pmatrix}
  \alpha & \beta \\
  -\beta^* & \alpha^*
\end{pmatrix}.$$

(i) Show that this is a representation of $SU(2)$.

(ii) Show that it is isomorphic to the spin-$j$ representation given in class.

(iii) Finally check that the representation is unitary with respect to the inner product,

$$\langle f, g \rangle = \frac{(2j + 1)!}{\pi} \int d^2z (1 + |z|^2)^{-2j-2} f(z)^* g(z).$$

5. Let’s learn something about fermions and related ideas. A bosonic creation operator, $a^\dagger$, satisfies:

$$[a, a^\dagger] = 1.$$ 

You can pick either $a$ or $a^\dagger$ to be a creation operator acting on a Fock vacuum $|0\rangle$ killed by all annihilation operators. In this convention,

$$a|0\rangle = 0.$$
The Hilbert space is built on this vacuum.

(i) Suppose we have \( M \) creation operators \( a_i^\dagger \) where \( i = 1, \ldots, M \) which commute with each other. Think of each of these operators as creating a species of boson i.e. phonons or photons, on the empty Fock vacuum. What does the operator \( \sum_i a_i^\dagger a_i \) measure? How big can it get?

(ii) A collection of fermionic creation operators, \( b_i^\dagger \), satisfy:

\[
\{ b_i, b_j^\dagger \} = \delta_{ij}, \quad \{ b_i^\dagger, b_j^\dagger \} = 0.
\]

Again, suppose we have \( M \) operators \( b_i^\dagger \). What is the dimension of the Hilbert space that you can build from these operators (start with a Fock vacuum killed by all \( b_i \))?

(iii) Consider the operator \( (-1)^{\sum_i b_i b_i^\dagger} \). What values can it take? Does it generate a symmetry for the Hamiltonian (any Hamiltonian containing these fermions)? If so, what is the symmetry group? What does \( \sum_i b_i^\dagger b_i \) measure? Is this operator necessarily part of the symmetry group of the Hamiltonian? If not, can you write down an interaction for some \( H \) as a counter-example?

(iv) Suppose we rewrite \( b = \psi_1 + i\psi_2 \) where \( \psi \) is real so \( \psi^\dagger = \psi \). What quantization condition does each \( \psi \) satisfy? Using this quantization law, let us turn to a more general setting: if we have three real fermions \( \psi_j \) where \( j = 1, 2, 3 \), can you find a matrix representation for each \( \psi_j \) that satisfies the quantization relations? Given \( M \) real fermions, what is the dimension of their Hilbert space? (Hint: there is a difference between \( M \) even and \( M \) odd.)

(v) Fermions bring us naturally to the notion of Clifford algebras. In \( D \) dimensions, consider a collection of vectors, \( e_i \), where \( i = 1, \ldots, D \). To each vector, we want to associate a Hermitian matrix which will be an element of the Clifford algebra. Let’s call this matrix \( \gamma_i \). We require that these matrices satisfy,

\[
\{ \gamma_i, \gamma_j \} = \delta_{ij}.
\]

What’s the smallest size for these matrices in \( D \) dimensions? (hint: Luke use the fermions!). How should the \( \gamma_i \) transform under rotations \( SO(D) \) for condition 3 to make sense? Can you explicitly construct matrices satisfying condition 3 for \( D = 3 \) and \( D = 4 \)? How about if we change from Euclidean space to Minkowski space where we impose the condition:

\[
\{ \gamma_i, \gamma_j \} = \eta_{ij}.
\]

Can you construct matrices for \( D = 3, 4 \) in this case?

These matrices are known as “gamma matrices” and make an appearance when we try to write Lagrangians for field theories with fermions. The reason should become clearer when we discuss spinor representations of groups.

**Extra credit:** in what dimensions can you construct \( \gamma \) matrices which are all real?