Problem Set 8
Physics 243
Due March 11

1. Let’s not forget about quantum circuits completely. Using the CNOT gate and the Hadamard gate, build a 3 qubit circuit that constructs the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle),$$

in analogy with the circuit we described to build Bell states.

2. Find the Schmidt decomposition of the following two qubit states:

$$\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle), \quad \frac{1}{2} (|00\rangle + |11\rangle + |01\rangle + |10\rangle), \quad \frac{1}{\sqrt{3}} (|00\rangle + |01\rangle + |10\rangle).$$

3. Prove that a state $|\psi\rangle$ of a composite system $AB$ is a product state iff it has Schmidt number 1. If the Schmidt number is greater than 1, the state is therefore entangled. Also prove that $|\psi\rangle$ is a product state iff the partial traces $\rho^A$ and $\rho^B$ are pure states.

4. Let’s explore some more properties of density matrices. A set $S$ is convex if for all points $x, y \in S$ the line segment joining the two points, given by

$$y\lambda + x(1-\lambda),$$

belongs to $S$ for $0 \leq \lambda \leq 1$.

(i) For a finite-dimensional Hilbert space of dimension $N$, show that the density operators form a convex subset of the space of Hermitian operators.

(ii) We introduced density matrices as a way to understand classical ensembles of quantum pure states, which are generally mixed states. In general, a density matrix can be expressed as a sum of other density matrices in many ways. The exception to this statement is a pure state. Show that the density matrix for a pure state cannot be written as a sum of two other density matrices.

(iii) Points in a convex set that cannot be expressed as a sum of other points are called extremal points. Pure states are extremal points in the set of density matrices. Show that only pure states are extremal points.

(iv) For a single qubit system, show that the density matrix for an arbitrary mixed state can be written in the form:

$$\rho = \frac{1}{2} (I + \vec{r} \cdot \vec{\sigma}),$$
where the real 3-dimensional vector $\vec{r}$ satisfies $|\vec{r}|^2 \leq 1$. This is called the Bloch vector for $\rho$. Show that the state is pure iff $|\vec{r}|^2 = 1$. Show that this definition of the Bloch vector coincides with the definition we gave at the beginning of the course.

(v) Systems that consist of two subsystems are called “bipartite systems.” Similarly, systems with 3 or more subsystems are called tripartite or multipartite systems. We originally studied entanglement for a bipartite system of 2 qubits. See, for example, question 5 of problem set 4. Most of our discussion has been for that case. Consider an arbitrary tripartite pure state $|\psi\rangle_{ABC}$ in $H_A \otimes H_B \otimes H_C$. Does a Schmidt decomposition exist? Namely, are there orthonormal bases $\{|i_A\rangle\}$, $\{|i_B\rangle\}$, $\{|i_C\rangle\}$ for $H_A$, $H_B$ and $H_C$ respectively so that

$$|\psi\rangle_{ABC} = \sum_i \lambda_i |i_A\rangle |i_B\rangle |i_C\rangle.$$

(vi) Lastly, we can ask about density matrices for multipartite states. What characterizes states without entanglement? By extending the discussion to density matrices, we can formulate a notion of entanglement for mixed states. Suppose we have a multipartite system with Hilbert space $H_1 \otimes H_2 \ldots \otimes H_n$. A separable pure state $|\psi\rangle$ is one that takes the form:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \ldots \otimes |\psi_n\rangle.$$

This clearly has no entanglement. Let’s generalize this to density matrices. A density matrix is separable if it can be written in the form,

$$\rho = \sum_k p_k \rho_1^k \otimes \rho_2^k \ldots \otimes \rho_n^k,$$

with $\sum_k p_k = 1$. Here each $\rho_i^k$ is a pure state of subsystem $i$.

It is generally very hard to determine whether a given $\rho$ is separable (in fact, it is NP-hard). So let’s examine a more tractable example of a bipartite system. Consider the density matrix

$$\rho = p |\beta_{11}\rangle \langle \beta_{11}| + (1-p) \frac{I_4}{4},$$

with $0 \leq p \leq 1$. For what values of $p$ is the state $\rho$ separable?