Problem Set 2

Physics 243

Due January 28

1. Now I know you love the Pauli matrices. Let’s connect last week’s exercises to some of the physics of relativity. Consider the operator

\[ P = p^0 \sigma^0 + \sum_i p^i \sigma^i \]

where the 4-momentum \( p = (p^0, p^1, p^2, p^3) \) is real. What is \( \det(P) \)? What is the physical meaning of \( \det(P) \)? What are the eigenvalues \( P \)? If you conjugate \( P \to U^\dagger PU \) by a unitary matrix \( U \), what happens to \( \det(P) \)? This is a nice way of mapping 4-momenta into an operator which can act on two component wavefunctions. We will later explore how Lorentz transformations, like boosts, acting on \( P \) are realized in this representation.

2. Let’s continue with an exercise involving Gram-Schmidt.

(i) Take the following three vectors in \( \mathbb{R}^3 \),

\[ |e_1\rangle = (1, 1, 1), \quad |e_2\rangle = (1, 1, 0), \quad |e_3\rangle = (1, 0, 1). \]

Construct an orthonormal basis using the Gram-Schmidt procedure starting with \( |e_1\rangle \).

(ii) Now let’s return to normalizable functions on the interval \([-1, 1]\). Consider the inner product

\[ \langle f | g \rangle = \int_{-1}^1 f^*(x)g(x)dx \]

on this space. We can expand these functions in the basis \( \{1, x, x^2, x^3, \ldots\} \) but this is not an orthonormal basis! We can build an orthonormal basis using Gram-Schmidt. For example, we can choose

\[ \phi_0 = \frac{1}{\sqrt{2}}, \quad \phi_1 = \sqrt{\frac{3}{2}} x, \]

for the first two basis elements. Please construct \( \phi_2 \) and \( \phi_3 \). These orthogonal polynomials are called Legendre polynomials.

(iii) Next let us modify the inner product space by introducing a weight function \( w(x) = e^{-x^2} \) and considering functions from \(( -\infty, \infty )\) rather than the interval. The new inner product is

\[ \langle f | g \rangle = \int_{-\infty}^{\infty} f^*(x)g(x)w(x)dx. \]

Please check that this defines an inner product on functions from \(( -\infty, \infty )\).

For functions to be normalizable with respect to this inner product, they must grow sufficiently slowly as \( |x| \to \infty \) to ensure that their norm is finite. Is \( x^2 \) a normalizable function with respect to this inner product? Compute the norm. Again starting with the basis
{1, x, x^2, x^3, ...}, construct the first 3 elements of an orthonormal basis i.e., \{\phi_0, \phi_1, \phi_2\}. These orthogonal polynomials are called Hermite polynomials and they play an important role in the study of simple harmonic oscillators.

3. Fun with quaternions. Let us define the quaternions in terms of real 4 \times 4 matrices \(I, J, K\) where

\[
I = \begin{bmatrix} 0 & -\sigma^1 \\ -\sigma^1 & 0 \end{bmatrix} \quad J = \begin{bmatrix} -i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{bmatrix} \quad K = \begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix}.
\]

(i) Show that \(I, J, K\) satisfy the quaternion algebra:

\[
I^2 = J^2 = K^2 = -1, \quad IJ = K.
\]

(ii) Show that unlike the case of complex numbers where multiplication is commutative, the quaternion algebra is non-commutative by evaluating the commutator:

\[
[q, q'] = qq' - q'q,
\]

where \(q\) is a quaternion \(q = 1q_1 + q_2I + q_3J + q_4K\).

(iii) Define the conjugate of \(q\) to be \(\bar{q}\) where

\[
\bar{q} = 1q_1 - q_2I - q_3J - q_4K.
\]

Note the analogy with complex numbers \(z\) and \(\bar{z}\). Show that \(q\bar{q}\) is positive. The set of unit quaternions \(q\) satisfying,

\[
q\bar{q} = 1,
\]

are particular interesting. This set provides a nice example of a group, which we defined in lecture. Namely, a group \(G\) is a set with a multiplication rule such that,

- \(a, b \in G\) \quad a \cdot b \in G,
- \((a \cdot b) \cdot c = a \cdot (b \cdot c),\)
- \(a \cdot e = e \cdot a = a\) so \(e\) is an identity element,
- For any \(a \in G\), there exists an \(a^{-1} \in G\) such that \(a \cdot a^{-1} = a^{-1} \cdot a = e.\)

The multiplication rule needs to be specified. It might be addition for the case of integers; please show that the integers form a group under the operation of addition. Or it could be the usual multiplication, or something else. Do the non-zero integers form a group under multiplication? How about the non-zero real numbers, \(\mathbb{R}\), under multiplication?

Groups appear in many physical contexts, but most commonly as symmetries of physical systems. For example, the translation symmetries of a crystal lattice form a group. Show that the unit quaternions form a group under multiplication.
(iv) Let us connect the unit quaternions to some more familiar concepts. A unitary matrix is an \( n \times n \) matrix \( U \) with complex entries satisfying,

\[
U^{-1} = U^\dagger.
\]

Consider a \( 2 \times 2 \) matrix,

\[
U = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}.
\]

If \( U \) is unitary with determinant one, what conditions must the \( z_i \) satisfy? This is just a variant of the problem from last week. Using these conditions, eliminate two of the \( z_i \) (say \( z_3 \) and \( z_4 \)) and list the condition on the remaining two variables. Show that all unitary matrices \( U \) with determinant one form a group under multiplication. The group of unitary determinant one matrices is the group \( SU(2) \) of special unitary matrices. For \( n \times n \) matrices, we get the group \( SU(n) \).

We can now connect with our earlier discussion. Let us take

\[
z_1 = q_1 + i q_2, \quad z_2 = q_3 + i q_4.
\]

Form a quaternion \( q \) by taking the combination \( q = q_1 + q_2 I + q_3 J + q_4 K \). What condition must \( q \) obey so that the matrix \( U \) formed from \( z_1, z_2 \) is an element of \( SU(2) \)?