Physics 243 Problem Set # 5 Solutions
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Problem 1

Alice needs to encode two classical bits, which can be in one of four states - 00, 01, 10, 11, into four distinct, orthonormal quantum states using unitary operators only on one qubit. Given that Alice and Bob start with a Bell pair, she can certainly do this because all four Bell states, which are orthonormal, are unitarily equivalent to each other under unitary transformations acting only on one bit. Here’s the protocol: Alice encodes the classical bits into Bell states by acting with unitaries on her qubit as expressed in the following table.

\[
\begin{align*}
00 & : |\beta_{00}\rangle \xrightarrow{I} |\beta_{00}\rangle \\
01 & : |\beta_{00}\rangle \xrightarrow{Z} |\beta_{01}\rangle \\
10 & : |\beta_{00}\rangle \xrightarrow{X} |\beta_{10}\rangle \\
11 & : |\beta_{00}\rangle \xrightarrow{ZX=iY} |\beta_{11}\rangle
\end{align*}
\]

She then sends her qubit to Bob, who can perform the inverse operation of constructing a Bell state to get the state of the classical bits.

Problem 2

The left circuit is the following set of operations:

\[(H \otimes H)\text{CNOT}(H \otimes H)\]

The matrix representation of \(H \otimes H\) is

\[
H \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} H & H \\ H & -H \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},
\]

while the matrix representation of the CNOT gate is given in the previous problem set. Multiplying out the matrices yield

\[
(H \otimes H)\text{CNOT}(H \otimes H) = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}
\]

The circuit on the right is a CNOT gate with the second qubit being the control bit. So its operation on the basis elements in the following way

\[
|00\rangle \mapsto |00\rangle, \quad |01\rangle \mapsto |11\rangle, \quad |10\rangle \mapsto |10\rangle, \quad |11\rangle \mapsto |01\rangle
\]

We can see that the matrix representation of this linear map is exactly what was obtained in the calculation.
Problem 3

To demonstrate this, let us first look at the result of the circuit. We start of with the state \( |0\rangle \otimes |\psi_{in}\rangle \). The first step in the circuit is a Haramard gate so this gives

\[
|0\rangle \otimes |\psi_{in}\rangle \mapsto \frac{1}{\sqrt{2}} (|0\rangle \otimes |\psi_{in}\rangle + |1\rangle \otimes |\psi_{in}\rangle)
\]

The second step is a \( CU \) gate so it only applies \( U \) to the second state if the control qubit is \( |1\rangle \). So this gives

\[
\frac{1}{\sqrt{2}} (|0\rangle \otimes |\psi_{in}\rangle + |1\rangle \otimes |\psi_{in}\rangle) \mapsto \frac{1}{\sqrt{2}} (|0\rangle \otimes |\psi_{in}\rangle + |1\rangle \otimes U|\psi_{in}\rangle)
\]

The last step is another Haramard gate so this gives

\[
\frac{1}{\sqrt{2}} (|0\rangle \otimes |\psi_{in}\rangle + |1\rangle \otimes U|\psi_{in}\rangle) \mapsto \frac{1}{2} (|0\rangle \otimes |\psi_{in}\rangle + |1\rangle \otimes |\psi_{in}\rangle + |0\rangle \otimes U|\psi_{in}\rangle - |1\rangle \otimes U|\psi_{in}\rangle)
\]

This can be rewritten as

\[
\frac{1}{2} (|0\rangle \otimes (1 + U)|\psi_{in}\rangle + |1\rangle \otimes (1 - U)|\psi_{in}\rangle)
\]

Since \( U \) is also Hermitian, we can write \( |\psi_{in}\rangle \) as a linear combination of the eigenvectors of \( U \). Let \( |\pm\rangle \) be the eigenstates with eigenvalue \( \pm 1 \), then

\[
|\psi_{in}\rangle = c_+ |+\rangle + c_- |\rangle
\]

Plugging this into the output of the circuit gives

\[
c_+ |0\rangle \otimes |+\rangle + c_- |1\rangle \otimes |\rangle
\]

So measuring the state of the first qubit will indicate which of the two eigenvalues of \( U \) the second qubit will give.
Problem 4

The Toffoli gate will flip the state of $c$ only if both $a$ and $b$ are in the state $|1\rangle$ and will otherwise leave the state unchanged. So the input/output table is

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>000\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>001\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>010\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>011\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>100\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>101\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>110\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>111\rangle$</td>
</tr>
</tbody>
</table>

To see if this can be done, let’s look at the input/output of this circuit. If $a$ were in the state $|0\rangle$, the states $b$ and $c$ would be unchanged since $V$ is unitary. If $a$ were in the state $|1\rangle$, the output of $b$ would be unchanged due to the two consecutive NOT gates. If $b$ were in the $|0\rangle$ state, then $c$ would be acted on by $V^\dagger$ and then $V$ leaving $c$ unchanged. If $b$ were in the $|1\rangle$ state, then $c$ would be acted on by $V$ and then $V$. Hence, (2) is equivalent to the following circuit:

![Circuit Diagram](image-url)