Problem 1 In the basis of the eigenvectors of $\sigma_3$, we have

\[ |\alpha^{(+)}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |\alpha^{(-)}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad |\beta^{(+)}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\beta^{(-)}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

Hence,

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \]

This is the Hadamard gate. To check that $U$ is unitary,

\[ U^\dagger U = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^2 = 1, \]

as desired.

Problem 2 For a finite dimensional Hilbert space with orthonormal basis \{\ket{e_i}\}, we have

\[ \text{tr}(M\ket{\psi}\bra{\psi}) = \sum_i \bra{e_i} M \ket{\psi} \bra{\psi} e_i \]

\[ = \sum_i \bra{\psi} e_i \bra{e_i} M \ket{\psi}, \]

But $\sum_i \ket{e_i}\bra{e_i}$ is just the identity operator. Hence,

\[ \text{tr}(M\ket{\psi}\bra{\psi}) = \bra{\psi} M \ket{\psi} \]
Problem 3
The first circuit is the operator \( H \otimes 1 \), where \( H \) is the Hadamard gate. We can evaluate the matrix elements by acting it on the basis vectors. We see that
\[
(H \otimes 1)|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle), \quad (H \otimes 1)|01\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)
\]
\[
(H \otimes 1)|10\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle), \quad (H \otimes 1)|11\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)
\]
Hence, the matrix representation is
\[
H \otimes 1 = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\]
The second circuit is the operator \( 1 \otimes H \)
Acting this operator on our basis vectors yield
\[
(1 \otimes H)|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle), \quad (1 \otimes H)|01\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle)
\]
\[
(1 \otimes H)|10\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle), \quad (1 \otimes H)|11\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)
\]
Hence, the matrix representation is
\[
1 \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & -1
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
H & 0 \\
0 & H
\end{pmatrix}
\]
In problem 1, we’ve showed that the Hadamard gate is unitary. Hence, both operators are unitary. You can also verify this by performing matrix multiplication with the blocks.

Problem 4
Taking the hermitian conjugate of the given expression gives
\[
T^{\dagger} = i \log U^{\dagger}
\]
\[
= i \log U^{-1}
\]
\[
= - i \log U
\]
\[
= T,
\]
thus verifying that the given expression is Hermitian. By taking the exponential of both sides, we get
\[
e^{iT} = e^{\log U} = U
\]
Problem 5

Suppose such a state $|\psi\rangle$ exists. We can express $|\psi_i\rangle$ in the eigenbasis of $\sigma_3$. This gives

$$|\psi\rangle = (\alpha_1|0\rangle + \alpha_2|1\rangle) \otimes (\beta_1|0\rangle + \beta_2|1\rangle)$$
$$= \alpha_1\beta_1|00\rangle + \alpha_1\beta_2|01\rangle + \alpha_2\beta_1|10\rangle + \alpha_2\beta_2|11\rangle$$

Equating this to the EPR pair yields:

$$\alpha_1\beta_2 = 0, \quad \alpha_2\beta_1 = 0, \quad \alpha_1\beta_1 = \frac{1}{\sqrt{2}}, \quad \alpha_2\beta_2 = \frac{1}{\sqrt{2}}$$

The solution to the first equation is

$$\alpha_1 = 0 \text{ or } \beta_2 = 0$$

But either solution is inconsistent with the third and fourth solution. Hence, we can conclude that no such state exists.