An Introduction to Neutrino Oscillation: A Quantum Mechanical Perspective

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Abstract

Neutrinos are one of the most elusive of the Standard Model particles known to physicists today. Despite the fact that crucial characteristics of these leptons remain largely a mystery, studying them can hold the key to fundamental insights for the field of particle physics and for our understanding of the universe as a whole. In order to learn why there is so much discussion within the physics community about these mysterious particles, its important to start at the beginning and connect one of their most important characteristics - that of oscillations between their types - with the subject of quantum mechanics. In fact, to engage with the subject of neutrino physics in general, beyond the surface-level qualitative approach, requires the quantum mechanical framework based heavily in linear algebra. Therefore, this paper will give an introduction to neutrino physics, specifically to their oscillations, through connecting the quantum mechanical formalism to current research within this area of particle physics. I plan to first introduce the subject of particle physics as a whole, review the necessary formalism, and then explain how this field has evolved both experimentally and theoretically. Finally, I will discuss what the future of it holds for understanding the nature of our universe.
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1 Introduction

1.1 The Particle Physics Background

In many ways, the story of the development of particle physics, of which neutrinos play an integral part, came directly from the groundwork laid out by the fathers of quantum physics - Planck, Heisenberg, Einstein, and Schrödinger. It was through their interpretation of a quantum regime that paved the way for scientists to probe the bounds of the then current understanding of subatomic particles.

By the mid 1960s, many of the subatomic particles that would eventually form what particle physicists call the Standard Model were already being discovered in accelerators. The interest in them grew from how it was learned through the intersection of physics and astronomy, that subatomic particles were crucial to the understanding of the formation of the universe [1]. On the theoretical side, quantum mechanics was the tool used to compliment the research done by experiments to further the work into establishing a compilation, similar to that of the Periodic Table, that characterized the particles known to be present in our universe.
Figure 1: The above shows the Standard Model of particles which includes the four forces - bosons - as well as the quarks and lepton elementary particles. (Source: https://www.quantumdiaries.org/2014/03/14/the-standard-model-a-beautiful-but-flawed-theory/)

Therefore, by the time that the Higgs Boson was formally discovered in 2012, the framework of the Standard Model was very well established [2]. It consisted, as shown in figure 1, of three known generations and a collection of bosons or forces - the weak force (w and z bosons), strong force (gluon), electromagnetic force (photon), and the Higgs Boson [1]. The first category of particles in the model are the quarks, the particles found to compose neutrons and protons. The other main type of particle is that of a lepton, which does not interact with the strong force and of which the neutrino makes up three - each type or flavor corresponding to its charged counterpart of the electron, muon, and tau variety [1].

An important classification that I will remark upon here, is that neutrinos are not only leptons, but also a particle classification known as a fermion. This label is given to particles that are spin - $\frac{1}{2}$ particles. It means that they a constrained by the Pauli Exclusion Principle.

To date, there are three flavors of neutrinos known to particle physicists that in fact correspond to the three generations of leptons within the Standard Model - electron, muon, and tau. The important characteristic that sets these particles apart from other Standard Model particles is that they only interact with the weak force and gravity [1]. Thus, despite the fact that they are a highly studied topic
in particle physics, this idea of not interacting much with the matter around them makes these leptons the least understood of the Standard Model particles.

1.2 The Neutrino

The idea that a particle like the neutrino existed came out of a study in beta decay dating back to the 1930s. It was discovered that a characteristic of what are called two body decays - a single nucleus decays into two products - was that the energy of the electron (assuming it to be one of the products) under the conservation of energy was found to be:

\[ E = \left( \frac{m_a^2 - m_b^2 + m_e^2}{2m_a} \right) c^2 \]

where \( m_a, m_b, m_e \) are the initial nucleus mass, one of the product masses, and the electron mass respectively [1]. However, this equation revealed that the electron’s energy would become fixed once the three masses were found, and in particular, it could only tell what the maximum energy would be emitted. Because this was such an unusual result, it led to scientists like Pauli and Fermi to propose that a new particle that was extremely light could account for the missing energy that seemed to have disappeared from this process.

Therefore, coined by Enrico Fermi in particular, the concept of a neutrino - little neutral one - was established. Nowadays, we can rewrite the beta decay process to explicitly include the neutrino:

\[ n \rightarrow p^+ e^- + \bar{\nu} \]

with the form of the neutrino given as \( \bar{\nu} \) indicating an antineutrino [1].

First came the discovery of the electron neutrino variety in 1950s by Reines and Cowan [3]. Then in 1962, Lederman, Steinberger, and Schwarts discovered that another flavor of neutrino existed - the muon version [4]. Finally, by 2001, Fermi National Accelerator Laboratory’s DONUT experiment confirmed the existence of the tau neutrino [5]. By the turn of the 21st century, we knew that there were three different types or flavors of neutrinos and that they all interacted via weak interactions only - i.e., they interacted only with the weak force and gravity. But now came even more puzzles such as their behavior at different distances and the fact that we have only observed left-handed versions. This idea of handedness is the relation to a particle’s spin and direction of its motion.

2 Neutrino Oscillations

2.1 The Discovery

It’s important, before diving into the quantum mechanical formalism, to introduce what led physicists to discover that neutrinos oscillated in the first place.
In long baseline experiments, neutrinos were found to change flavor i.e., they would start off as an electron neutrino - known through a cross section measurement, propagate a certain distance in an experiment, and then when the cross section was remeasured it was found to have become a new flavor [6]. This was true for atmospheric, solar, reactor, and accelerator experiments. Therefore, this prompted the idea that certain types of experiments were sensitive to different neutrino types.

It also led to something known as the solar neutrino problem in which there was a discrepancy between the predicted versus the measured solar neutrinos from the sun [7]. However, both of these ideas were solved, along with the fact that neutrinos are not massless particles, with the joint discovery from Takaaki Kajita and Arthur B. McDonald on neutrino oscillations [8]. It won them the Nobel prize in 2015.

2.2 Standard Formalism

Moving on from the history to the mathematical formalism of neutrino physics that is quantum mechanics, let’s start off with the notion of ”mixing”. Originally, neutrinos were thought to be massless as they are very difficult to detect. However, with the advent of the discovery of neutrino oscillations (the subject of a 2015 Nobel prize) they have since been proven to have mass, albeit very small [8]. In particular, they must have mass and we will also say “mix” in order to agree with results in current neutrino experiments.

To expand a little on what is meant by ”mixing,” let’s begin by borrowing a concept from standard linear algebra or an introductory quantum mechanics course, that of the superposition of eigenstates. In this case, the “mixing” refers to the fact that each neutrino flavor eigenstate (α=e,μ,τ) is a superposition of its mass eigenstates (i = 1,2,3) by:

\[ |v_\alpha> = \sum_i U^{*}_{\alpha,i} |v_i> \]

with U being:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix}
\begin{pmatrix}
c_{13} & 0 & s_{13} e^{-i\delta_{CP}} \\
0 & 1 & 0 \\
-s_{13} e^{i\delta_{CP}} & 0 & c_{13}
\end{pmatrix}
\begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

where the “U” refers to the PMNS matrix which is nothing more than a unitary transformation that houses information on both the mixing angles and the charge-parity (CP) phase. The mixing angles are simply a measure on the degree of mass each flavor contributes, and the CP phase denotes how much neutrinos violate CP symmetry through their interactions with the weak force [9]. In particular, CP symmetry is something that relates to how particles are exchanged with their antiparticle counterpart as their spatial coordinates as reversed. Note that c_{ij} and s_{ij} are the cosine and sine terms with angle theta as the mixing angle.
Also, note that the “P” refers to another yet matrix, where this one contains the criteria to constrain the neutrino masses further, depending on whether these leptons are Majorana or Dirac particles. This is an ongoing debate that can help to shed light on the nature of the universe, something that I will go more in depth on in the extension section on solving the matter asymmetry problem with neutrinos.

Now, in most common literature regarding this formalism, the reference to massive neutrinos specifies the mass eigenstates themselves and not the flavor states [10]. In particular, one takes them to be orthogonal:

\[ <v_i|v_j> = \delta_{ij} \]

and note also that the U matrix defined above, by definition of being unitary implies that:

\[ U^\dagger U = UU^\dagger = 1 \]

But as the U relates the mass eigenstates directly to the flavor eigenstates, one can say that the flavor states must also be orthogonal [10]. Note however, that we have not assumed U to be diagonal as seen in its full representation above.

I should also note that the masses (1,2,3) are eigenstates of the Hamiltonian governing this situation exactly as we’ve always seen it in an introductory quantum course:

\[ H |v_i> = E_i|v_i> \]

where the \( E_i \) eigenvalues in the ultra-relativistic limit are Taylor expanded to be, to first order:

\[ E_i = \sqrt{p^2 + m^2_i} \approx E + \frac{m^2_i}{2E} \]

This approximation that we made will come into play later when I show an example of the two neutrino mixing [10].

One final consideration that must be made for our deviation is to explain the overarching goal of this exercise. We use quantum mechanics in particular, to find the likelihood or probability that a neutrino has oscillated into one type of another in a given amount of time. Therefore, it is important to think about the Time Dependent case of Schrödinger’s equation.

This final concept should be very familiar to students of quantum mechanics. In the case of the Time Dependent Schrödinger equation, we think of it as an evolution:

\[ i\hbar \frac{d}{dt}|v_i(t)> = H |v_i(t)> \]

where \( |v_i(t)> \) is related to the time independent case via \( e^{-iE_it}|v_i(0)> \) [10]. Now switching up notation briefly from Dirac to a wavefunction form for clarity, consider that the transition probability of say an electron neutrino, in a certain time \( t \) will oscillate into a muon variety. This process can be analytically written in the language of quantum mechanics, as:

\[ P_{\psi_e \rightarrow \psi_\mu} = |A_{\psi_e \rightarrow \psi_\mu}|^2 \]
where $A$ is the probability amplitude [10]. The value of $A$ for our case is constructed as

$$A_{\psi_e \rightarrow \psi_\mu}(t) = \langle \psi_\mu | \psi_e(t) \rangle$$

as the inner product form of the transition from the initial to final state [10]. This can be also further rewritten to accommodate the unitarity structure, therefore,

$$\langle \psi_\mu | \psi_e(t) \rangle = \sum_{i,j} U_{e,i}^{*} U_{\mu,i} U_{e,j}^{*} U_{\mu,j} e^{-iE_j t}$$

after we employ the kronecker delta property [10]. Now, plugging this term back into the explicit form of the probability amplitude, we get an expression for the propagation of a neutrino to time $t$ as:

$$P_{\psi_e \rightarrow \psi_\mu}(t) = \sum_{i,j} U_{e,i}^{*} U_{\mu,i} U_{e,j}^{*} U_{\mu,j} e^{-i(E_j - E_i) t}$$

The last full formalism I will mention is a bit of a leap, but it concerns how to account for the detector distance in the neutrino probability. Conceptually this makes sense to include as the varying lengths that the neutrino travels greatly alters its likelihood of oscillating from one flavor to another. Therefore, applying the approximation we made on the neutrino energy, and by saying that we can equate the distance between source and the detector with the time it takes for that neutrino to be detected i.e., $l \approx t$, then the probability amplitude is:

$$P_{\psi_e \rightarrow \psi_\mu}(l, E) = \sum_{i,j} M_{ij}^{e\mu} e^{-\frac{\Delta m^2_{ij} l}{2E}}$$

where the $\Delta m^2_{ij}$ term is called a mass-squared difference, something I will touch on in the later portion of this paper [10]. Note that this above derivation only assumes a vacuum space i.e., more realistic affects within matter were not considered.

To see an explicit answer for this probability amplitude, let’s take a brief look at a two neutrino system - opposed to the usual three.

### 2.3 Two Neutrino Mixing Case

In the two particle case, consider a simple rotation matrix that relates the flavor eigenstates with those of the mass eigenstates of two neutrinos. This is just:

$$
\begin{pmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{pmatrix}
$$

From here, it can be easily seen to take the form of the probability equation for the probability amplitude above, were I simply plug into $U$ and $U^*$ the $2x2$ matrix. This means:

$$P_{\psi_e \rightarrow \psi_\mu}(l, E) = \sum_{i>j} M_{ij}^{e\mu} e^{-i\frac{\Delta m^2_{ij} l}{2E}} + \sum_{i<j} M_{ij}^{e\mu} e^{i\frac{\Delta m^2_{ij} l}{2E}} + \sum_{i=j} M_{ij}^{e\mu}$$

where I’ve taken into account separately the terms each in the sum for when $i=j$. 

i,j, and i=j [10]. Note that I also, to save time in writing everything out, replaced the term $U_{e,i}U_{\mu,i}U_{e,j}U_{\mu,j} = M_{ij}^{\mu\mu}$. Now it is only a matter of expanding everything out so for each sum:

$$= -\cos^2(\theta)\sin^2(\theta)\left(e^{-i\frac{\Delta m_{ijl}^2}{2E}}+e^{i\frac{\Delta m_{ijl}^2}{2E}}\right)+2\cos^2(\theta)\sin^2(\theta)$$

$$= 2\cos^2(\theta)\sin^2(\theta)(1 - \cos(\frac{\Delta m_{ijl}^2}{4E}))$$

$$= \sin^2(\theta)\sin^2(\frac{\Delta m_{ijl}^2}{4E})$$

Therefore, in the end, at least for this simple 2x2 mixing case in a vacuum, the probability that an electron neutrino oscillating into a muon neutrino at a distance of "l" is given directly above. As a small aside, let also note that as I briefly mentioned before, we have a parameter within the full PMNS matrix defined as the degree of CP (charge-parity) violation. But this is something, on a basic level akin to looking at the difference say between the probability of an electron neutrino becoming a muon neutrino versus an electron antineutrino becoming a muon antineutrino. In other words: $P_{\psi_e\rightarrow\psi_\mu} - P_{\bar{\psi}_e\rightarrow\bar{\psi}_\mu}$ [10].

I will finish off this section by stating a few final thoughts. First, in a more realistic case where one considers the matter effects is that the mixing angles change to account for scattering - another topic prevalent in quantum mechanics. And this means that from a Hamiltonian that contains these properties, the matrix $U$ defined say in the 2x2 case above, is diagonalized in a different way. Finally, I’ve gone into detail of what is known as the Dirac formalism notion as I believe that it is much more illuminating than in the context of a wavepacket. However, both are acceptable and correct ways to understand the quantum mechanics behind neutrino physics and in particular neutrino oscillations.

## 3 Impact on Future Physics

### 3.1 Importance of the Mass-Squared Difference

We’ve seen the motivation for why the development of the physics - in the language of quantum mechanics - for neutrino oscillations is needed, but why do we still care to know about it today. After all, the oscillations were already discovered. Well as defined back in the derivation of the probability amplitude, there is a term called $\Delta m_{ij}^2$ otherwise known as the mass-squared difference. This was not just a clever choice of a change of variable to simplify some calculation, but rather, it is a parameter that is directly probed by many neutrino experiments. In fact, this parameter is part of the key to unlocking how neutrinos fit into our understanding of the universe.

Therefore, both experimentalists and theorists who perform research in neutrino physics care about this parameter greatly, as well as that of the mixing angle $\theta$ and the CP phase $\delta_{CP}$. In fact, to briefly explain why this is so important, let me go back to considering the three neutrino mixing as defined in the PMNS matrix
at the beginning of this paper. Now, there are really two different mass squared-differences which relate the three masses 1, 2, and 3 by the difference $\Delta m^2_{12}$ and $\Delta m^2_{23}$ as seen in the figure 2 below.

**normal hierarchy**  

$$(m_3)^2$$

$$(\Delta m^2)_{atm}$$

$$(\Delta m^2)_{sol}$$

**inverted hierarchy**

$$(m_2)^2$$

$$(m_1)^2$$

$$(m_3)^2$$

$$(\Delta m^2)_{atm}$$

$$(\Delta m^2)_{sol}$$

$$(\Delta m^2)_{sol}$$

Figure 2: The above shows the two proposed neutrino mass hierarchies where the normal indicates a larger $m_3$ while the inverted case assumes the opposite - $m_3$ as the smallest. (Source: arXiv:hep-ph/0411274)

In fact, these differences change depending on whether the actual mass eigenstate of $m_1$ or $m_3$ is larger in size relative to the other two. Remember, this is directly correlated to the fact that the masses themselves are super-positions of the three flavors, i.e., the fact that they mix. From there, we define what are known as the two hierarchies under consideration in which the ordering of the mass-squared differences can be based [11]. These are referred to as normal or inverted, again see figure 2. This notion of probing parameter space to justify which is the true hierarchy is one of the important questions that I said is circulating within the physics community today as it will help to further characterize neutrinos. And while it has been probed further in recent experiments, there is another concept that experiments probe through neutrino oscillations and that is the idea of a sterile neutrino.
3.2 Sterile Neutrinos

Concepts that require ideas beyond the scope of the Standard Model are called Beyond the Standard Model Physics or BSM. As mentioned prior to this, there is an active search within the neutrino physics community for a fourth possible neutrino called the sterile neutrino. This addition was originally introduced to the neutrino physics community in order to explain experimental anomalies, specifically in the oscillation process. These neutrinos behave differently from their known three flavor counterparts because, while they would participate in oscillations, in this “sterile” phase they would only interact with gravity - not the weak force [12]. This implies that one can treat it as an extra phase in the full oscillation process, but given that they interact only with gravity, this means it’s even harder to study them.

Therefore, while still a relatively theoretical particle, it is the subject of a growing number of experiments, from the current short Baseline (SBN) program at Fermi National Accelerator Laboratory and the future long Baseline experiment DUNE, to small reactor experiments like KamLAND, these also will help to contribute to further constraining the parameter space with which one could detect a sterile neutrino [13].

Figure 3: The above shows the graphic of possible cases for a sterile neutrino of different masses. This uses constraints from cosmology. The x axis indicates the mass of the sterile neutrino while on the left vertical axis is the cosmological parameter $N_{\text{eff}}$ that tells us how many flavors there could be and the right vertical axis has the neutrino density in the universe. (Source: Planck Collaboration - 2018 cosmological parameters via a Neutrino University talk given by A. de Gouvea at Fermilab in 2019.)
One final thought on the sterile neutrino case is that this would mean adding in an extra factor into our standard 3x3 PNMS matrix for oscillations. This is very good evidence to suggest that sterile neutrinos exist and this actually comes from astrophysics. Above is a plot that shows a correlation between a parameter in cosmology in particular known as $N_{\text{eff}}$, or in other words, the effective number of neutrinos. All this is, is a factor that should tell the total number of flavors. However, experimentally, this is found to be more than 3 - actually its 3.046 to be exact [14]. So in figure 3 above, this shows a comparison of how well the mass of a given sterile neutrino - if found - relates to this parameter. The different shades of color simply refer to what is on the right side vertical axis, a parameter denoting the density - amount - of neutrinos in the universe corresponding again to the different possible sterile neutrino masses.

It is a good visual representation that there can be a sterile neutrino especially since the parameter space with the most color is around the value of $N_{\text{eff}}$. Note that in this part of the space the potential fourth neutrino would have a very small mass - even smaller than the current predictions of the overall mass pertaining to the three standard flavor case.

It’s another interesting question in the neutrino physics community, and in order to solve it, we need neutrino oscillation experiments. But how do we explain the physics behind such a tool - that would be quantum mechanics.

### 3.3 Extension: Neutrinos and the Universe

In discussing neutrinos and their oscillations, it seems only fitting to end our story by briefly mentioning how the neutrino fits into our study of the universe as a whole. We know that, as with the study of particle physics in general, there is an interrelation between the subatomic and cosmic realms of study in physics. However, neutrinos in particular could hold the key to explaining a fundamental aspect of cosmological theory that basically tells us why the stars, planets, and even humans are here. In this final part of the paper, I will discuss whether neutrinos behave as Dirac or Majorana particles.

Neutrinos like other fermions, have two orientations of spin which we call right-handed and left-handed. This is akin to the topic on spin - $\frac{1}{2}$ particles that are discussed in every quantum course. However, only left-handed neutrinos have been detected within the neutrino experiments to date. So, to account for this, we say that there is a distinction between a Dirac or Majorana neutrino. Dirac neutrinos generate masses via the Higgs mechanism, which simply implies that their mass must be close to other particles. This is not as widely accepted as the strength of neutrino interactions with the Higgs Boson is too small as compared to those with the top quark [15].

In the case of Majorana neutrinos, their main characteristic is that these particles are their own antiparticle and in the case, neutrinos follow what is called the see-saw mechanism, where in essence, it explains how massive right-handed neutrinos at the early stages of the universe must then have decayed into the lighter left-
handed ones we see in experiments. Further, the decaying process would produce left-handed neutrinos and right-handed antineutrinos, which we know from other experiments, do decay at slightly different rates [16]. Since they would decay at different rates, this could explain why, despite that at the time of the Big Bang there were equal amounts of matter and antimatter, they didn’t annihilate each other.

Considering the whole history of the universe, it’s incredible that we are here studying the physics of subatomic particles today. It’s also incredible that a particle, whose physics is routed directly in something every undergraduate physics major learns - the quantum mechanical formalism - can be studied readily by both the theoretical and experimental community, and its characteristics can help us to understand the nature of the universe.

4 Conclusion

I have discussed neutrino oscillations in the context of their quantum mechanical formalism as well as its motivation for the understanding of the particle physics and by extension, the universe. In many ways, neutrino physics as a whole is a multifaceted field that extends from the subatomic up to the cosmic scales of our universe. Also, it is very active, both in a theoretical and in an experimental way, where this collaboration relies on the quantum mechanical framework that describes the physics behind one of the most important known characteristics of neutrinos. It’s that property - neutrino oscillations - that has opened the door to further exploring the field for decades to come.

References


