A Brief Introduction to Superconducting Charge Qubits

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1 Why Superconducting Qubits

Over the course of recent decades, there has been an increasing push for creating more efficient and reliable quantum computers with the goal of achieving quantum supremacy. Superconducting qubits are one of the most widely used physical realizations of quantum computing due to their scalability [6] and the success researchers have had in creating a universal set of quantum logic gates for superconducting qubits [1]. Both Google and IBM have built functioning superconducting quantum computers with 72 and 53 qubits respectively, demonstrating the feasibility of constructing a superconducting quantum computer. Finally, superconducting qubits are based on circuit systems that can be reliably constructed in a modular fashion [2] making them an excellent foundation on which to build quantum computers.

2 Quantum LC Circuits

The following sections describing the theory underlaying quantum LC circutis and superconducting qubits are drawn from Nathan Langford’s [4] and Steven Givrin’s [2] lecture notes on superconducting qubits.

Superconducting qubits are constructed out of what amount to classical circuit elements that are supercooled and made small enough that the law of Quantum Mechanics apply. This means that we will have to develop a means of constructing quantum Hamiltonians for systems which are usually treated and understood classically. A solution to this problem is given by circuit QED which uses the correspondence principle to quantize the classical Hamiltonians of circuit elements. To understand how this is done will look at one of the simplest circuits, an LC circuit. A classical LC circuit consists only of a capacitor and an inductor connected in series and exhibits a type of resonance as energy oscillates between being stored in the capacitor and the inductor in a way analogous to
a spring system. Typically, the dynamics of this circuit are expressed in the differential equation

\[ \ddot{Q} + \frac{1}{LC}Q = 0 \]

where \( Q \) is the charge stored in the capacitor. However, the equations governing the circuit can also be expressed through the use of a Lagrangian. Despite the fact that Lagrangians are typically viewed through the lens of classical mechanics and are used to describe the motion of systems of particles with potential and kinetic energies, a Lagrangian is ultimately a mathematical tool that can be used to describe any system. So long as the Lagrangian is chosen carefully enough that its Euler-Lagrange equations produce the equations governing the system, Lagrangians can be used to describe systems in any areas of physics. Lagrangian’s are often used to describe gravity, fluid motion and even electromagnetic fields. In the case of the LC circuit, the Lagrangian is the difference of the energy stored in the capacitor and inductor which comes out to

\[ \mathcal{L} = \frac{1}{2}L\dot{Q}^2 - \frac{1}{2C}Q^2 \]

The Euler-Lagrange equations for this Lagrangian are then

\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{Q}} \right) = \frac{\partial \mathcal{L}}{\partial Q} \]

which yields our differential equation

\[ \frac{d}{dt} (L\dot{Q}) = -\frac{1}{C}Q \]

\[ \ddot{Q} + \frac{1}{LC}Q = 0 \]

Hence this Lagrangian accurately models the circuit’s dynamics. However, in order to construct a Hamiltonian for the quantum version of this system this Lagrangian must be turned into a classical Hamiltonian. After we have a classical Hamiltonian for the system we will be able to use canonical quantization to construct the Hamiltonian for the quantum analogue of this system. To find the classical Hamiltonian, we will need to find the canonical momentum for this system which is the conjugate variable to \( Q \) which we are taking as the canonical position for the Lagrangian. The canonical momentum is given by

\[ \Phi = \frac{\partial \mathcal{L}}{\partial \dot{Q}} = L\dot{Q} \]

Then the Hamiltonian is given by

\[ H = Q\Phi - \mathcal{L} = \frac{1}{2L}\Phi^2 + \frac{1}{2C}Q^2 \]
Converting this classical Hamiltonian to a quantum Hamiltonian is then quite easy. Much like in the case of constructing a quantum Hamiltonian for a single particle, we promote the canonical position and momentum coordinates to operators, requiring them to obey the canonical commutation relation. That is we take

\[
\begin{align*}
Q & \rightarrow \hat{Q} \\
\Phi & \rightarrow \hat{\Phi} \\
[\hat{Q}, \hat{\Phi}] & = i\hbar
\end{align*}
\]

This is done much in the same way when constructing a Hamiltonian for a single particle where we take

\[
\begin{align*}
x & \rightarrow \hat{x} \\
p & \rightarrow \hat{p} \\
[\hat{x}, \hat{p}] & = i\hbar
\end{align*}
\]

The Hamiltonian for the quantum LC circuit is then given by

\[
H = \frac{1}{2L}\hat{\Phi}^2 + \frac{1}{2C}\hat{Q}^2
\]

where the first term corresponds to the Hamiltonian of the inductor and the second to the Hamiltonian of the capacitor. Now that we have a Hamiltonian that can describe the circuit elements that will be used to build our qubit we can turn to a brief description of superconductivity.

### 3 Superconductivity

As spin $\frac{1}{2}$ particles, electrons are fermions which means that no two electrons can occupy the exact same state. This means that in a material with a large number of electrons, the overall wavefunction for all electrons is incredibly complicated and the system’s Hilbert space has a large number of degrees of freedom. However, when a material is cooled below its critical temperature, the vibrations of atoms in the material’s lattice become incredibly weak, allowing for positive nuclei in the lattice to slightly attract to a nearby electron in the lattice and group up around it creating a region of positive charge. A distant electron might then be attracted to this positive charge buildup leading it to be indirectly attracted to the first electron forming a bound state called a Cooper pair. Cooper pairs behave as though they were single particles with spin 1 which means that they follow Bose-Einstein statistics and, unlike electrons, can occupy the same state.

This has the effect of reducing the ground state of all electrons in the material a state where all electrons are paired up into Cooper pairs which exist at the same low-energy state. Hence the Hilbert space consisting of all electrons in the metal can be reduced to a space consisting of single states $|N\rangle$ describing the number of Cooper pairs in the material since all other properties of the
low-energy ground state of all Cooper pairs remain constant. This is incredibly beneficial to the construction of a qubit out of a "superconducting island" of material isolated from its surroundings since it greatly simplifies the description of that system.

Figure 1: Sketch of the attraction between Cooper pairs in a supercooled lattice, taken from Padavić [5]

4 Josephson Junctions and a Prelude to Charge Qubits

A Cooper pair is a system constructed out of a metallic superconducting island connected by a thin insulating barrier, called a Josephson Junction, to a superconducting electron reservoir (or in some cases a second superconducting island). A Cooper pair box can be used to build a type of superconducting qubit known as a charge qubit where the state of the qubit is determined by the number of Cooper pairs that have tunneled across the Josephson Junction to the superconducting island.

The Josephson junction acts simultaneously as a tunneling element and a capacitor so the Hamiltonian constructed to describe it must include terms to describe both aspects of the Junction. We can use our previous discussion of quantum LC circuits to describe the capacitive term in the Josephson Junction, but the tunneling term cannot be constructed in analogy to a classical system since it is a purely quantum phenomenon. However, the tunneling element is fairly simple and can be built out of the orthonormal basis states \( |N\rangle \) that denote the number of Cooper pairs on the superconducting island. The tunneling element of the Josephson Junction is then given by

\[
H_T = -\frac{1}{2} E_J \sum_N \left( |N\rangle \langle N+1| + |N+1\rangle \langle N| \right)
\]

where \( E_J \) is called the Josephson coupling energy and is a measure of the difficulty for Cooper pairs to tunnel across the junction. To then get the full Hamiltonian for the Josephson Junction we add the tunneling term to the term \( \hat{Q}/2JC \), which was derived earlier and describes a quantum capacitor with capacitance \( C_J \). Thus the full Hamiltonian a physical Josephson Junction is
\[ H = \frac{\hat{Q}^2}{2C_J} - \frac{1}{2} E_J \sum_N \left( \langle N | \langle N + 1 | + \langle N + 1 | \langle N | \right) \]

In this Hamiltonian the capacitive term accounts for the energy associated with adding Cooper pairs to the superconducting island while the tunneling term accounts for the energy associated with Cooper pairs tunneling across the junction. However, this Hamiltonian cannot be used to fully describe a charge qubit since it does not include any terms that allow us to manipulate the state of the Cooper pair box and hence the qubit.

5 Voltage Biased Cooper Pair Boxes

A method of controlling the qubit is to apply a gate voltage to an electrode near the superconducting island so there is a capacitive interaction between the electrode and the island which we can control shown in Figure 2. We can imagine this as a circuit where the superconducting island is connected with wires, on one side to a capacitor which is itself is connected to the voltage source and on the other side is connected to the superconducting reservoir through the Josephson Junction which has a tunneling and a capacitive element. This amounts to placing the "Josephson Junction capacitor" and this gate capacitor in parallel.

If we chose the gate voltage to induce a constant charge \( Q_g \) on the electrode then the total charge on the parallel capacitive system is \( Q - Q_g \) allowing us to construct the modified Hamiltonian
\[
H = \frac{(\hat{Q} - Q_g)^2}{2(C_J + C_g)} - \frac{1}{2} E_J \sum_N \left( |N\rangle \langle N + 1| + |N + 1\rangle \langle N| \right)
\]

Effectively this means that applying a positive charge to the gate makes it energetically easier to place more Cooper pairs, which are negatively charged, onto the superconducting island. Note that \(Q_g\) is not an operator since the charge on the electrode is constant which means it acts only as an offset to the Hamiltonian. Because charge is quantized in units of Cooper pairs then we can write the charge operator \(\hat{Q}\) in terms of a number operator \(\hat{N} = \sum_N N |N\rangle \langle N|\) which gives the number of Cooper pairs on the island when applied to a state. We then have that \(\hat{Q} = 2e\hat{N}\) so then defining the charging energy as

\[
E_C = \frac{4e^2}{2(C_J + C_g)}
\]

and

\[
N_g = \frac{Q_g}{2e} (\text{since the charge on the electrode does not have to come in units of Cooper pairs})
\]

we can rewrite the Hamiltonian as

\[
\begin{align*}
H &= E_C (\hat{N} - N_g)^2 - \frac{1}{2} E_J \sum_N \left( |N\rangle \langle N + 1| + |N + 1\rangle \langle N| \right) \\
H &= E_C \sum_N (N - N_g)^2 |N\rangle \langle N| - \frac{E_J}{2} \sum_N |N\rangle \langle N + 1| + |N + 1\rangle \langle N|
\end{align*}
\]

6 Superconducting Charge Qubits

The Hamiltonian described above can now finally be used to construct a charge qubit system. We will operate in the regime where \(E_J << E_C\) because this provides good separation between energy levels in the system since energy scales quadratically with \(N_g\).

Proper separation between the energies of different states in the system is important to make sure we can cleanly measure the state of our qubits and interact with them. If the difference between the energies of different states remained constant or increased very slowly then state transition are equally likely to be induced at any energy level by voltage biasing rather than just between states \(|0\rangle\) and \(|1\rangle\) i.e. it would be equally likely for the system to transition from \(|0\rangle\) to \(|1\rangle\) as it would be for it to transition from \(|1\rangle\) to \(|2\rangle\) which would make it incredibly difficult to construct reliable qubit gates. Working in the \(E_J << E_C\) regime and setting \(N_g = \frac{1}{2} + \Delta g\) for \(\Delta g << 1\) we can then write out the system’s Hamiltonian, discarding terms of \(|N\rangle|N\rangle\) with \(N > 1\) to get

\[
H = E_C \left( \frac{1}{4} + \Delta g \right)^2 |0\rangle \langle 0| + E_C \left( \frac{1}{4} - \Delta g \right)^2 |1\rangle \langle 1| - \frac{E_J}{2} \left( |0\rangle \langle 1| + |1\rangle \langle 0| \right)
\]

\[
H = E_C \left( \frac{1}{4} + \Delta g \right) I + E_C \Delta g \sigma_z - \frac{E_J}{2} \sigma_x
\]

If \(E_J = 0\) then the states \(|N\rangle\) would be the energy eigenstates \(|E\rangle\) of the system so then in the regime \(E_J << E_C\) we have that \(|N\rangle \approx |E\rangle\). Then as
desired, the $|2\rangle$ state has an energy at least $9$ times higher than the $|0\rangle$ and $|1\rangle$ states which lends to a system with easy to manipulate qubits.

## 7 General Qubit Operations

Though there is a great amount of ongoing research on constructing gates for superconducting qubits, I will discuss some simple examples of gates and system manipulations that can be easily done with the qubit system discussed in the earlier section.

**State Preparation:** The most basic operation, state preparation, can be achieved quite easily in this system. For a sufficiently low temperature, the system will naturally relax into the ground state $|0\rangle$ so long as you set $N_g = 0$ if you wait long enough. This allows you to guarantee, with very high probability, that the system will be in its ground state before running any computations or experiments.

**Fast Single Qubit Gate:** Fast gates are named so because they rely on changing the Hamiltonian "instantaneously" in order to produce a change in the system. A simple example of this can be seen as follows. Set the system to the ground state $|0\rangle$ with $N_g = 0$ and at $t = 0$ "instantly" increase the gate voltage so $N_g = \frac{1}{2}$. This will leave the Hamiltonian in the form

$$H = -\frac{E_J}{2}\sigma_x$$

This means that the time-translation operator will be

$$U = \exp(i\frac{E_J}{2\hbar}\sigma_x t)$$

Therefore, the system’s wavefunction will evolve according to

$$|\psi(t)\rangle = \exp(i\frac{E_J}{2\hbar}\sigma_x t)|0\rangle$$

Noting that

$$\exp(i\alpha \sigma_x) = I \cos \alpha + i \sigma_x \sin \alpha$$

our state will evolve like

$$|\psi(t)\rangle = (I \cos (\frac{E_J}{2\hbar}t) + i \sigma_x \sin (\frac{E_J}{2\hbar}t))|0\rangle$$

$$|\psi(t)\rangle = \cos (\frac{E_J}{2\hbar}t)|0\rangle + \sin (\frac{E_J}{2\hbar}t)|1\rangle$$

Thus by carefully choosing at which time $t$ to measure the system or make the system interact with another qubit or gate, we can chose the exact state of the system.
Adiabatic Single Qubit Gate: The adiabatic or slow gate operates through the opposite mechanism as the fast qubit gate relying on the Adiabatic Theorem. The Adiabatic Theorem states that if the Hamiltonian of a system is changed slowly enough and the system starts in an eigenstate of the Hamiltonian, then it will end in a corresponding eigenstate of the transformed Hamiltonian. A simple example of this type of gate is to again start with the system in the state $|0\rangle$ with $N_g = 0$ and slowly bring $N_g$ up to $\frac{1}{2}$ so that the final state of the system when $N_g = \frac{1}{2}$ will be $|+\rangle$.

8 Noise and the Transmon Regime

Though our earlier assumption that $E_J << E_C$ was incredibly helpful in simplifying our system and allowing us to easily construct a qubit, it makes the system fairly susceptible to noise. Because in the $E_J << E_C$ regime the energy of the $|0\rangle$ and $|1\rangle$ eigenstates varies according to $N_g^2$, small changes in the bias voltage can heavily impact the system. Thus noise in the bias electrode or charge accumulations in the environment heavily impact the qubit system. To remedy this, experiments often work with transmon qubits which are charge qubits in the $E_J >> E_C$ regime where the energy eigenstates of the Hamiltonian no longer are heavily dependent on $N_g$. However, in this regime we can no longer implement gates by modifying the bias voltage since the system is now fairly independent of the bias voltage. Though a full description of the implementation of gates in this regime is fairly complicated and beyond the scope of this paper, I will briefly describe how these gates work in principle.

In the Transmon Regime qubit gates can be constructed by coupling the Cooper pair box to an optical cavity with a resonant frequency in the microwave range that is significantly different from the optical resonant frequencies of the Cooper pair box. Such a cavity can trap photons at its resonant frequency inside the cavity in the form of a standing wave. If microwaves are emitted at the cavity’s resonant frequency, they excite the cavity allowing for measurement of the state of the qubit based on measuring the amplitude and phase of the photons. Conversely, if microwaves are emitted at the Cooper pair box’s resonant frequency, they leave the cavity unaffected thus not performing a measurement on the system, but rather rotate the qubit by inducing state transitions in the system. The cavity can even be connected to multiple qubits in order to implement multi-qubit gates.

9 Future Research in Superconducting Qubits

Despite the fact that tremendous advances have been made in the construction and manipulation of superconducting qubits, there is still great potential for improvement. For example, superconducting qubits, have fairly low coherence
times compared to other types of qubits [6]. The coherence time is the average time it takes for a qubit’s preset state to change due to interaction with the environment. Higher coherence times make the implementation of gates easier and make it so error correction can be applied less frequently [6], so the versatility of superconducting quantum computers and their ease of fabrication would be greatly improved by higher coherence times. Another area of improvement comes from the fact that the Transmon qubits described earlier suffer from having low separation between energy eigenstates which means that it is fairly easy for system excitations to leave the two-state computation subspace [3]. This is can be remedied by using fluxonium qubits which couple the superconducting island to the reservoir through an inductor rather than a capacitor and can combine the benefits of the traditional Cooper-pair box and Transmon qubits while avoiding their drawbacks [3]. However, fluxonium qubits have not been successfully incorporated into large-scale circuits and their operation is far more difficult, requiring further research [3].
References


