1. The Pauli matrices are a beautiful and fundamental set of matrices in physics. The reason for their importance is that they are used to describe how spatial rotations in three dimensions act on fermions in nature; for example, on electrons. They are also critical in describing unitary operations on qubits.

(i) These matrices are given by

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Compute the commutator \([\sigma_i, \sigma_j]\) and the anti-commutator \(\{\sigma_i, \sigma_j\}\).

(ii) Which of the Pauli matrices are unitary? Which are Hermitian?

(iii) Please find the eigenvalues and eigenvectors of each Pauli matrix.

(iv) Add the identity matrix \(\sigma_0 = 1\) to these three matrices and show that any \(2 \times 2\) matrix \(M\) can be written in the form

\[ M = \sum_{i=0}^{3} \alpha^i \sigma_i \]

for some complex \(\alpha^i\). Under what conditions on \(\alpha^i\) is \(M\) unitary? Under what conditions is \(M\) Hermitian?

2. Just in case you didn’t get enough of the Pauli matrices, let’s take an orthonormal basis \(\{|0\rangle, |1\rangle\}\) for a two-dimensional Hilbert space. With respect to this basis, assume we have 3 operators that correspond to the Pauli matrices. Assume that the basis \(\{|0\rangle, |1\rangle\}\) is an eigenbasis for \(\sigma_3\) with eigenvalues \(+1\), \(-1\) respectively. Can you write each of those \(\sigma_i\) operators in “ket-bra” notation. Namely as linear sums of operators like \(\{|0\rangle\langle 0|, |0\rangle\langle 1|, \ldots\}\).

3. As a final exercise, let’s understand what it means to exponentiate a matrix or operator.

(i) What is the Taylor series definition of the function \(e^x\)?

(ii) Take the generator of rotations in two dimensions given by \(T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\). The operator \(1 + \epsilon T\) generates an infinitesimal rotation of the vector \(\begin{pmatrix} x \\ y \end{pmatrix}\) by angle \(\epsilon\). To get a finite rotation, we will want to exponentiate the matrix using the Taylor series definition. So evaluate \(e^{\theta T}\) using the Taylor series definition above. What matrix do you get?