QUANTUM GRAVITY AND NEUTRON STARS

SRIHARI NARAYANAN

Abstract. In this expository paper, we study quantized gravity and neutron stars as two disparate topics and investigate the role quantum gravity plays in neutron star cores. Specifically, we introduce quantum gravity by explaining the clash between quantum mechanics and general relativity and, through a gedankenexperiment proposed by Mari et al. [1] and analyzed by Belenchia, Wald, et al. [2], justify the need for elements of quantum gravity such as vacuum fluctuations in a gravitational field (i.e. minimal positional uncertainty) and quantized gravitational radiation. We then introduce in parallel basic properties of neutron stars and the Tolman-Oppenheimer-Volkoff equation. Finally, mainly following [3], we marry these two topics by solving the TOV equation while accounting for quantum gravity, which we do by using the equation of state of an ultra-relativistic Fermi gas governed by minimal positional uncertainty.

Contents

1. QM vs. GR: Quantum Foam 1
2. The gedankenexperiment of [1] 2
2.1. Electromagnetic version 3
2.2. Gravitational version 5
3. Neutron stars and the TOV equation 6
4. Putting it all together: Quantum gravity in neutron star cores 7
Acknowledgements 9
References 9

1. QM vs. GR: Quantum Foam

As we know, quantum mechanics becomes relevant at subatomic scales. In particular, we recall the energy-time uncertainty principle—that is, for any observable $Q$,

$$\sigma_H\sigma_Q \geq \frac{\hbar}{2} \left| \frac{d\langle Q \rangle}{dt} \right|$$

By setting $\Delta E = \sigma_H$ and $\Delta t = \sigma_Q \left| \frac{d\langle Q \rangle}{dt} \right|^{-1}$, we get $\Delta E \Delta t \geq \frac{\hbar}{2}$ (where $\Delta t$ is observable-specific). That is, if we have small energy uncertainty in our system, the expectation value of all observables is slow to change (by 1 standard deviation), and if $\Delta t$ is large for any observable, then we have large energy uncertainty. This implies the occurrence of spontaneous particle-antiparticle pairs at subatomic scales.

Date: March 24, 2020.
and at extremely short time scales (i.e., when $\Delta t$ is large for any observable). This is a quantized description of vacuum fluctuations in a quantum field, also known as quantum foam, pictured below:

![Quantum foam](https://via.placeholder.com/150)

**Figure 1.** Quantum foam (Source: Google Images)

On the other hand, general relativity becomes relevant at high masses and large scales. This comes from the postulate of general relativity that accelerating reference frames are locally indistinguishable from gravity, which gives the $\gamma$-factor of general relativity

$$\gamma = \left(1 - \frac{v_{\text{esc}}^2}{c^2}\right)^{-1/2} = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2}$$

where $v_{\text{esc}}$ represents escape velocity.

General relativity generally implies a flat spacetime, which we realize breaks down at subatomic scales, as seen with quantum foam. More formally, although we easily have an entirely consistent quantum field theory of linearized gravity (that is, when we replace the spacetime metric tensor with a first-order approximation), wholly uniting quantum mechanics and general relativity is extremely difficult. Therefore, it has been suggested that a theory of quantized gravity might require a radical reformulation of quantum mechanics [4]. However, by following the analysis in [2] of the gedankenexperiment in [1], we realize the need of a) vacuum fluctuations in a gravitational field and b) quantized gravitational radiation in resolving the otherwise paradoxical results obtained.

2. **THE GEDANKENEXPERIMENT OF [1]**

We consider two individuals, Alice and Bob, who control two nonrelativistic particles with charge $q_A$, $q_B$ and with mass $m_A$, $m_B$. Sometime long before the beginning of the experiment, $t = 0$, Alice uses a Stern-Gerlach apparatus to put her particle (with nonzero spin) in an equal superposition of states $|L\rangle_A$ and $|R\rangle_A$ separated by distance $d << D$ (see diagram below), so that $|\psi\rangle_A = (1/\sqrt{2}) (|L\rangle_A |\downarrow\rangle + |R\rangle_A |\uparrow\rangle)$. More formally, we have that $\langle x|L\rangle_A = \phi(x)$ and $\langle x|R\rangle_A = \phi(x - d)$ for some wave function $\phi$. Additionally, before $t = 0$, Bob puts his particle in a finite square well with an arbitrarily large potential step $V_0$, so that, while in the well, Bob’s particle has negligible interaction with Alice’s particle.
Now, starting at $t = 0$, Bob randomly chooses whether or not to release his particle from the well. If he releases his particle, it will interact with Alice’s superposition, so that, after a time $T_B$, Bob will measure a path difference $\delta x$ from whether his particle interacts with the state $|L\rangle_A$ or $|R\rangle_A$. Bob thus effectively measures the path of Alice’s particle. In parallel, starting at time $t = 0$, Alice attempts to reverse her superposition by sending her particle through a “reverse” Stern-Gerlach apparatus within a time $T_A$. We realize that, since Bob obtains path information about Alice’s particle when he releases his particle, Alice’s particle becomes entangled with his, so that Alice’s particle is now in a mixed state, which we see by taking the partial trace over Bob’s particle/qubit of the corresponding density matrix. Therefore, if Bob releases his particle, Alice fails to recover her original pure state (that is, she fails to recohere her particle) and conversely. Alice thus knows whether or not Bob released his particle from the trap, based on the success of her experiment. Because of this complementarity between coherence and entanglement, this seems to work even in the case where $T_A < \frac{\hbar}{c}$ and $T_B < \frac{\hbar}{c}$, implying superluminal communication from Bob to Alice, a violation of causality! On the other hand, if the recoherence of Alice’s particle is not affected by whether or not Bob released his particle from the trap, then complementarity is violated.

Following [2], we first analyze the electromagnetic version of this situation, where we assume the gravitational field is negligibly weak compared to the electromagnetic field. We then use the results here to obtain an analogous resolution to the gravitational version—that is, where we set $q_A = q_B = 0$. Before we begin, we note that, for both versions, both causality and complementarity are independently satisfied by the properties of an electromagnetic/gravitational field (i.e. that they behave causally) and by quantum theory—the paradox arises when we consider them together in the scenario above. For the analysis, we set $\hbar = c = G = 1$ (so that, for instance, $\ell_p = 1$).

2.1. Electromagnetic version. We first define the electromagnetic field states in Alice’s superposition. Setting $\psi_L = \langle x|L\rangle_A$ and $\psi_R = \langle x|R\rangle_A$, we put $|\alpha_L\rangle$ to be the coherent state of the electromagnetic field with charge density $\rho_L = q_A |\psi_L|_2$ and current density $j_L = (q_A/m_A) \text{Im}(\overline{\psi}_L \nabla \psi_L)$ and define $|\alpha_R\rangle$ similarly. Alice’s
superposition is thus of the form

$$\frac{|L\rangle_A |\alpha_L\rangle + |R\rangle_A |\alpha_R\rangle}{\sqrt{2}}$$

We first consider vacuum fluctuations in the electromagnetic field. For a spacetime region with dimensions $R$ and $R/\ell$ ($= R$), we claim that fluctuations in the electric field $E$ are of the order $\Delta E \sim R^{-2}$, the explanation of which we pass to [2]. We now consider the behavior of a particle in our electric field with charge $q$ and mass $m$ over a timescale $R$. We have that $\ddot{x} = qE/m$. Since the electric field fluctuations over this particle’s worldline are random and comparatively small, $E$ is mostly constant. Therefore, integrating both sides of this equation twice w.r.t. $t$ gives $x = qER/m$ (assuming our particle starts at rest), so that we have $\Delta x = qER^2/m$ and $\Delta x \sim q/m$. We thus realize that, for the path difference $\delta x$ in Bob’s particle, we need $\delta x \gg q/\ell_B$ for Bob to successfully obtain information about the path of Alice’s particle.

We now calculate $\delta x$ using standard E&M arguments. We see that Alice’s superposition represents an effective dipole of $D_A = qa$, much in the same way that two oppositely charged particles with charge $\pm q$ and distance create a dipole $q(q/2) - (-q(q/2)) = qd$. Multipole expansion helps us obtain a first order approximation of the potential $\phi(D) = -\vec{D}_A \cdot \nabla(D^{-1})$ (note that $4\pi\ell_0 = 1$). Taking the negative gradient of this yields

$$E(D) \approx \frac{2D_A}{D^3}$$

By simple kinematics, we have that $\delta x = (\frac{1}{2})aT_B^2$, with $a$ being the acceleration caused by the effective dipole. We thus have that

$$\delta x = \frac{q_B E}{2m_B} T_B^2 \approx \frac{q_B D_A}{m_B D^3} T_B^2$$

Since we need $\delta x > q/\ell_B$, we obtain $D_A T_B^2 > D^3$ as a necessary condition for Bob to successfully measure the path of Alice’s particle.

We now consider the phenomenon of quantized electromagnetic radiation. When Alice attempts to reverse her superposition, her effective dipole $D_A$ vanishes. Therefore, if $T_A$ is small enough, Alice emits some electromagnetic radiation, so that $|\alpha_L\rangle$ and $|\alpha_R\rangle$ do not return to the initial vacuum state $|0\rangle$ of the electromagnetic field (that is, the state that vanishes under the Hamiltonian, $H |0\rangle = 0$) regardless of what Bob does, since even a single photon will give our system a nonzero energy level. By following the derivation of the energy flux [5] (up to equation 1094), we see that the energy flux radiated is $\sim (\vec{D}_A)^2$. Therefore, the total energy radiated across $T_A$ is on the order of

$$\mathcal{E}_A \sim \left(\frac{D_A}{T_A^2}\right)^2 T_A$$

Since $T_A$ is the time taken for the electromagnetic field state to change from the stationary state represented by an equal superposition of $|\alpha_L\rangle$ and $|\alpha_R\rangle$, $T_A$ must be the period of the E&M waves emitted, so that the energy $\mathcal{E}_A$ is represented by photons of frequency $\sim T_A^{-1}$. Therefore, noting that $h = (2\pi)^{-1}$ gets canceled out by a factor of $2\pi$ in equation 1093 of [5], the number of photons $N$ emitted must
be around

\[ N \sim \left( \frac{D_A}{T_A} \right)^2 \]

To get \( N < 1 \) (so that Alice can recohere her original particle), we require \( D_A < T_A \). However, if this condition holds, we must also have that \( D_A T_B^2 < D^3 \) since we have \( T_B < D \). Therefore, in this case, Bob cannot distinguish between the two states in Alice’s superposition \( |L>_A \) and \( |R>_A \), so that Alice always succeeds in recovering her original pure state without violating complementarity (and thus does not know if Bob chose to release his particle from the trap, satisfying causality). On the other hand, if \( D_A > T_A \), then Alice always fails to recohere her particle, so that she again does not learn anything about Bob’s choice. We thus achieve a resolution to the electromagnetic version of Mari et al.’s gedankenexperiment [1] when we consider vacuum fluctuations in the electromagnetic field and quantized electromagnetic radiation.

### 2.2. Gravitational version.

Motivated by the above analysis, we assume the existence of vacuum fluctuations in the gravitational field and quantized gravitational radiation. We take from [2] the result that \( \delta x > \ell_p = 1 \), where \( \ell_p \) is the Planck length. As in the electromagnetic case, we appropriately define \( |\alpha_L> \) and \( |\alpha_R> \) to be the corresponding coherent states of the gravitational field. However, since we are in an almost flat spacetime (that is, \( ds^2 \approx -c^2 dt^2 + d|\vec{r}|^2 \)), the center of mass of Alice’s system must follow an invariant inertial trajectory. Therefore, when Alice initially put her particle in the superposition between \( |L>_A \) and \( |R>_A \), her laboratory would have also shifted to compensate (remember that, for the electromagnetic case, we assumed gravity to be negligible). Therefore, we must also consider the laboratory states \( |\beta_L> \) and \( |\beta_R> \), so that her superposition is of the form

\[ |\beta_L> |\beta_L> |\alpha_L> + |\beta_R> |\beta_R> |\alpha_R> \]

\[ \sqrt{2} \]

We thus note that \( |L>_A |\beta_L> \) and \( |R> \beta_R> \) have the same center of mass, so that Alice does not have a mass dipole, unlike in the electromagnetic case. Alice, however, does have a mass quadrupole \( Q_A = 2m_A(3(\ell/2)^2 - (\ell/2)^2) = m_A d^2 \). The gravitational potential of this quadrupole is

\[ V = \frac{GQ_A}{2D^3} = \frac{Q_A}{2D^3} \]

so that the specific gravity is of the order of \( Q_A/D^4 \). Therefore, similarly to the electromagnetic case, we get that

\[ \delta x \sim \frac{Q_A}{D^4} T_B^2 \]

giving us \( Q_A T_B^2 > D^4 \) as a necessary condition for Bob’s successful measurement. Similarly to how we have an extra factor of \( D \), we have extra factors of \( T_A \) in the equation for the energy radiated by Alice, so that

\[ \mathcal{E}_A \sim \left( \frac{Q_A}{T_A^3} \right)^2 T_A \]

and

\[ N \sim \left( \frac{Q_A}{T_A} \right)^2 \]
To get $N < 1$, we require $Q_A < T_A^2$. Therefore, if Alice successfully reverses her superposition, as before, we have that $Q_A T_B^2 > D^4$ since $T_B < D$, so that Bob cannot distinguish between Alice’s two states. Alice therefore succeeds in recovering her pure state regardless of what Bob does. If $Q_A > T_A^2$, then Alice fails to recohere her particle regardless of what Bob does. In either case, Alice cannot know whether or not Bob released his particle from the well, so that causality is satisfied without violating complementarity. Through this gedankenexperiment, we realize the need to incorporate vacuum fluctuations in the gravitational field and quantized gravitational radiation in any theory that attempts to fully reconcile quantum mechanics and general relativity.

3. Neutron stars and the TOV equation

For this section, we mainly follow [6]. Neutron stars are amongst the densest and most exotic objects in our universe. Along with black holes, neutron stars are the result of core-collapse (types Ib and II) supernovae, the end-of-life scenario for main-sequence stars (stars in the primary stage of their lifetime) with mass greater than $8 M_\odot$, where $M_\odot$ represents the mass of the Sun. This $8 M_\odot$ limit corresponds to an upper limit of $1.44 M_\odot$ for a white dwarf, which is the end-of-life scenario for main-sequence stars with mass less than $8 M_\odot$. This maximum mass limit of white dwarfs is known as the Chandrasekhar limit.

We now briefly describe the mechanics of core-collapse supernovae. Stars with mass above the $8 M_\odot$ have extremely high core pressure, so that not only does the star fuse hydrogen in its core (which all main-sequence stars do) but also helium through the triple-$\alpha$ process and carbon through the CNO (carbon-nitrogen-oxygen) cycle, all the way up to iron. Then, the rapid neutron capture process, or r-process, occurs, where iron nuclei in the star’s core rapidly capture free neutrons, so that $\beta^-$-decay does not have enough time to occur. This causes the core to reach a maximum density, triggering a core-collapse supernovae.

Because of the nature of core-collapse supernovae, neutron stars are extremely dense—although neutron stars tend to have masses between $1.44 M_\odot$ and $2 M_\odot$. 

![Figure 3. Rapid neutron capture process (r-process)](image-url)
they have diameter of around $10 - 30$ km, which pegs them at around the size of a large city. The upper mass limit of neutron stars is called the Tolman-Oppenheimer-Volkoff limit, which comes from the fact that main-sequence stars that are heavy enough to result in neutron stars with mass above this limit would instead collapse into a black hole. Historically, the TOV limit has varied from $1.5M_\odot - 3.0M_\odot$, but modern estimates place it at around $2M_\odot$. Because of their high density, the internal pressure of neutron stars is extremely high, to the point where typical atomic structure breaks down and where $\beta^+$ decay is forced in protons (hence the name). Neutron stars are also very energetic—they spin with a period that can vary from $10^{-3} - 10$ s and have extremely high surface temperatures, at around $10^5 - 10^6$ K.

We now take a step back and focus on stellar structure. Central to all of Newtonian astrophysics is the concept of hydrostatic equilibrium. We consider a small rectangular prism of matter within a star with base area $A$ and height $dr$. There are three forces acting on this block of matter: the force from the pressure of the material above this block of matter $-AP_{\text{above}}$, the weight of this block of matter $-\rho(r)Ag(r)dr$, and the force from the pressure of the material below this block of matter $AP_{\text{below}}$. Summing up all three of these forces and setting $dP = P_{\text{above}} - P_{\text{below}}$ gives

$$0 = A(P_{\text{below}} - P_{\text{above}} - \rho(r)g(r)dr)$$

$$\frac{dP}{dr} = -\rho(r)g(r)dr$$

which is the equation representing hydrostatic equilibrium. Generally, one would solve the equation of hydrostatic equilibrium for given boundary conditions and an equation of state, which are equations that relate state variables such as pressure, density, and temperature. For neutron stars, the same concept applies; however, since neutron stars are so dense, one must account for general relativity. These are the Tolman-Oppenheimer-Volkoff equations:

$$\frac{dP}{dr} = -(\rho + \rho/\gamma^2) \frac{Gm(r) + 4\pi Gr^3(P/\gamma^2)}{r(r - 2G(m(r)/c^2))}$$

We note the $\gamma^2$ factor present in this equation. We also observe that, when $P(r) << c^2$ and when $2Gm(r) << c^2r$, the TOV equation reduces to the equation for hydrostatic equilibrium.

4. Putting it all together: Quantum gravity in neutron star cores

In this section, we mainly follow [3]. Taking quantum gravity into account, we state a modified commutation relation between position and momentum:

$$[x, p] = i\hbar(1 + \beta p^2)$$

where $\beta = \beta_0(\ell_p^2/h^2)$ for a dimensionless constant $\beta_0$ that will indicate the region where quantum gravity begins to have a significant effect. Plugging this into the generalized uncertainty principle yields

$$\sigma_x\sigma_p \geq \frac{\hbar}{2}[1 + \beta(p^2)] = \frac{\hbar}{2}[1 + \beta(\sigma_p)^2]$$

which implies the minimal positional uncertainty $\sigma_x \geq \sqrt{\beta_0}\ell_p$. 
As $r \to 0$ in a neutron star, it is theorized that neutrons in turn decay into quarks, so we solve the TOV equations for the equation of state of an ultra-relativistic Fermi gas, which is an ideal gas composed entirely of quarks and leptons. These equations of state are

$$\frac{N}{V} = \frac{8\pi}{(hc)^3} E_H^3 f(\kappa) \quad \rho = \frac{8\pi}{c^2(hc)^3} E_H^3 h(\kappa) \quad P = \frac{8\pi}{(hc)^3} E_H^4 g(\kappa)$$

where we have

$$f(\kappa) = \frac{1}{8} \left[ \frac{\kappa(\kappa^2 - 1)}{(1 + \kappa^2)^2} + \tan^{-1}(\kappa) \right] \quad h(\kappa) = \frac{1}{4} \frac{\kappa^4}{(1 + \kappa^2)^2} \quad g(\kappa) = \kappa f(\kappa) - h(\kappa)$$

In these equations, we have that $\kappa = \varepsilon_F/E_H$, where $E_H = \sqrt{\beta}$ is called the Hagedorn energy and where $E_F$ is the Fermi energy, defined for a gas composed of $N$ fermions by

$$N = \frac{8\pi V}{(hc)^3} \int_0^{E_F} \frac{E^2 dE}{(1 + \beta E^2/c^2)^3}$$

We now make the substitutions $\tilde{r} = r/\beta_0\ell_p$, $\tilde{m} = m/m_0$, $\tilde{m} = m/m_0$, and $\tilde{\rho} = \eta/\rho_0$, where we have

$$\frac{Gm_0}{c^2\beta_0\ell_p} = 1 \quad \rho_0 = \frac{m_0}{4\pi(\beta_0\ell_p)^3} \quad P_0 = \rho_0 c^2$$

We thus obtain a modified TOV equation:

$$\frac{d\tilde{P}}{d\tilde{r}} = -\left(\tilde{\rho} + \tilde{P}\right) \tilde{m} + \frac{\tilde{r}^3 \tilde{P}}{(\tilde{r}^2 - 2\tilde{m})}$$

We now solve for the stated equation of state to obtain the following solutions for $\tilde{r} \to 0$:

$$\tilde{m}(\tilde{r}) = \frac{\tilde{r}^3}{12} \quad \kappa(\tilde{r}) = \frac{32}{\tilde{r}^2} \quad \tilde{P}(\tilde{r}) = \frac{2}{\tilde{r}^3}$$

We compare with the solutions obtained without accounting for quantum gravity (that is, when we use the equation of state $P = \eta/3c^2$):

$$\tilde{m}(\tilde{r}) = \frac{3\tilde{r}}{14} \quad \tilde{P}(\tilde{r}) = \frac{1}{14\tilde{r}^2}$$

These two solutions are fairly different! For one, we see that, as $\tilde{r} \to 0$, the mass decreases much more slowly when we account for quantum gravity. Additionally, while the pressure increases similarly, we have a much larger scalar factor when we account for quantum gravity. Therefore, we see the importance of accounting for quantum gravity in the study of neutron stars, and, with the relatively straightforward example of neutron stars (as compared to, say, black holes), we realize the significance of a fully consistent quantized theory of gravity not only for theoretical physics but also for the experimental study of exotic objects such as neutron stars and black holes.
Figure 4. Cool depiction of the layers and core of a neutron star

ACKNOWLEDGEMENTS

This paper was written for the final course project of PHYS 24300/1 [23215]: Advanced Quantum Mechanics. Accordingly, I’m grateful to Professor Savdeep Sethi for the inspiration towards this topic and for his suggestions on starting references. In general, I would like to thank Professor Sethi and our graduate student TA, Umang Mehta, for an amazing quarter.

REFERENCES