The Fractional Quantum Hall Effect

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1 Introduction

In 1980, a team of scientists lead by Klaus Von Klitzing conducted an experiment to observe the properties of a quasi-two-dimensional electron gas (2DEG). A plate shaped and probed like in the figure above is subjected to a strong magnetic field perpendicular to the plane of the plate. This experiment occurs at low temperatures. This effect yielded quantized planes of resistivity in the plate. These planes do not suffer from irregularities in the semiconductor like impurities. Moreover, the resistivity could be placed completely in terms of fundamental constants. This is particularly notable since the resulting measurements for the quantization is extremely accurate[1].

This effect has been directly responsible for two Nobel prizes. The first one was for the “Integer Quantum Hall Effect,” which was discovered in 1980[2] resulted in a Nobel prize in 1985. In addition to the Integer Quantum Hall Effect, in 1982, the “Fractional Quantum Hall Effect” was discovered by Stormer, Tsui, and Gossard[3]. However, due to the additional complexity with this phenomena, it wasn’t until 1998 that the Nobel Prize was awarded to Stormer, Tsui, and Laughlin. After the discovery of the Fractional Quantum Hall Effect, implications to quantum computing was found in the form of topological quantum computing, which have the ability to resist outside influences or “faults.”
2 The Integer Quantum Hall Effect

In the figure above, a typical plot of the bulk and edge reactivities in a hall bar[4]. As is clear in the above image, there are several plateaus. Each plateau corresponds to a quantized resistance value, which has a quantum number $\nu$. As Von Klitzing discovered, these values are in terms of physical constants as is the magnetic field in the center of these plateaus. Using the vector guide provided in the figure above, we can determine these values pretty easily by using the Classical Lagrangian\(^2\), $\mathcal{L}$.

Given $\vec{B} = B\hat{z} = \nabla \times \vec{A} \rightarrow V = e\hat{x}\vec{A}$

$\rightarrow \mathcal{L} = T - V = \frac{1}{2} \dot{x}^2 - e\hat{A}\dot{x} \rightarrow p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} - eA$

For the schrodinger equation, we need the Hamiltonian

$\rightarrow \mathcal{H} = \dot{p} - \mathcal{L} = \frac{1}{2} m\dot{x}^2 + eA\dot{x} \rightarrow \mathcal{H} = \frac{1}{2m}(p + eA)^2$

In the symmetric gauge, given $\vec{B} = B\hat{z} = \nabla \times \vec{A} \rightarrow \vec{A} = B\frac{2}{2}\hat{y}\hat{x} + B\frac{2}{2}\hat{x}\hat{y}$

Using a simple gauge transformation $\vec{A} = A^{sym} + \nabla f = B\hat{y}\hat{x}$ for $f = \frac{B}{2} xy$

$\rightarrow$ Our Quantum Hamiltonian becomes $\mathcal{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2m}(\hat{p}_y - eBx)^2$

In the y direction we have a simple free particle, so

$\hat{p}_y \psi = \hbar k \psi \rightarrow \mathcal{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2m}(\hbar k - eBx)^2 \rightarrow \mathcal{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2m}x = \frac{m eB}{m eB}$

The Cyclotron Frequency is given by $\omega_c = \frac{eB}{m} \rightarrow \mathcal{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}\omega_c^2(x - \frac{\hbar k}{m \omega_c})^2 \rightarrow \text{let } x_0 = \frac{\hbar k}{m \omega_c}$

$\rightarrow \mathcal{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}\omega_c^2(x - x_0)^2$

\(^1\)The labeling scheme is sort of backwards. The highest resistivity longer plateaus correspond to lower $\nu$ values. The rightmost one is $\nu = 2$ then 3, then 4, and so and so forth.

\(^2\)I'm using MKS units for this derivation. This is based of the solutions from problem set 5 and a bit of modification on my own part.
This Hamiltonian is quite familiar to us as it is almost identical to that of the harmonic oscillator with the only difference being that the potential is shifted by $x_0$. Starting from

$$\mathcal{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}\omega_x^2(x-x_0)^2$$

$$\rightarrow \mathcal{H}\psi = E\psi$$ Since this is just a quantum harmonic oscillator

$$E = \hbar\omega_c(n + \frac{1}{2})$$

Combining the free particle and the harmonic oscillator equation gives

$$\psi(x, y)_{n,k} = e^{iky}\psi_n(x-x_0)$$

where $\phi(x-x_0)$ is just the standard QHO formula

Next, we need to find the degeneracy of each state. Given the sheet nature of the hall bar, suppose that we define the constraints as $L_x$ by $L_y$. As is the case in general, $k$ can be quantized in units of $\frac{2\pi}{L_y}$. Additionally, $x_0$ must be less than $L_x$.

$$0 \leq x_0 \leq L_x$$

$$\rightarrow x_0 = \frac{\hbar k}{m\omega_c} = \frac{2\pi h N}{m\omega_c L_y} = \frac{2\pi h N}{eB L_y}$$

$$\rightarrow 0 \leq \frac{2\pi h N}{eB L_y} \leq L_x$$

$$\rightarrow 0 \leq N \leq \frac{eB A}{2\pi h} \text{ for } A = L_x L_y$$

Let $\phi_0 = \frac{2\pi h}{e}$.

This is called the “Quantum of Flux”

$$\rightarrow N = \frac{BA}{\phi_0} = \frac{\phi}{\phi_0}$$

This Quantum of Flux is based off of a variable $l_B^2$, which is known as the magnetic length. This can be defined from $x_0 = kl_B^2$. If this is factored in, the Quantum of Flux is the magnetic flux through a circle of radius $l_B$. We define states $\nu$, which each have space for $N$ electrons. To get the conductivity, we look at the first $\nu$ filled states. Classically, in the y direction, $I = e\dot{y}$.

Recall, $m\dot{y} = p_y + eA \rightarrow \dot{I}_y = \frac{e}{m}(-i\hbar \frac{\partial}{\partial y} + eBx) = \frac{e}{m}(\hbar k + eBx)$

$$\rightarrow <\psi|I_y|\psi> = -\frac{e}{m} \sum_{n=1}^{\nu} \sum_{k} <\psi_{n,k}|\hbar k + eBx|\psi_{n,k}>$$

From Tong’s notes this sum becomes[5]

$$-\frac{e}{m} \sum_{n=1}^{\nu} \sum_{k} <\psi_{n,k}|\hbar k + eBx|\psi_{n,k}> = evN \frac{E}{B} = evAE \phi_0$$

$$\rightarrow$$ Using Ohm’s Law $J = \frac{I}{A} = ev \frac{E}{\phi_0} = \sigma_{xy}E$

$$\rightarrow \rho_{xy} = \frac{1}{\sigma_{xy}} = \frac{\phi_0}{ev} = \frac{2\pi h}{e^2 \nu}$$

$$\rho_{xy} = \frac{2\pi h}{e^2 \nu}$$
This result is actually quite remarkable. As it turns out, this almost perfectly corresponds to the plateaus in the resistivity plot. So much so, that when Von Klitzing ran the experiment, he found for $\nu = 4 = \frac{2\pi h}{e^2}$, the sample had a resistance of $6453.17 \pm 0.02\Omega[2]$. Using, The National Institute of Standards database of physical constants[6], I got a value of $6453.20\Omega$. Even in 1980, this experiment was able to give almost exact results to the theoretical result. A more recent article[7] got a result of $12906.4035\Omega$ for $\nu = 2$ vs a theoretical result of $12906.4037\Omega$.

The next question that arises is when do these plateaus occur. In the earlier section, we found that there were several energy levels corresponding to a harmonic oscillator like wavefunction. These levels are what are called Landau Levels. It turns out that when a new level is filled, a new plateau occurs. However, at high magnetic fields, this eventually leads to a very complicated situation known as the fractional quantum hall effect.

### 3 The Fractional Quantum Hall Effect

![Figure 3: A Typical Fractional Quantum Hall Resistivity Plot[8]](image)

In 1982[3], a group of physicists lead by Tsui and Stormer discovered a $\nu = \frac{1}{3}$ state. This means they found a state in which the resistivity was given by $\frac{2\pi h}{e^2} \frac{1}{3} = \frac{3 \times 2\pi h}{e^2}$ as shown in the figure above. In general, it was discovered over time that there were several states with fractional values that had plateaus corresponding to values of quantum hall resistivity that correspond to the $\nu$’s. This was confusing as it made no sense how these states could occur since $\nu = 1$ should be the lowest possible state given the scheme found by the landau levels above. Moreover, it was quite strange since some fractions would occur and others would not. For example there was a $\nu = \frac{1}{4}$ but not a $\nu = \frac{1}{2}$. There’s a $\nu = \frac{3}{5}$, but there is no $\nu = \frac{3}{4}$. In general, it was found that no state had an even denominator.
3.1 Laughlin States

In 1983, in an attempt to explain this restriction with the information he had, Robert Laughlin attempted to find a many-body wave function for the quantum Hall states. To solve this normally, it would be quite difficult given this is a many-body quantum mechanical problem. In order to deal with this issue, Laughlin used the variational method to obtain a wave function\(^9\). In 1983, in an attempt to explain this restriction with the information he had, Robert Laughlin attempted  

\[ \mathcal{H} = \sum_{j} \left( \frac{1}{2m} \left[ \frac{\hbar}{i} \nabla_{j} - \frac{e}{c} A(r_{j}) \right]^{2} + V_{\text{ion}}(r_{j}) \right) + \sum_{j<k} v(r_{j} - r_{k}) \]

Starting with a variational Hamiltonian

In later parts, \( v(r) \) will be generic, but for this part \( v(r) \) is a coulombic interaction, so

\[ \mathcal{A}(r_{j}) = \frac{B}{2} (x_{j}y_{j} - y_{j}x_{j}) \]

Note: in any subsequent functions, the variable \( l \) is the magnetic length. (Unless it is used as a subscript)

\begin{align*}
\psi_{m}(z_{1}, ..., z_{N}) &= \prod_{j<k} (z_{j} - z_{k})^{m} \exp \left[ -\frac{1}{4l} \sum_{j<k} |z_{j}|^{2} \right] \\
\text{Note: } z_{j} &= x_{j} + iy_{j} \\
\end{align*}

In order for this to hold true, \( m \) must be odd, or in other words, fractional quantum Hall states must have an odd denominator\(^5\). That is to say

\[ \psi_{m}(z_{1}, ...z_{j}, z_{k}, ..., z_{n}) = -\psi_{m}(z_{1}, ...z_{k}, z_{j}, ..., z_{n}) \]

In order for this to hold true, \( m \) must be odd, or in other words, fractional quantum Hall states must have an odd denominator\(^5\). However, this is not the only thing this shows

Starting with the wave function, let’s find its probability distribution

\[ |\psi_{m}|^{2} = \prod_{j<k} (z_{j} - z_{k})^{m} \exp \left[ -\frac{1}{4l} \sum_{j} |z_{j}|^{2} \right]^{2} \]

\[ \text{let } \Phi = -\sum_{j<k} 2m^{2} \ln |z_{j} - z_{k}| + \frac{m}{2l^{2}} \sum_{j} |z_{j}|^{2} \]

\[ \rightarrow |\psi_{m}|^{2} = e^{-\frac{1}{4l} \Phi} \]

This result is important as it turns out the number density of a plasma satisfies the condition for a potential \( \Phi \) that

\[ n_{e} \propto e^{-q\beta \Phi} \]

\(^{3}\)Derivation is based off his Nobel Prize Lecture

\(^{4}\)While in the larger context of things, this \( \frac{1}{m} \) description might not seem useful as it doesn’t give descriptions for states like \( \nu = \frac{2}{3} \), it is important to remember that this function was derived based off of Stormer and Tsui’s paper the previous year which found a \( \nu = \frac{1}{3} \) state.

\(^{5}\)Note: in any subsequent functions, the variable \( l \) is the magnetic length. (Unless it is used as a subscript)

\(^{6}\)Note: The electron density is now \( \frac{1}{2\pi m^{2}} \).
While, this wavefunction appears to explain a good amount of the phenomena seen thus far, it has not been checked yet. In order to do this, Laughlin calculated the overlap between his wavefunction and the real wavefunction. To do so, he calculated the lowest energy state of an angular momentum of $3m$ with $V=0$ coupled with the potentials listed above. The data above was with a simulation of 3 particles. Given that the overlap for all of these are nearly one, it strongly indicates that the wavefunction above is of the correct form[10].

### 3.2 Quasiholes, Quasiparticles, and The Hierarchy Model

Now that the variational method has yielded a ground state wavefunction, the next step is to observe the excited states. For this, imagine taking an infinitely thin solenoid and using it to pierce the fluid at a point $z_0$. After this, imagine using an adiabatic process to change the magnetic flux by $\Delta \Phi = \frac{hc}{e}$. This will cause the wavefunction to shift

$$(z - z_0)^m \exp \left[ -\frac{1}{4l} |z|^2 \right] \rightarrow (z - z_0)^{m+1} \exp \left[ -\frac{1}{4l} |z|^2 \right]$$

Using this there are two approximate cases generated

**Quasihole:**

$$\psi_m^{+z_0} = A_{z_0} \psi_m = \prod_i (z_i - z_0) \prod_{j<k} (z_j - z_k)^m \exp \left[ -\frac{1}{4l} \sum_j |z_j|^2 \right]$$

**Quasiparticle:**

$$\psi_m^{-z_0} = A_{z_0}^{+} \psi_m = \prod_i \left( \frac{\partial}{\partial z_i} - \frac{z_0}{l^2} \right) \prod_{j<k} (z_j - z_k)^m \exp \left[ -\frac{1}{4l} \sum_j |z_j|^2 \right]$$

$$\rightarrow |\psi_m^{+z_0}|^2 = e^{-\beta \Phi'} \rightarrow \Phi' = \Phi - 2ln \sum_i |z_i - z_0|$$

Once again, Laughlin using a similar process as the one for the ground state, found an overlap of .998 and .994 for the $\nu = \frac{1}{5}$ and $\nu = \frac{1}{9}$ quasihole functions and .982 for the $\nu = \frac{1}{3}$ quasiparticle function. These excited states are quite important as they lead to a Hierarchy model for finding the other fractional quantum hall states[10]. The quasihole is meant as a representation of a “hole” in the electron liquid of the quantum hall state. There can be multiple quasiholes and each hole adopts a charge of $+\frac{e}{m}$. In contrast the quasiparticle is has the opposite charge of $-\frac{e}{m}$. It should be noted that while have determined that there are independent particles with fractional charge in these experiments, the total charge still adds up to integer amounts of charge as one would expect from this system[5].
The Hierarchy Model\(^\text{7}\) comes out of this quite easily. In the situation of excited states laid out above, an imaginary solenoid is used to excite the system by changing the magnetic field. In a sense, this is akin to changing the magnetic field in the quantum hall system leading to other plateaus. The only difference being is that the excited states are generated through a local change in field whereas the change in plateaus is through a uniform change. By doing so, we could imagine that some quasiholes or quasiparticles would be generated in the situation of a uniform change.

To do this, consider that there is a \(\pm\) factor for quasiparticles and quasiholes. Keeping this in mind, Halperin made a modification to the Laughlin wavefunction changing the polynomial terms so that

\[
\psi = P(z_k)Q(z_k)\exp\left[-\frac{1}{4l} \sum_j^N |z_j|^2\right]
\]

\[
\rightarrow Q(z_k) = \prod_{j<k} (z_j - z_k)^{\pm}
\]

Q represents the quasihole and quasiparticle excitation

We write P as a symmetric polynomial given by

\[
\prod_{j<k} (z_j - z_k)^{2p}
\]

These two conditions make this change as an anyon under the interchange instead of a fermion. This results in the end function of

\[
\psi = \prod_{j<k} (z_j - z_k)^{2p\pm \frac{1}{2}} \exp\left[-\frac{1}{4l} \sum_j^N |z_j|^2\right]
\]

Taking the max angular momentum to be \(N(2p \pm \frac{1}{m})\rightarrow A = 2\pi N(2p \pm \frac{1}{m})(ml^2)\).

\[
\rightarrow N = \frac{\Phi}{\Phi_0} = \frac{BA}{\Phi_0} = (2p \pm \frac{1}{m})m^2N
\]

\[
\rightarrow \nu_{\text{quasi}} = \pm \frac{1}{2pm^2 \pm m} = \frac{1}{m \pm \frac{1}{2p}}
\]

Using this equation to represent the effects of the quasiholes and quasiparticles, this allows us to get the not \(1/m\) states. For example if \(m=3\). Then, using \(p=\pm 1\), we get \(\nu = \frac{2}{3}\) and \(\nu = \frac{2}{5}\). Of course, if we keep with this logic, we should be able to cascade this by adding quasiholes and quasiparticles, which gives a final result of[13]

\[
\nu = \frac{1}{m \pm \frac{1}{2p \pm \frac{1}{2p}}}
\]

This allows us to get other states such as \(\nu = \frac{3}{7}, \frac{4}{9}\) and \(\frac{5}{11}\).

### 3.3 Composite Fermions

This leads into the last major description of the standard fractional quantum hall states. This is the composite fermion model developed by Jainendra Jain in 1989. The premise of this was developing the concept of a structure in which one fermion combines with vortex structures. With this, he was able to qualitatively explain the fractional states and provide a new wavefunction for the fractional quantum hall effect[14]. Qualitatively, this approach says that the fractional quantum hall effect is just an integer quantum hall effect with composite fermions instead of regular fermions. To see this look at the integer and fractional effect graphs overlayed.

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\(^7\)This section uses some stuff about anyons. A brief explanation as to what they are is provided in the Appendix One: Anyons.
From a cursory glance, it looks like the red lines representing the resistivity on the edges of the bar are nearly identical. The points in which there are dips, and there are plateaus corresponding these dips are also nearly identical. From this, Jain concluded that composite fermions act like fermions in the integer effect. By that, he meant that, $\nu = \frac{1}{3}$ corresponds to $\nu = 1$, $\nu = \frac{2}{5}$ corresponds to $\nu = 2$, etc. This goes on until $\nu = \frac{1}{2}$ which acts like the integer effect at zero magnetic field. It should also be noted that the reverse can be done for states for $\frac{1}{2} < \nu < 1$. Instead, $\frac{2}{3}$ corresponds to $\nu = 1$, $\frac{3}{5}$ corresponds to $\nu = 2$ etc[15].

In the above figure, the comparison between the integer and fractional is provided. In the left most part, the hall bar is at B=0, and the electrons are just floating as a Fermi sea. When the magnetic field is turned up to a state like $\nu = 3$, the landau levels are filled with fermions, which results in a resistivity plateau at that field. As the field hits $\nu = 1$, only the lowest landau level is filled, which represents the last integer effect plateau. Now, for the fraction, the $\nu = \frac{1}{2}$ represents the equivalent for B=0 and the integer effect. By that, it means that there is just a sea of these composite fermions. As one deviates from $\nu = \frac{1}{3}$ level to say $\nu = \frac{3}{7}$, a state equivalent to $\nu = 3$ occurs. Instead of fermions, the first $\nu$ landau levels are filled with composite fermions[15].
However, unfortunately, while this looks nice and all, there needs to be some proof. Luckily, Jain wrote out a trial function, which was quite accurate as shown in the above table. As shown above, the overlap in Jain’s model was precise to the third decimal place for N=8 to 12 particles. With this a good qualitative idea of the mechanics behind the Fractional Quantum Hall Effect.

3.4 Summary of the Fractional Effect

From the many-body wave function approach, a few conclusions can be made as summarized by Stormer and Tsui in 1987[11].

1. The wave function is stable at Landau-level filling factors $\nu = \frac{1}{m}$ and $\nu = 1 - 1/m$ with $m =$ odd integers.
2. It shows the existence of a liquid state of fractional charges.
3. It indicates the existence of quasiholes and quasiparticles with fractional charges.
4. For $m=10$, the quantum fluid is expected to crystallize into a solid.
5. A hierarchical model using the quasiholes and quasiparticles to act on the standard $\nu = \frac{1}{3}$ to get the other states for the fractional effect.
6. A new type of structure called a composite fermion can be used to equate the integer and fractional quantum hall effect by using filled composite fermion landau levels instead of electron filled landau levels[15]

4 Moore-Read States

In 1987[16]8, something that defies all of the previous findings in the previous parts occurred. Namely, Willet discovered an even denominator state at $\nu = \frac{5}{2}$. Obviously this significantly deviates from the results that had been found previously. Moreover, it was later discovered that there is also a $\nu = \frac{7}{2}$ state[17]. Luckily, there is an explanation that has been made for these plateaus in the form of Moore-Read states and the Pfaffian. The background for this is deeply rooted in several difference areas of study including super

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8One thing to note is that this state occurs under extremely ideal conditions. To get this the team had to drop the temperature below 100mK.
conductors. The goal of this section is to explain the states without dealing with too much of a tangent on the origins of the math outside of the scope of this paper. The first step is recognizing that another way of looking at $\nu = \frac{5}{2}$ and rewriting it as $\nu = 2 + \frac{1}{2}$. While this may seem insignificant, it is an important re-framing given that we have been equating fractional states to integer states with the composite fermion model. More importantly, this tells us we should look at the properties of the $\nu = \frac{1}{2}$ state even though there is not a plateau at this location. Luckily, in 1991, a variational function for this was already found by Gregory Moore and Nick Read[18]. To start, like with many variational Fractional Quantum Hall Effect wave functions, we start with Laughlin’s wave function, but remove the antisymmetric condition from it to accommodate the even denominator state with m=2. This gives us a preliminary term of

$$\phi_{\frac{1}{2}} = \prod_{j<k}(z_j - z_k)^2 exp \left[ -\frac{1}{4l} \sum_{j} |z_j|^2 \right]$$

We want a new function for this state given by

$$\psi_{\frac{1}{2}} = f_{\frac{1}{2}} \phi_{\frac{1}{2}}$$

For this, we are now dealing with a boson wavefunction instead of our standard fermion situation. Given this, it makes sense to look at the results of Freeman Dyson and BCS theory, which is a theory around superconductors, which end up having bosonic properties[19]. For this type of situation, Dyson that the equation for a definite number of particles incorporates a function called the Pfaffian where the Pfaffian is defined using a few parameters[5]

Consider an antisymmetric NxN matrix M

$$det(M) = Pf(M)^2$$

This relation makes it a polynomial of degree N/2 which they use to get

$$\rightarrow Pf(M) = \frac{1}{2^N (\frac{N}{2})!} \sum_{\sigma} sign(\sigma) \prod_{k} M_{\sigma(2k-1)\sigma(2k)}$$

For example, in our contexts, using four particles

$$Pf(\frac{1}{z_1 - z_2}) = \frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} + \frac{1}{z_1 - z_3} \frac{1}{z_4 - z_2} + \frac{1}{z_1 - z_4} \frac{1}{z_2 - z_3}$$

This builds up terms very fast though. For N=6, there are 12 terms, and N=8 gives 105 terms. This leaves us with a result of

$$\psi_{\frac{1}{2}} = Pf(\frac{1}{z_1 - z_2}) \phi_{\frac{1}{2}} = Pf(\frac{1}{z_1 - z_2}) \prod_{j<k}(z_j - z_k)^2 exp \left[ -\frac{1}{4l} \sum_{j} |z_j|^2 \right]$$

Now, at this point, Greiter used this to argue the existence of a $\nu = \frac{1}{2}$ state, and he stated there was a fermion liquid at this state as well[19]. This was done through numerical estimates as usual. However, given the last several years of data, the idea of a $\nu = \frac{1}{2}$ state is not particularly realistic. However, this is likely applicable to the $\nu = \frac{5}{2}$ as numerical data suggested consistency with this state. It was also determined that the $\nu = \frac{1}{2}$ state vanishes in the thermodynamic limit. However, in the case of the $\nu = \frac{5}{2}$ state, it is adiabatically connected with a spin-polarized coulomb potential, which connects it to the Moore-Read state in the thermodynamic limit. [20]. It can be found that since composite fermions all have the same spin, their differences must come from the angular momentum. This leads to the combination of the BCS theory and composite fermion theory. The idea is that the fermions form a Fermi liquid, which collapses to superconductivity due to the pair instabilities modeled by the Pfaffian[10][21][5].

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9Note: Greiter cites a result from a book on superconductors by J. R. Schrieffer for this bit

10In general BCS theory refers to unstable pairs collapsing to super conductivity, which matches this case given how the Pfaffian’s Bosonic nature mirrors that of BCS theory

11Even though it is not explicitly discussed in any of these papers, it intuitively makes that $\nu = \frac{5}{2}$ fits under the same banner as it is just a $\nu = 3 + \frac{1}{2}$

12For more information on superconductivity read Theory of Superconductivity by J. R. Schrieffer
5 Read-Rezayi States

In general, with Moore-Read states, and now, Read-Rezayi states, we’ve been looking at fractional effects past the first excited Landau level (ν = 2). In the case of Read-Rezayi states, the case Read and Rezayi were look at were \( \nu = 2 + \frac{2}{5} \) and \( \nu = 2 + \frac{3}{5} \). As one would expect given the region, this will be based off of the approach to Moore-Read states. The main difference will be in the choice of toy Hamiltonian. In the previous section, there is a property that a Moore-Read State will vanish when three or more particles come together, which yields a toy Hamiltonian of

\[
H = V \sum_{i<j<k} \delta^2(z_i - z_j)\delta(z_j - z_k)
\]

However in this case, the Hamiltonian is going to be more generalized to handle \( k+1 \) particles before the state vanishes. This means that our new Hamiltonian will be

\[
H = V \sum_{1<i_1<i_2<...<i_{k+1}} \delta^2(z_{i_1} - z_{i_2})\delta(z_{i_2} - z_{i_3})...\delta^2(z_{i_k} - z_{i_{k+1}})
\]

At this point, we can make it more generalizable by dividing into two groups. Let’s call them v and w. Next to further generalize, we switch from the Pfaffian to a general symmetrization operator \( S_n \). This allows us to work over all possible ways to divide the electrons into two groups and ensures that our wavefunction is still bosonic\([23][5]\). This yields

\[
\Psi_{MR} = S_n \left[ \prod_{i<j} (v_i - v_j)^2(w_i - w_j)^2 \right] \prod_{i<j} (z_i - z_j)^{m-1}
\]

To further generalize, rather than divide into two groups, we set \( N = pd \) where there are \( p \) groups of \( d \) particles. This gives us a general variational result of\([5][23]\)

\[
\Psi_{RR} = S_n \left[ \prod_{i<j} (v_i^{(1)} - v_j^{(1)})^2...\prod_{i<j} (v_i^{(p)} - v_j^{(p)})^2 \right] \prod_{k<l} (z_k - z_l)^{m-1}
\]

In general, this function only vanishes if \( p+1 \) particles coincide as opposed to the specific Moore-Read case, in which, it only required 3. In general with this function, if \( m \) is even, it is describing fermions, and if \( m \) is odd, it is describing bosons. This gives us a filling factor of

\[
\nu = \frac{p}{p(m-1) + 2}
\]

Using a similar composite fermion based method as with the Moore-Read States, setting \( p \) to 3 and \( m \) to 2 give \( \nu = \frac{3}{5} \). Then taking \( \nu = 1 - \frac{3}{5} \) gives \( \nu = \frac{2}{5} \). If this is added on to the \( \nu = 2 \), we get the \( \nu = 2 + \frac{2}{5} \) and \( \nu = 2 + \frac{3}{5} \) states, which we were originally interested in.

6 Non-Abelian Anyons in Read-Rezayi States

Something that hasn’t been discussed much since the Hierarchy model is the concept of quasiholes. In general we have only been dealing with the ground states of these wave functions, so what happens if we add quasiholes. Recall that in the case of the Laughlin model, the quasihole function was

\[
\psi^{+z_0}_m = A_{z_0} \psi_m = \prod_{i=1}^{N} (z_i - z_0) \prod_{j<k}^{N} (z_j - z_k)^m \exp\left[ -\frac{1}{4\ell} \sum_{j=1}^{N} |z_j|^2 \right]
\]

In this case, the idea was to put an excitation at a point \( z_0 \). In this case, since things have been divided into \( p \) groups, it would make sense that there would be \( p \) quasiholes involved. So, simply adding the holes at a
point $z_a$ gives

$$S_n \left[ \prod_{a=1}^{p} \prod_{i=1}^{N} (v_i^{(a)} - z_a)^2 \prod_{a=1}^{p} \prod_{i<j}^{N} (v_i^{(a)} - v_j^{(a)})^2 \prod_{k<l}^{m} (z_k - z_l)^{m-1} \right]$$

Now, if we recall from the Laughlin states, Fractional Quantum Hall States have factional charge $e^*$, but the total charge is still in terms of $e$ given by $q = \nu e$. Using this we get

$$e^* = \frac{\nu}{p} = \frac{1}{p(m-1) + 2}$$

Now if $p=3$, then we have $3n$ quasiholes, which as it turns out has $d_{3n-2}$ independent states where $d_i$ is an index in the Fibonacci sequence. Also, recall, anyons have fractional charge\(^{13}\), which means that our states in this system are anyons. But that’s not all, this $p=3$ is also the requirement we set for the $\nu = 2 + \frac{5}{2}$ and $\nu = 2 + \frac{3}{2}$ wave functions. This means we have anyons which follow a Fibonacci sequence in terms of number of states. These are known as Fibonacci Anyons, and they are Non-Abelian\(^5\).

## 7 Conclusion

In general, the Quantum Hall Effect is full of exotic physical phenomena. It has directly resulted in two Nobel Prizes. The Fractional Quantum Hall Effect in particular has been a subject of numerous experiments and theories due to its mysterious nature. It has multiple interesting features such as a resistivity entirely in physical constants, which translates to highly accurate experimental vs. theoretical data. In the case of the fractional effect, there are new states of matter, fractional charge, and anyons. Even today, this problem is technically unsolved as the Moore-Read states and Read-Rezayi states are not actually proven results. So, the $\nu = \frac{5}{2}$ state is not definitively a solved problem yet. While these elements may seem arbitrary or just some neat things that some physicists found, the Quantum Hall Effect has a very important application in quantum computing. In the last section, we discussed the idea of Fibonacci anyons in the Read-Rezayi states. It turns out the Non-Abelian properties of these anyons allow for one to create topological quantum computers. These are quantum computers which have systems that are more resistant to outside influences than regular quantum computing using what are known as braid topologies. They also have the added benefit of still being capable of doing anything a regular quantum computer can do \(^{24}\). Unfortunately, due to the sensitive nature of the Moore-Read and Read-Rezayi states in terms of temperature conditions, a topological qubit has not been made, so Topological Quantum Computing only exists as a theoretical construct at the moment.

## 8 Appendix One: Anyons

The premise behind Anyons is quite simple. In general, we say that particles are either bosons or fermions. Mathematically, that means that

**Bosons:** $\psi(1, 2) = \psi(2, 1) \rightarrow$ Symmetric

**Fermions:** $\psi(1, 2) = -\psi(2, 1) \rightarrow$ Antisymmetric

However, another way to look at this is that for bosons, $\psi(1, 2) = e^{i \pi m} \psi(2, 1) = e^{i \pi 2} \psi(2, 1)$, and for fermions, $\psi(1, 2) = e^{i \pi} \psi(2, 1)$. Then in general, we could define particles such that

Anyons: $\psi(1, 2) = e^{i \pi \alpha} \psi(2, 1)$ for $\alpha \neq a \in \mathbb{Z}$

Now, consider a quasihole. If an anyon traverses a closed path $c$ in the quasihole, it will give a shift of $e^{i \gamma}$ for $\gamma = \frac{e^\phi}{m}$, which is given by the area of $\frac{1}{2 \pi m}$. This is just the phase shift given by the The Aharonov–Bohm Effect\(^5\). This means that the generic anyon has a charge of $\frac{e}{m}$.

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\(^{13}\)See Appendix One: Anyons
9 Appendix Two: The Aharonov–Bohm Effect

In general for gauge transformations in the classical sense

\[ \phi' = \phi - \frac{\partial \Lambda}{\partial t} \quad \text{and} \quad A' = A + \nabla \Lambda \]

The main goal of this is to illustrate the results of the discoveries by Aharonov and Bohm. In 1959, they discovered that a vector potential can affect a particle in a quantum system even if the particle is in a region where the field is zero.

In the general A-field case

\[ \mathcal{H} = \frac{1}{2m}(\hat{p} - qA)^2 + q\phi \]

In order to satisfy this and apply a gauge transformation

\[ \psi' = e^{i\frac{q\Lambda}{\hbar}} \psi \]

In the figure above[12], a solenoid with a current I and radius a is orbited by a bead at radius b. For a long solenoid, B is uniform inside the solenoid and the field outside is zero, but the vector potential outside is non-zero. It adopts the standard condition \( \nabla \cdot A = 0 \), and \( A = \frac{\Phi}{\pi a} \hat{\phi} \). This is letting \( \Phi = B \ast A = \pi a^2 B \).

Since the solenoid itself is uncharged, so the scalar potential, \( \phi = 0 \). Putting all of this together gives

\[ \frac{d^2\psi}{d\phi^2} - 2i\beta \frac{d\psi}{d\phi} + \epsilon \psi = 0 \]

For \( \beta = \frac{q\Phi}{2\pi \hbar} \) and \( \epsilon = \frac{2mb^2E}{\hbar^2} - \beta^2 \)

Which gives solutions in the form of \( \psi = Ae^{i\lambda\phi} \)

\[ \text{for } \lambda = \beta \pm \sqrt{\beta^2 + \epsilon} = \beta \pm \frac{b}{\hbar} \sqrt{2mE} = n \]

\[ \rightarrow E_n = \frac{\hbar^2}{2mb^2} \left( n - \frac{q\Phi}{2\pi \hbar} \right) \]

The most important conclusion of this is that despite the particle being outside the field, the particle’s
energies are still dependent on the field. Now let’s look at a static A field.

Using the same Hamiltonian

\[ H = \left( \frac{1}{2m} (-i\hbar \nabla - qA)^2 \right) \psi = i\hbar \frac{\partial \psi}{\partial t} \]

This can be made even simpler by changing the wavefunction to

\[ \psi = e^{iz\psi'} \quad \text{for} \quad g(r) = \frac{q}{\hbar} \int r A(r') \, dr' \quad \text{for arbitrary point} \ O \]

This yields a simple

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi' = i\hbar \frac{\partial \psi'}{\partial t} \]

Figure 9: The Setup Proposed by Aharonov and Bohm

Above is an experiment proposed by Aharonov and Bohm. This experiment entailed shooting a beam at a long solenoid, which results the beam splitting in two around the solenoid and then recombining. Since they only experience an A potential and not a B field, the beams split into two different phases. This results in

\[ g = \frac{q}{\hbar} \int A \, dr = \pm \frac{q\Phi}{2\hbar} \]

\[ \rightarrow \text{The phase difference is} \quad \frac{q\Phi}{\hbar} \]

This result is quite important with regards to anyons as it shows that they have fractional charge[12].

References


14This whole section is based off of the 3rd edition of Griffiths’ section on the The Aharonov–Bohm Effect


