Notes on Matrices and Linear Transformations

Physics 221

Let’s recall that vectors are objects independent of which coordinates we choose to describe them. For example, the vector $v$ depicted below exists regardless of the coordinates that you or I might choose to describe it.

So let’s imagine two coordinate systems. The first is the familiar $\hat{i}, \hat{j}$ system where

\[ \hat{i} = (1, 0), \quad \hat{j} = (0, 1). \] (1)

With respect to this basis, $v$ is described by coordinates $(2, 3)$ or

\[ v = 2\hat{i} + 3\hat{j}. \] (2)

On the other hand, an equally good basis is given by the two linearly independent vectors $e_1, e_2$. What are the coordinates in this basis? To answer this question, we need to determine the linear transformation $S$ that relates the two bases. First we observe that

\[ e_1 = 4\hat{i} + 2\hat{j}, \quad e_2 = -\hat{i} + 3\hat{j}. \] (3)

Note that the dot product $e_1 \cdot e_2 = -4 + 6 = 2$ so this basis does not consist of orthogonal vectors unlike our starting basis. You can see this in the picture.

We can also invert this relation to solve for $(\hat{i}, \hat{j})$ in terms of $(e_1, e_2)$:

\[ \hat{i} = \frac{3e_1 - 2e_2}{14}, \quad \hat{j} = \frac{4e_2 + e_1}{14}. \] (4)
What this teaches us is that a general vector expressed in the \((\hat{i}, \hat{j})\) basis has coordinates
\[
v = \alpha_1 \hat{i} + \alpha_2 \hat{j}
\] (5)
which we can express in the \((e_1, e_2)\) basis using (4) as follows
\[
v = \alpha_1 \hat{i} + \alpha_2 \hat{j} = \alpha_1 \frac{3e_1 - 2e_2}{14} + \alpha_2 \frac{4e_2 + e_1}{14}
= \left( \alpha_1 \frac{3}{14} + \alpha_2 \frac{1}{14} \right) e_1 + \left( -\alpha_1 \frac{2}{14} + \alpha_2 \frac{4}{14} \right) e_2
= \beta_1 e_1 + \beta_2 e_2.
\] (6)
This relates the \(\alpha\) coordinates to \(\beta\) coordinates with respect to the new basis.

Let’s write this as a linear transformation \(S\) acting on \(\alpha\) to give \(\beta\)
\[
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\end{pmatrix} = \frac{1}{14} \begin{pmatrix}
3 & 1 \\
-2 & 4 \\
\end{pmatrix} \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\end{pmatrix}.
\] (7)
This change of basis matrix determines everything. First it must have non-zero determinant since it maps a basis to a basis and hence must be invertible. It’s easy to check that \(\det(S) = 1/14\). This means that \(S\) is an element of the group \(GL(2, \mathbb{R})\) of invertible \(2 \times 2\) matrices.

The inverse is also easy to determine either by using the general form for the inverse of a \(2 \times 2\) matrix or by inspection:
\[
S^{-1} = \begin{pmatrix}
4 & -1 \\
2 & 3 \\
\end{pmatrix}.
\] (8)
So \(v\) is described with respect to the \(\alpha\) coordinates by \((2, 3)\) but using (7) we see that \(v\) is also described in the \(\beta\) coordinates by \((9/14, 8/14)\).

Once you have \(S\), you can determine the matrix representing a linear transformation in this new basis. For example, suppose the linear transformation \(T\) is represented by a \(2 \times 2\) matrix \(M\) in the \((\hat{i}, \hat{j})\) basis so that
\[
T \cdot v \rightarrow M \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\end{pmatrix}.
\] (9)
Well, we can determine how \(T\) is represented in the \((e_1, e_2)\) basis by inserting the identity matrix in a clever way:
\[
M \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\end{pmatrix} = M (S^{-1}S) \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\end{pmatrix} = (MS^{-1}) S \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\end{pmatrix}
= \left( MS^{-1} \right) \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\end{pmatrix}.
\] (10)
Now this still gives the output result with respect to the \(\alpha\) coordinates which we need to convert to \(\beta\) coordinates using (7). This amounts to another application of \(S\) so with respect to the \((e_1, e_2)\) basis, we see that \(T\) is represented by
\[
T \rightarrow SMS^{-1}.
\] (11)
This is conjugation of $T$ by $S$. Conjugation amounts to a change of basis which is why we are interested in the issue of whether matrices can be diagonalized by conjugation.