

# Notes on the Dirac Delta and Green Functions

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## 1 The Dirac Delta

One can not really discuss what a Green function is until one discusses the Dirac delta “function.” There are different ways to define this object. I will first discuss a definition that is rather intuitive and then show how it is equivalent to a more practical and useful definition.

### 1.1 Intuitive Definition

The Dirac delta “function,” denoted  $\delta(x - x')$ , is not really a function in the conventional sense, but it can be defined as the *limit* of a *sequence of functions*,  $(f_n)_{n=1}^{\infty}$ . The  $f_n(x)$  can not be totally random functions, of course. They must satisfy two key properties. (1) They must be such that their limit goes to zero for all  $x \neq x'$  and goes to  $\infty$  at  $x = x'$ . (2) The integral of all of the  $f_n(x)$  (over whatever space we are considering) must be equal to 1. Thus

$$\delta(x - x') = \lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0, & x \neq x' \\ \infty, & x = x' \end{cases}, \quad (1)$$

where the  $f_n(x)$  are such that

$$\int_{-\infty}^{\infty} dx f_n(x) = 1, \quad \forall n. \quad (2)$$

Here I assumed we are in one dimension, so I integrated over the whole real line. One can define the Dirac delta,  $\delta(\mathbf{x} - \mathbf{x}')$ , in arbitrary dimensions by making the  $f_n(\mathbf{x})$  such that their integral over the total space in question is equal to 1. Note that you should think of  $x$  as the variable, while  $x'$  is some fixed position.

Now, there are still many sequences of function that have these required properties. Here I will list two. There are the box functions

$$f_n(x) = \begin{cases} n, & |x - x'| \leq \frac{1}{2n} \\ 0, & |x - x'| > \frac{1}{2n} \end{cases} \quad (\text{box functions}) \quad (3)$$

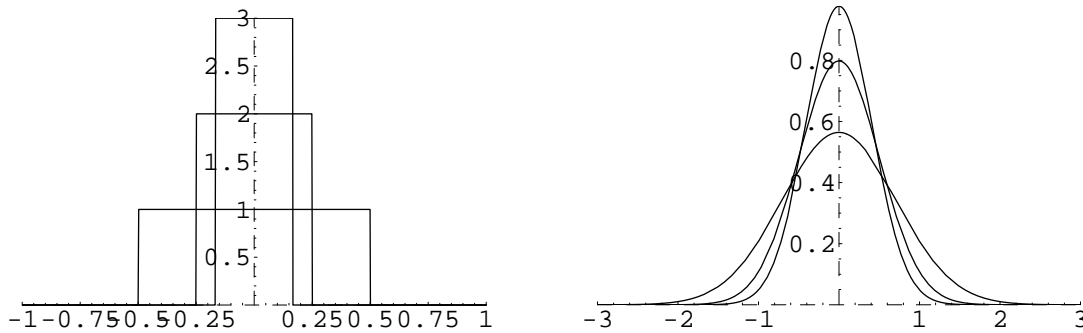


Figure 1: The first three functions of the box function sequence and the Gaussian sequence.

which get progressively taller and skinnier around  $x = x'$  in such a way that the area of the box is always 1; and there is a sequence of Gaussians

$$f_n(x) = \sqrt{\frac{n}{\pi}} e^{-n(x-x')^2} \quad (\text{Gaussians}) \quad (4)$$

which become more and more peaked around  $x = x'$ . See Figure 1 for a plot of the first few functions in each of these sequences, where I take  $x' = 0$  so that they are centered about the origin. In the case of the Gaussians, you can think of the width of the Gaussian as going like  $1/\sqrt{n}$ , while the height is going like  $\sqrt{n}$ .

Though there are many different sequences that have the limit (1), they all define the same object  $\delta(x - x')$ . To show the equivalence between this definition and the more practical definition in the following subsection, I will use the box functions because they are simple to work with.

## 1.2 Practical Definition

The Dirac delta can also be defined as a map from *functions* to *numbers*, that acts in the following way: If  $g(x)$  is some arbitrary function then

$$g \mapsto \int_{-\infty}^{\infty} dx \delta(x - x') g(x) = g(x'). \quad (5)$$

In words, the Dirac delta,  $\delta(x - x')$ , takes a function  $g$  to the number  $g(x')$ . The way in which it acts is via the integral<sup>1</sup>. The equality on the right-hand-side of this statement is not “automatic,” but rather *follows* from the definition of  $\delta(x - x')$  given in the previous subsection. What I am saying is that, writing “ $g \mapsto \int_{-\infty}^{\infty} dx \delta(x - x') g(x)$ ” alone already defines a map from functions to numbers, since the result of doing a definite integration is always a number. However, this definition by itself

<sup>1</sup>As an aside, such a map is a particular type of functional called a “distribution” in mathematics. This is what the Dirac delta really is. Note that any function  $f(x)$  can be thought of as a distribution, since I can always consider the map  $g \mapsto \int f(x)g(x)$ , but distributions are more general and include things that are not functions, such as the Dirac delta.

is not very useful unless you know how to actually evaluate what the number is—ie do the integral. I will now show that the definition in the previous subsection enables one to evaluate the integral and obtain the claimed result.

We have

$$\int_{-\infty}^{\infty} dx \delta(x - x') g(x) = \int_{-\infty}^{\infty} dx (\lim_{n \rightarrow \infty} f_n(x)) g(x) = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dx f_n(x) g(x), \quad (6)$$

where I will take  $f_n(x)$  to be the box functions, and I assumed that it was kosher to interchange the processes of integration and taking the limit. This assumption boils down to an assumption that  $g(x)$  is a “nice enough” function around  $x = x'$ . (We will assume this since it is always the case of interest in physics, though one can generalize slightly to allow  $g(x)$  to be non-smooth or even have a finite discontinuity at  $x'$ ). Then plugging in what  $f_n(x)$  is,

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dx f_n(x) g(x) = \lim_{n \rightarrow \infty} \left( n \int_{x' - \frac{1}{2n}}^{x' + \frac{1}{2n}} dx g(x) \right). \quad (7)$$

Now, since  $g(x)$  is assumed to be a nice function at  $x = x'$ , we can Taylor expand it around that point:

$$g(x) = g(x') + g'(x')(x - x') + \frac{1}{2} g''(x')(x - x')^2 + \dots. \quad (8)$$

Let us plug this in and integrate term by term. Changing variables to  $y = x - x'$  we have

$$\begin{aligned} \int_{x' - \frac{1}{2n}}^{x' + \frac{1}{2n}} dx g(x) &= \int_{x' - \frac{1}{2n}}^{x' + \frac{1}{2n}} dx (g(x') + g'(x')(x - x') + \frac{1}{2} g''(x')(x - x')^2 + \dots) \\ &= \int_{-\frac{1}{2n}}^{\frac{1}{2n}} dy (g(x') + g'(x')y + \frac{1}{2} g''(x')y^2 + \dots) \\ &= \left( g(x')y + \frac{1}{2} g'(x')y^2 + \frac{1}{6} g''(x')y^3 + \dots \right) \Big|_{-\frac{1}{2n}}^{\frac{1}{2n}} \\ &= \frac{1}{n} g(x') + 0 + g''(x') \frac{1}{24n^3} + \dots, \end{aligned} \quad (9)$$

where the terms in the last line involve higher and higher powers of  $1/n$ . Plugging this result back into (7) we find

$$\begin{aligned} \int_{-\infty}^{\infty} dx \delta(x - x') g(x) &= \lim_{n \rightarrow \infty} \left[ n \left( \frac{1}{n} g(x') + \frac{1}{24n^3} g''(x') + \dots \right) \right] \\ &= g(x') \end{aligned} \quad QED. \quad (10)$$

As a corollary, note the following useful result. By taking  $g(x)$  to be the constant function  $g(x) = 1, \forall x$ , we learn that

$$\int_{-\infty}^{\infty} dx \delta(x - x') = 1. \quad (11)$$

This is also intuitively clear from the fact that  $\delta(x - x')$  is the limit of a sequence of functions that also has this property.

I call (5) the practical definition because this is how the Dirac delta is typically used in physics. Also, this is the definition you want to apply in order to do problem 3 on the last homework set. Specifically, suppose we have two different expressions involving Dirac delta functions. Let us denote these expressions  $\mathcal{F}(\delta)$  and  $\mathcal{G}(\delta)$ . In the homework,

$$\mathcal{F}(\delta) = \delta(ax), \quad \mathcal{G}(\delta) = \frac{1}{|a|}\delta(x), \quad (12)$$

where  $a$  is a real number not equal to zero. Then in order to show  $\mathcal{F}(\delta) = \mathcal{G}(\delta)$  you must show

$$\int_{-\infty}^{\infty} dx \mathcal{F}(\delta)g(x)dx = \int_{-\infty}^{\infty} \mathcal{G}(\delta)g(x)dx, \quad (13)$$

where  $g(x)$  is an arbitrary function. This is done by explicitly evaluating each side, using the rule (5), and showing that you get the same result.

### 1.3 Physical Interpretation

The Dirac delta function is used in physics to represent a “point source.” An example comes from electrostatics. A continuous charge distribution in 3-dimensional space is described by a *charge density*, typically denoted  $\rho(\mathbf{x})$ . The total charge of the distribution is given by integrating the charge density of all of space:  $Q = \int d^3x \rho(\mathbf{x})$ . Now suppose that I have a single *point charge*,  $q$ , at position  $\mathbf{x}'$ . What is the charge density of a point charge? Well, it should be zero everywhere except at  $\mathbf{x} = \mathbf{x}'$ , since there is no charge anywhere except at this point. On the other hand, at  $\mathbf{x}'$ , we have a finite charge in an infinitely small volume, so the density should be infinite there. Finally, it must satisfy  $q = \int d^3x \rho(\mathbf{x})$ , since  $q$  is the total charge. These requirements are uniquely satisfied by

$$\rho(\mathbf{x}) = q\delta(\mathbf{x} - \mathbf{x}') \quad (\text{charge density of a point charge } q \text{ at } \mathbf{x}'). \quad (14)$$

One would have similar expressions for the (mass) density of a point mass  $m$ .

## 2 Green Functions

### 2.1 Definition

The question “What is the Green function?” only makes sense after you’ve been given two items. The first item is a *linear differential operator*. An example of a linear differential operator is the Laplacian  $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ , or minus the Laplacian. (This, in particular, is the Laplacian in three dimensions). The second item of information is the *boundary conditions* of the functions on which the linear operator acts. Once you have been given both of these items, you can ask a well

defined question: “What is the Green function associated with a given operator,  $\mathcal{L}$ , and a given set of boundary conditions?” The answer is the following: The Green function  $G(x, x')$  is the solution to

$$\mathcal{L}G(x, x') = \delta(x - x'), \quad (15)$$

satisfying the given boundary conditions. Since  $\mathcal{L}$  is a differential operator, this is a differential equation for  $G$  (or a partial differential equation if we are in more than one dimension), with a very specific source term on the right-hand-side: the Dirac delta function. Note again that  $x$  is the variable while  $x'$  is a parameter, the position of the point source. When we write “ $G(x, x')$ ” we are indicating that the Green function will be a function of the variable  $x$ , and it will also depend on the parameter  $x'$ .

As a specific example, consider the question on the homework set. The given operator is

$$\mathcal{L} = -\nabla^2 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}. \quad (16)$$

This operator acts on functions  $\phi(x, y, z)$  defined in a cube of sides  $L$  that satisfy the boundary conditions

$$\phi(0, y, z) = \phi(L, y, z) = 0, \quad \partial_y \phi(x, 0, z) = \partial_y \phi(x, L, z) = 0, \quad \phi(x, y, 0) = \phi(x, y, L) = 0. \quad (17)$$

Then the Green function  $G(\mathbf{x}, \mathbf{x}')$ , (where  $\mathbf{x} = (x, y, z)$ ,  $\mathbf{x}' = (x', y', z')$ ), is the solution to the partial differential equation

$$-\frac{\partial^2 G}{\partial x^2} - \frac{\partial^2 G}{\partial y^2} - \frac{\partial^2 G}{\partial z^2} = \delta(\mathbf{x} - \mathbf{x}'), \quad (18)$$

which satisfies the boundary conditions (17).

There are three good questions that you could ask at this point. (1) How do you actually obtain the Green function? It is defined as the solution to some partial differential equation, so this is equivalent to the question of how do we solve such an equation. (2) What is the physical interpretation/significance of the Green function? (3) What is it useful for? I will briefly answer the latter two questions first and then return to the question (1), which is the most relevant for the homework set.

## 2.2 Physical Significance

The laws of physics are written in terms of differential equations—Newton’s Law is a differential equation for the position of an object as a function of time, Maxwell’s equations are differential equations for the electric and magnetic fields as functions of space and time, etc. Generally the quantity we are interested in is some function  $u$  and the physics tells us which differential operator,  $\mathcal{L}$ , to consider. The right-hand-side of the differential equation  $\mathcal{L}u(x) = f(x)$  is the *source term*  $f(x)$ . It is the force if we are dealing with Newton’s laws and the charge and current densities if we

are dealing with electrodynamics. It is usually considered as a given. When the source term is a Dirac delta function, the quantity of interest,  $u$ , becomes the Green function of the operator  $\mathcal{L}$ . In other words, *the Green function characterizes the response of a system to the presence of a point source*.

As an example, consider electrostatics. The laws of physics tell us that the divergence of the electric field  $\mathbf{E}$  is proportional to the charge density  $\rho(\mathbf{x})$ :  $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho(\mathbf{x})$ . Furthermore the electric field can be expressed as the gradient of an electric potential  $V(\mathbf{x})$ :  $\mathbf{E} = -\nabla V(\mathbf{x})$ . Putting these together we find

$$-\nabla^2 V(\mathbf{x}) = \frac{1}{\epsilon_0} \rho(\mathbf{x}). \quad (19)$$

The differential operator is  $-\nabla^2$ , the source is the charge density, and the quantity of interest is the electric potential. Now, if we let our source be a point charge  $q$  at position  $\mathbf{x}'$ , then from what we said earlier,  $\rho(\mathbf{x}) = q\delta(\mathbf{x} - \mathbf{x}')$ , and so the potential generated by a point charge at position  $\mathbf{x}'$ ,  $V_{p.c.}(\mathbf{x}, \mathbf{x}')$ , satisfies

$$-\nabla^2 \left( \frac{\epsilon_0}{q} V_{p.c.}(\mathbf{x}, \mathbf{x}') \right) = \delta(\mathbf{x} - \mathbf{x}'). \quad (20)$$

Hence, the electric potential generated by a point charge *is the Green function of  $-\nabla^2$* ! (up to some constants of proportionality).

## 2.3 Usefulness

The usefulness of the Green function is evident once you make the following realization. *Any* distribution of source (ie charge density for instance) can be written as a sum, or integral in the continuous case, of point sources. Therefore, if we know how the system reacts to a point source, then we should be able to determine how it reacts to any distribution of source, since we can sum up all the contributions. Note it is absolutely critical here that the differential operator is linear—that's the whole point.

To make this concrete, suppose we have some linear differential equation (and specified boundary conditions) for an unknown function  $u(x)$  with an arbitrary source term:

$$\mathcal{L}u(x) = f(x). \quad (21)$$

Now suppose that we are handed the Green function  $G(x, x')$  corresponding to  $\mathcal{L}$  and the specified boundary conditions. Then I claim that the solution to this equation is:

$$u(x) = \int dx' G(x, x') f(x'). \quad (22)$$

Note that we are integrating over the parameter  $x'$ , so the result of the integration will indeed be a function of  $x$ . To see that this is the solution we use the well established method of plug-in-and-check:

$$\mathcal{L}u = \mathcal{L} \int dx' G(x, x') f(x') = \int dx' (\mathcal{L}G(x, x')) f(x'). \quad (23)$$

In this step we have made use of the following two facts. First,  $\mathcal{L}$  is a differential operator with respect to the variable  $x$ —it doesn't care about  $x'$  so we can move it in past the integral over  $x'$ . Second, only the Green function depends on the variable  $x$ , because I am evaluating the source function at  $x'$ . Now use the property (or rather definition) of the Green function:  $\mathcal{L}G(x, x') = \delta(x - x')$ , and then let the Dirac delta function do what it was born to do:

$$\mathcal{L}u = \int dx' \delta(x - x') f(x') = f(x). \quad (24)$$

So it works. By uniqueness of solutions to such differential equations with specified boundary conditions, we can conclude that (22) is *the* solution.

Thus, once you have the Green function, you can immediately find the solution corresponding to any source term, just by doing an integral, which is much easier than solving the differential equation directly. Now that we've decided the Green function is a good thing to have, it's time to turn to the question of how to get it.

## 2.4 Constructing the Green Function

There are different methods of solving the (partial) differential equation  $\mathcal{L}G(x, x') = \delta(x - x')$ . The method I present here is the one that will be useful for the second question on the problem set. It makes use of the eigenvalues and eigenfunctions of the operator  $\mathcal{L}$  to construct the Green function. In order for this method to be useful, we need to restrict the class of operators that we consider. Henceforth I will assume that our operator  $\mathcal{L}$ , with our specified boundary conditions<sup>2</sup>, is *Hermitian*, and that it has *no zero eigenvalues*.

Switching now to the bra-ket notation, we denote the *normalized* eigenfunctions of  $\mathcal{L}$  as  $|\phi_k\rangle$  and the corresponding eigenvalues as  $\lambda_k$ . The eigenfunctions and eigenvalues are obtained by solving the (partial) differential equation  $\mathcal{L}|\phi\rangle = \lambda|\phi\rangle$  and imposing the boundary conditions. For example, on the homework this amounts to solving (by the separation of variables technique)

$$-\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = \lambda \phi, \quad (25)$$

and imposing the boundary conditions (17). This method leads to a discrete (though infinite) set of allowed eigenvalues  $\lambda_k$  and eigenfunctions  $\phi_k(x, y, z)$ . Here the index  $k$ , as you should find, runs over all possible triplets of natural numbers:  $k = (n_x, n_y, n_z)$ , where  $n_x, n_y, n_z \in \mathbb{N}$ . Note also that you will need to normalize the eigenfunctions that you find by requiring  $\langle \phi_k | \phi_k \rangle = 1$ . For the problem on the homework, the innerproduct is  $\langle f | g \rangle = \int_{cube} d^3x f^* g$ .

Now then, let us suppose that we have obtained all of the (normalized) eigenfunctions and allowed eigenvalues, and let us try to solve  $\mathcal{L}G(x, x') = \delta(x - x')$ . Since  $\mathcal{L}$  is Hermitian, the spectral theorem guarantees that the eigenfunctions form an orthonormal basis. Hence *any* function satisfying the given boundary conditions can be expanded as a sum of eigenfunctions and furthermore

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<sup>2</sup>Recall from from the last problem on homework set 4 that determining whether or not a differential operator is Hermitian depends crucially on the boundary conditions of the functions on which it acts.

$\langle \phi_k | \phi_m \rangle = \delta_{km}$ . In particular, we can expand both the Green function and Dirac delta function:

$$|G\rangle = \sum_k g_k |\phi_k\rangle, \quad |\delta\rangle = \sum_k d_k |\phi_k\rangle. \quad (26)$$

Here  $g_k, d_k$  are constants—all  $x$  dependence is in the eigenfunctions  $\phi_k(x)$ . However,  $g_k, d_k$  can depend on the parameter  $x'$ . In fact, using orthonormality, we can solve for  $d_k$ :

$$\langle \phi_m | \delta \rangle = \langle \phi_m | \sum_k d_k |\phi_k\rangle = \sum_k d_k \langle \phi_m | \phi_k \rangle = \sum_k d_k \delta_{mk} = d_m. \quad (27)$$

Hence

$$d_k = \langle \phi_k | \delta \rangle = \int dx \phi_k(x)^* \delta(x - x') = \phi_k(x')^*. \quad (28)$$

Here we have used the definition of the innerproduct and then the property of the Dirac delta function to do the integral. On the other hand, by plugging in the expansions (26) into the differential equation we have

$$\mathcal{L}\left(\sum_k g_k |\phi_k\rangle\right) = \sum_k d_k |\phi_k\rangle. \quad (29)$$

Now on the left use the linearity of  $\mathcal{L}$  and the fact that the  $|\phi_k\rangle$  are eigenfunctions to obtain

$$\mathcal{L}\left(\sum_k g_k |\phi_k\rangle\right) = \sum_k g_k \mathcal{L}|\phi_k\rangle = \sum_k g_k \lambda_k |\phi_k\rangle. \quad (30)$$

By equating this with the right-hand-side, and using the *linear independence* of the  $|\phi_k\rangle$ , we conclude that

$$\lambda_k g_k = d_k, \quad \forall k \quad \text{or} \quad g_k = \frac{1}{\lambda_k} d_k, \quad \forall k. \quad (31)$$

Here we are using the assumption that none of the  $\lambda_k$  are zero. Plugging in what  $d_k$  is from above, we have just solved for the coefficients of the Green function:

$$g_k = \frac{1}{\lambda_k} \phi_k(x')^*. \quad (32)$$

Thus the Green function itself is given by

$$|G\rangle = \sum_k \frac{1}{\lambda_k} \phi_k(x')^* |\phi_k\rangle, \quad \text{or} \quad G(x, x') = \sum_k \frac{1}{\lambda_k} \phi_k(x')^* \phi_k(x). \quad (33)$$

So, if you know the allowed eigenvalues and corresponding normalized eigenfunctions of a linear operator  $\mathcal{L}$ , equation (33) gives the Green function.