Automorphic Black Hole as Probes of Extra Dimensions

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Basic Question

- **Vague question**: How much information about the structure of spacetime can be extracted from black holes?

- **More concrete question**: Given a black hole what geometric information can be extracted from its entropy?

This question leads to

- **Two leitmotifs**: automorphic black hole entropy and automorphic spacetime.
LM1: Automorphic Black Holes

In recent years explicit examples of automorphic black holes have been discovered:

- Early start by Dijkgraaf-Verlinde-Verlinde in 1996;
- Jatkar-Sen05 extended this work to prime CHL\(_p\) models.
- Extension of Jatkar-Sen05 by Govindaraja-Krishna09 to non-prime CHL\(_N\) models.

This is good for several reasons:
Automorphic forms are highly restricted, forming finite dimensional function spaces.
Hence a finite amount of computation determines the whole function.
The second leitmotif involves automorphic geometry.

- The problem of whether geometry is automorphic in some sense was dates back to Felix Klein in the late 19th century.

- It was made popular in the 1950s by Taniyama and Shimura for elliptic curves. Solved for thes CYs of dimension 1 by Wiles, Taylor and others in the 1990s in the context of the proof of the Fermat conjecture.

- The web of Langlands conjectures suggests a Galois interpretation of automorphic forms & raises the hope that automorphy generalizes, but no proof is in sight in higher dimensions.
In general the rank of the cohomology is too high, leading to automorphic forms on high rank groups: $H^r(X)$ may be automorphic, but is in general not an irreducible object.

Needed are subgroups $H(M) \subset H^r(X_n)$ spanned by some irreducible "geometric particles" $M$ (irreducible motives).

Constructive strategy: consider examples of automorphic Calabi-Yau varieties via the construction of certain types of motives. RS 0603/0812; Kadir-Lynker-RS1012
Universal CY examples: $\Omega$–motives $M_\Omega$ are associated to the holomorphic $n$–forms of CY varieties:

$$H(M_\Omega) \subset H^n(X_n)$$

Modular $\Omega$–motives of CYs lead to forms $f(M_\Omega, q)$ of weight

$$w(f(M_\Omega, q)) = \dim_{\mathbb{C}} X + 1.$$ 

In general $\Omega$–motives are automorphic, associated to $GL(n)$, $n > 2$. 
LM2: Automorphic Varieties

String theoretic extension of Langlands:

- Whenever physics provides automorphic forms we can ask whether we can construct a geometry that induces these forms.

- Emergent spacetime application:
  Consider automorphic forms on the string worldsheet $\Sigma$ and construct Calabi-Yau varieties $X$ induced by those automorphic forms. [RS hep-th0603/0812]

This program allows the explicit construction of CYs $X$ from the worldsheet $\Sigma$ via

$$\{f_i(\Sigma, q)\} \rightarrow \{f(M_j, q)\} \rightarrow X.$$
Automorphic black holes

Black hole application:
Given that both black holes and string compactification varieties are determined by automorphic forms (at least sometimes) we can ask the natural

**Question:**
What, if anything, do automorphic black holes encode about automorphic geometry (motives)?

The goal of this talk is to explore this question in the context of an explicit class of models.
CHL\textsubscript{N} Models

- **Concrete framework:** CHL\textsubscript{N} models Chaudhuri-Hockney-Lykken 95

\[
\text{CHL}_N : \text{Het}(T^6/\mathbb{Z}_N) \cong \text{IIA}(K3 \times T^2/\mathbb{Z}_N).
\]

- **Duality group:** \( G_N = \Gamma_1(N) \times O(6, r_N - 6, \mathbb{Z}) \), with \( \Gamma_1(N) \subset \text{SL}(2, \mathbb{Z}) \) defined by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \text{ mod } N.
\]

- **Electric and magnetic charges of rank \( r_N \):** \((Q_e, Q_m)\)

- **T-duality invariants:** \(Q_e^2, Q_m^2, Q_e \cdot Q_m\)

- **S-duality invariant:** \(Q_e^2 Q_m^2 - (Q_e Q_m)^2\)
CHL Models

- Charge conjugate variables of the Siegel upper half plane:

\[(\tau, \sigma, \rho) \cong Z = \begin{pmatrix} \tau & \rho \\ \rho & \sigma \end{pmatrix} \in \mathcal{H}_2\]

- Automorphic forms for CHL$_N$ black holes are Siegel modular form of genus 2

\[\Phi^N \in S_w(\Gamma_0^{(2)}(N))\]

of weight

\[w = \left\lceil \frac{24}{N + 1} \right\rceil - 2.\]

- The level $N$ is determined by the congruence subgroups

\[\Gamma_0^2(N) \subset \text{Sp}(4, \mathbb{Z})\]

that are acting on the Siegel upper halfplane $\mathcal{H}_2$. 
For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(4, \mathbb{Z})$ the form $\Phi^N$ transforms as

$$\Phi(MZ) = \det(CZ + D)^w \Phi^N(Z).$$

The Fourier coefficients of the Siegel form partition function

$$\frac{1}{\Phi^N(\tau, \sigma, \rho)} = \sum_{k, \ell, m} d^N(k, \ell, m) q^k r^\ell s^m$$

with $q = e^{2\pi i \tau}$, $r = e^{2\pi i \sigma}$, $s = e^{2\pi i \rho}$ and

$$(k, \ell, m) = \left( \frac{1}{2} Q_e^2, \frac{1}{2} Q_m^2, Q_e Q_m \right).$$
The degeneracies $d^N(Q_e, Q_m)$ lead to the microscopic entropy

$$S^N_{\text{micro}}(Q_e, Q_m) = \ln \tilde{d}^N(Q_e, Q_m).$$

in terms of multiplicities $\tilde{d}^N$ related to the $d^N$. Jatkar-Sen hep-th/0510
The Question Reformulated

The basic question can now be formulated in a more concrete way:

*Are the Siegel modular forms associated to (motives of) the compact dimensions?*

The answer is that naively there seems to be no relation.

Siegel modular forms are quite intricate, and are difficult to deal with in a motivic/cohomological/geometric context.

But: the Siegel modular forms of CHL$_N$ are not of the most general Siegel type — they are lifts of ordinary modular forms.
The Maaß-Skoruppa lift

Key structure: CHL\(_N\) black hole Siegel forms \(\Phi^N(\tau, \sigma, \rho)\) can be constructed from ordinary modular forms \(f^N(\tau)\)

\[
S_{w+2}(\Gamma_1(N)) \ni f^N(\tau) \xrightarrow{\text{MS}} \Phi^N(\tau, \sigma, \rho) \in S_w(\Gamma_0^{(2)}(N))
\]

via a 2-step construction.

1\(^{\text{st}}\) Step: the Skoruppa lift.
Construct Jacobi forms via a map that lifts ordinary cusp forms \(f_{w+2}\) to 2-variable forms on \(\mathcal{H} \times \mathbb{C}\), where \(\mathcal{H}\) is the upper half plane

\[
S_{w+2}(\Gamma_1(N)) \ni f_{w+2}^N(\tau) \xrightarrow{\text{SL}} \varphi_w(\tau, \sigma) \in J_w
\]

with \(\Gamma_1(N) \subset \text{SL}(2, \mathbb{Z})\) as defined earlier.
The Maaß-Skoruppa lift

The Skoruppa lift is constructed via a special form

\[ K(\tau, \sigma) = \frac{\psi_1(\tau, \sigma)}{\eta^3(\tau)}, \]

where \( \psi_1(\tau, \sigma) \) is a particular Jacobi form and \( \eta(\tau) \) is the Dedekind function:

\[ f_{w+2}^N \rightarrow \varphi_w^N := K^2 f_{w+2}^N \]

The resulting Jacobi form can be expanded as

\[ \varphi_w^N(q, r) = \sum_{k, \ell} c_w^N(k, \ell) q^k r^\ell = \sum_{k, \ell} c_w^N(4k - \ell^2) q^k r^\ell. \]
The Maaß-Skoruppa lift

2nd Step: the Maaß lift

For any (weak) Jacobi form $\varphi_N^w(\tau, \sigma)$ one can define the additive lift as

$$\Phi_N^w(\tau, \sigma, \rho) = \sum_{k, \ell, m} g^N(k, \ell, m)e^{2\pi i (k\tau + \ell\sigma + m\rho)},$$

where $\chi$ is the Legendre symbol $(\pm 1)$ with

$$g^N(k, \ell, m) = \sum_{d | (k, \ell, m)} \chi(d)d^{w-1}c^N\left(\frac{k\ell}{d^2}, \frac{m}{d}\right).$$

A redefinition of variables leads to

$$\tilde{\Phi}_N(\tilde{\tau}, \tilde{\sigma}, \tilde{\nu})$$

and coefficients $g(m, n, r)$. 
The Maaß-Skoruppa lift

Combining the Skoruppa lift from ordinary modular forms to Jacobi forms

\[ f^N(q) \xrightarrow{SL} \varphi^N(q, r) \]

with the Maaß lift

\[ \varphi^N(q, r) \xrightarrow{ML} \Phi^N(q, r, s) \]

leads to the Maaß-Skoruppa lift of ordinary modular forms to Siegel modular forms of genus 2

\[ f^N(q) \xrightarrow{SL} \varphi^N(q, r) \xrightarrow{ML} \Phi^N(q, r, s). \]
The Maaß-Skoruppa lift

Key property of the CHL$_N$ Maaß-Skoruppa lift:

The only ingredient of the construction of $\tilde{\Phi}_N$ that is sensitive to the structure of the CHL$_N$ model is the ordinary modular form

$$f^N_{w+2}(\tau) \in S_{w+2}(\Gamma_1(N))$$

determined by the symmetry group $\mathbb{Z}_N$ of the CHL$_N$ model.

Calling this form the Maaß-Skoruppa root of the Siegel modular form $\Phi^N$, we can ask a better

**Question:**

What geometric/motivic information is contained in the Maaß-Skoruppa roots of the CHL$_N$ black hole Siegel forms, if any:

$$f^N_{w+2}(\tau) \overset{?}= f(M, q).$$
Motivic modular forms

To answer this question we have to recall something about motivic modular forms.

- The $\Omega$–motives $M_\Omega$ with

\[ H(M_\Omega) \subset H^n(X_n) \]

are the only universal motives associated to any Calabi-Yau manifold in any dimension.

- It is therefore natural to ask about the modular properties of these objects. If the $\Omega$–motive is modula the weight $w(f)$ of the associated motivic modular form $f(M_\Omega, q)$ is given by

\[ w(f(M_\Omega, q)) = \dim_{\mathbb{C}} X + 1. \]
\( f^N \) is not naturally motivic

Considering the weights

\[
w + 2 = \left\lceil \frac{24}{N + 1} \right\rceil
\]

of the Maaß-Skoruppa lifts shows that these forms have weights in the range

\[
w + 2 \in \{3, \ldots, 12\},
\]

in general incompatible with the weight of CY threefold.

The weight obstruction to a direct geometric interpretation leads to the following
A further reduction

**Question.**
Can the black hole modular form

\[ f^N(q) \in S_{w+2}(\Gamma_1(N)), \quad w + 2 = \left\lceil \frac{24}{N + 1} \right\rceil \]

given by the Maaß-Skoruppa root be constructed in terms of simpler forms that admit a motivic/geometric interpretation?

The answer to this is yes, involving two different types of lifts for these modular forms because they fall into two different classes:

- Class I: Modular forms \( f^N \) without complex multiplication
- Class II: Modular forms \( f^N \) with complex multiplication
Class I elliptic lifts

**Claim.**
For each Maaß-Skoruppa root $f^N(q) \in S_{w+2}(\Gamma_1(N))$ of type I (without complex multiplication) there exists an modular cusp form of weight two

$$f_2^{(\tilde{N})}(q) \in S_2(\Gamma_1(\tilde{N})), $$

of some level

$$\tilde{N} = \tilde{N}(N)$$

such that the Maaß-Skoruppa root is given by

$$f^N(q) = f_2^{\tilde{N}}(q^{1/m})^m.$$ 

Here

$$m = \frac{1}{2} \left\lceil \frac{24}{N+1} \right\rceil.$$
The geometric/motivic interpretation is then obtained by elliptic motives such that

$$H^1(M) = H^1(E)$$

since for each form $f_2^\tilde{N}(q) \in S_2(\Gamma_1(\tilde{N}))$ of this type there exists an elliptic curve $E_{\tilde{N}}$ of conductor $\tilde{N}$ such that

$$f_2^\tilde{N}(q) = f(E_{\tilde{N}}, q).$$
Consider the case of CHL$_3$ with Maaß-Skoruppa root

\[ f^3(q) = \eta(q)^6 \eta(q^3)^6 \in S_6(\Gamma_1(3)) \]

The elliptic form of weight two is given by

\[ f_2^3(q) = \eta(q^3)^2 \eta(q^9)^2 \]

and

\[ m = \frac{1}{2} \left\lceil \frac{24}{4} \right\rceil = 3 \quad \text{and} \quad \tilde{N} = 27. \]

Furthermore there exists an elliptic curve $E_{27}$ of conductor $\tilde{N} = 27$ such that

\[ f_2^3(q) = f_2(E_{27}, q). \]
Class II reduction

The forms of type II admit complex multiplication, i.e. they are admit a symmetry.

► One reflection of this symmetry is that the modular forms $f^N(q)$ are more sparse than generally possible: their $q$–expansions

$$f^N(q) = \sum_n a_n q^n$$

vanish at certain prime exponents $p \in \mathbb{N}$.

► The class of primes for which this happens depends on the symmetry algebra, which can be viewed as the ring of algebraic integers

$$\mathcal{O}_K \subset K = \mathbb{Q}(\sqrt{-D})$$

of some imaginary quadratic field $K$. 
Class II reduction

- The key consequence of CM by some imaginary field $K$ is that the modular form is determined by so-called Hecke character $\Psi_K$ associated to $K$. These are maps defined on the set $\mathcal{I}_m$ of (fractional) ideals prime to an integral ideal $m \subset \mathcal{O}_K$

$$\Psi : \mathcal{I}_m \longrightarrow \mathbb{C}$$

- Class II reduction is best described in terms of the L-series associated to modular forms $f(q)$:

$$f(q) = \sum_n a_n q^n \longrightarrow L(f, s) = \sum_n \frac{a_n}{n^s}.$$
Class II reduction

- Associated to Hecke characters $\Psi_K$ are Hecke L-series

$$L(\psi, s) = \sum_n \frac{b_n(\psi)}{n^s},$$

where the coefficients $b_n$ are determined by the ideals of the field $K$

$$b_n(\psi) = \sum_{(a, m) = 1, N\alpha = n} \psi(a).$$

Claim.

For each CHL$_N$ model of class II there exists an algebraic Hecke character $\Psi^N$ such that the L-series $L(f^N, s)$ of the Maaß-Skoruppa roots $f^N$ are given by

$$L(f^N, s) = L(\Psi^N, s).$$
Example of class II reduction

Example:
For $\text{CHL}_N$ with $N = 4$ the Maaß-Skoruppa root is given by

$$f^4(\tau) = \eta(\tau)^4 \eta(2\tau)^2 \eta(4\tau)^4 \in S_5(\Gamma_0(4), \chi_{-1})$$

This modular form can be constructed in terms of an algebraic Hecke character $\Psi_4$ associated to the Gauss field

$$K = \mathbb{Q}(\sqrt{-1})$$

with

$$L(f^4, s) = L(\Psi_4^4, s).$$

The character itself $\Psi_4$ is elliptic, associated to the elliptic curve $E_{32}$ of conductor 32

$$L(E_{32}, s) = L(\Psi_4, s).$$
Black holes as motivic probes

The result:
All Maaß-Skoruppa roots $f^N(\tau)$ of the CHLN black hole Siegel modular forms $\Phi^N(\tau, \sigma, \rho)$ can be constructed from modular cusp forms of weight 2 associated to elliptic curves $E$.

This means that the black hole entropy of the black holes considered so far ”sees” only an elliptic structure.

In turn this means that these black holes provide probes that are not fully sensitive to the compactification manifold of the CHL models

$$X = K3 \times E / \mathbb{Z}_N.$$
More sensitive probes

The Siegel modular forms constructed by Jatkar-Sen and Govindarajan-Krishna apply only to black holes with special charges that satisfy the torsion constraint Dabholkar-Gaiotto-Nampuri0702

\[ t = \gcd\{Q^i_e Q^j_m - Q^i_e Q^j_m| i, j = 1, ..., n\} = 1 \]

It would be interesting to check whether black holes with more general charges, leading to

\[ t > 1 \]

lead to more refined probes

\[ t > 1 \quad \rightarrow \quad \Phi^N_t(\tau, \sigma, \rho) \quad \rightarrow \quad f^{BH}_t(q) \]
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