Gravity as the Square of Gauge Theory

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based mainly on arXiv:1004.0693 with Zvi Bern, Tristan Dennen, Yu-tin Huang
### KLT relations

- **4-point:**

\[ \mathcal{M}_4(1, 2, 3, 4) = -is_{12}A_4(1, 2, 3, 4)\tilde{A}_4(1, 2, 4, 3). \]

- **5-point:**

\[ \mathcal{M}_5(1, 2, 3, 4, 5) = is_{12}s_{34}A_5(1, 2, 3, 4, 5)\tilde{A}_5(2, 1, 4, 3, 5) \]
\[ + is_{13}s_{24}A_5(1, 3, 2, 4, 5)\tilde{A}_5(3, 1, 4, 2, 5). \]
KLT relations

- **4-point:**

\[ \mathcal{M}_4(1, 2, 3, 4) = -i s_{12} A_4(1, 2, 3, 4) \tilde{A}_4(1, 2, 4, 3). \]

- **5-point:**

\[ \mathcal{M}_5(1, 2, 3, 4, 5) = i s_{12} s_{34} A_5(1, 2, 3, 4, 5) \tilde{A}_5(2, 1, 4, 3, 5) \]
\[ + i s_{13} s_{24} A_5(1, 3, 2, 4, 5) \tilde{A}_5(3, 1, 4, 2, 5). \]

- **n-point?**
KLT relations, \( n \)-point

\[
\mathcal{M}_n(1, 2, \ldots, n) = i(-)^{n+1} \left[ A_n(1, 2, \ldots, n) \sum_{\text{perms}} f(i_1, \ldots, i_j) \bar{f}(l_1, \ldots, l_j) \right. \\
\left. \tilde{A}_n(i_1, \ldots, i_j, 1, n-1, l_1, \ldots, l_j, n) \right] \\
+ \mathcal{P}(2, \ldots, n-2).
\]

with

\[
\{i_1, \ldots, i_j\} \in \mathcal{P}(2, \ldots, \lfloor n/2 \rfloor), \quad \{l_1, \ldots, l_j\} \in \mathcal{P}(\lfloor n/2 \rfloor + 1, \ldots, n-2).
\]

and

\[
f(i_1, \ldots, i_j) = s_{1,i_j} \prod_{m=1}^{j-1} \left( s_{1,i_m} + \sum_{k=m+1}^{j} g(i_m, i_k) \right),
\]

where \( g(i, j) = s_{ij} \) for \( i > j \) and \( g(i, j) = 0 \) otherwise.
Problem

KLT relations in this form express the unordered gravity amplitude in terms of color-ordered gauge theory amplitudes!
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Possible Solutions

- Use “ordered” gravity amplitudes
  ⇒ Drummond, Spradlin, Volovich, Wen [arXiv:0901.2363]
Problem

KLT relations in this form express the unordered gravity amplitude in terms of color-ordered gauge theory amplitudes!

Possible Solutions

- Use “ordered” gravity amplitudes
  ⇒ Drummond, Spradlin, Volovich, Wen [arXiv:0901.2363]
- Use full unordered gauge-theory amplitude
  ⇒ Bern, Carrasco, Johansson [arXiv:0805.3993]
How to Square Gauge Theory?

BCJ duality

The gauge theory amplitude can be written as

$$A_n = \sum_{\text{diags. } i} \frac{n_i c_i}{\prod s_{\alpha_i}},$$

where

- $n_i$ are the numerators
- $c_i$ are the cubic vertices
- $s_{\alpha_i}$ are the denominators

Jacobi-like relations ("BCJ duality")

$$c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0.$$
The gauge theory amplitude can be written as

\[ A_n = \sum_{\text{diags. } i} \frac{n_i c_i}{\prod s_{\alpha_i}}, \]

- diagrams \( i \) only contain cubic vertices:

\[ + \]

\[ + \]

\[ + \ldots \]
BCJ duality

The gauge theory amplitude can be written as

\[ \mathcal{A}_n = \sum_{\text{diags. } i} \frac{n_i c_i}{\prod s_{\alpha_i}} , \]

- diagrams \( i \) only contain cubic vertices:

\[
\begin{align*}
\text{Diagram 1} & \quad + \quad \text{Diagram 2} & \quad + \quad \ldots \\
\end{align*}
\]

- numerators \( n_i \) satisfy Jacobi-like relations ("BCJ duality"):

\[ c_i + c_j + c_k = 0 \quad \Rightarrow \quad n_i + n_j + n_k = 0 . \]
conjectured BCJ squaring relations

Gauge theory amplitudes

\[ A_n = \sum_{\text{diags. } i} \frac{n_i c_i}{\prod s_{\alpha_i}}, \quad \tilde{A}_n = \sum_{\text{diags. } i} \frac{\tilde{n}_i c_i}{\prod s_{\alpha_i}} \]

with numerators satisfying Jacobi-like relations:

\[ c_i + c_j + c_k = 0 \implies n_i + n_j + n_k = 0, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0. \]
conjectured BCJ squaring relations

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\[ A_n = \sum_{\text{diags. } i} \frac{n_i c_i}{\prod s_{\alpha_i}}, \quad \tilde{A}_n = \sum_{\text{diags. } i} \frac{\tilde{n}_i c_i}{\prod s_{\alpha_i}} \]

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Gravity amplitude:

\[ -iM_n = \sum_{\text{diags. } i} \frac{n_i \tilde{n}_i}{\prod s_{\alpha_i}}. \]
conjectured BCJ squaring relations

Gauge theory amplitudes

$$A_n = \sum_{\text{diags. } i} \frac{n_i c_i}{\prod s_{\alpha_i}}, \quad \tilde{A}_n = \sum_{\text{diags. } i} \frac{\tilde{n}_i c_i}{\prod s_{\alpha_i}}$$

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$$c_i + c_j + c_k = 0 \implies n_i + n_j + n_k = 0, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0.$$ 

Gravity amplitude:

$$-i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i \tilde{n}_i}{\prod s_{\alpha_i}}.$$ 

Why do these squaring relations hold?

What are the implications?
Important related work

Stringy approach/generalizations of BCJ

- Bjerrum-Bohr, Damgaard, Vanhove [0907.1425]
- Stieberger, Mafra, Schlotterer [0907.2211, 1104.5224]
- Tye, Zhang [1003.1732]
- Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove [1003.2396, 1003.2403]
- Bern, Dennen [1103.0312]

Applications of Squaring Relations at loop level

- Bern, Carrasco, Johansson [1004.0476]
- Vanhove [1004.1392]
1. Generalized Gauged Invariance

2. Field Theory Derivation of the Squaring Relations

3. The Squaring Relations from a Lagrangian Viewpoint

4. BCJ at loop level
1. Generalized Gauged Invariance

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Generalized Gauge Invariance

**Generalized Gauge Transformations**

Gauge theory amplitude

\[ \mathcal{A}_n = \sum_{\text{diags. } i} \frac{n_i c_i}{\prod s_{\alpha_i}} \]

is invariant under

\[ n_i \rightarrow n_i + \Delta_i \]

with

\[ \sum_{\text{diags. } i} \frac{\Delta_i c_i}{\prod s_{\alpha_i}} = 0. \]
Generalized Gauge Invariance

Generalized Gauge Transformations

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with

\[ \sum_{\text{diags. } i} \frac{\Delta_i c_i}{\prod s_{\alpha_i}} = 0. \]

- \( \Delta_i \) “move around” contact terms, can be local or non-local
- Preserves Jacobi-like relations if

\[ \Delta_i + \Delta_j + \Delta_k = 0. \]
Generalized Gauge Invariance

The Squaring Relations

\[ n_i + n_j + n_k = 0, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \quad \Rightarrow \quad -i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i \tilde{n}_i}{\prod s_{\alpha_i}}. \]
Generalized Gauge Invariance

The Squaring Relations

\[ n_i + n_j + n_k = 0, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \quad \Rightarrow \quad -i \mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i \tilde{n}_i}{\prod s_{\alpha_i}}. \]

Generalized Gauge Transformation of the Squaring Relations

\[ n_i \rightarrow n_i + \Delta_i \quad \text{with} \quad \sum_{\text{diags. } i} \frac{\Delta_i c_i}{\prod s_{\alpha_i}} = 0, \quad \Delta_i + \Delta_j + \Delta_k = 0 \]

Gravity amplitude transforms as

\[ -i \mathcal{M}_n \rightarrow -i \mathcal{M}_n + \sum_{\text{diags. } i} \frac{\Delta_i \tilde{n}_i}{\prod s_{\alpha_i}}. \]
Generalized Gauge Invariance

The Squaring Relations

\[ n_i + n_j + n_k = 0, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \quad \Rightarrow \quad -iM_n = \sum_{\text{diags. } i} \frac{n_i \tilde{n}_i}{\prod s_{\alpha_i}}. \]

Consistency Identity

If \( \Delta_i, \tilde{n}_i \) satisfy Jacobi-like relations:

\[
\sum_{\text{diags. } i} \frac{\Delta_i c_i}{\prod s_{\alpha_i}} = 0 \quad \Rightarrow \quad \sum_{\text{diags. } i} \frac{\Delta_i \tilde{n}_i}{\prod s_{\alpha_i}} = 0.
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Generalized Gauge Invariance

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\]

Origin: \( c_i \) are color factors any gauge group

\[ \Rightarrow \text{identity only relies on algebraic properties of } c_i \]

\[ \Rightarrow \text{must work for } c_i \rightarrow \tilde{n}_i \]
Generalized Gauge Invariance

The Squaring Relations

\[ n_i + n_j + n_k = 0, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \quad \Rightarrow \quad -iM_n = \sum_{\text{diags. } i} \frac{n_i \tilde{n}_i}{\prod s_{\alpha_i}}. \]

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If \( \Delta_i, \tilde{n}_i \) satisfy Jacobi-like relations:

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\sum_{\text{diags. } i} \frac{\Delta_i c_i}{\prod s_{\alpha_i}} = 0 \quad \Rightarrow \quad \sum_{\text{diags. } i} \frac{\Delta_i \tilde{n}_i}{\prod s_{\alpha_i}} = 0.
\]

Origin: \( c_i \) are color factors any gauge group
\( \Rightarrow \) identity only relies on algebraic properties of \( c_i \)
\( \Rightarrow \) must work for \( c_i \rightarrow \tilde{n}_i \)
\( \Rightarrow \) \( \Delta_i \) actually do not need to satisfy Jacobi-like relations!
1. Generalized Gauged Invariance

2. Field Theory Derivation of the Squaring Relations

3. The Squaring Relations from a Lagrangian Viewpoint

4. BCJ at loop level
Deriving the Squaring Relations

\[ n_i + n_j + n_k = 0, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \quad \Rightarrow \quad -iM_n = \sum_{\text{diags. } i} \frac{n_i \tilde{n}_i}{\prod \alpha_i}. \]
Deriving the Squaring Relations

\[ n_i + n_j + n_k = 0, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \quad \Rightarrow \quad -i \mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i \tilde{n}_i}{\prod s_{\alpha_i}}. \]

Strategy

- Squaring relations trivial at 3-point:
  
  \[ -i \mathcal{M}_3 = A_3 \times \tilde{A}_3. \]
Deriving the Squaring Relations

\[ n_i + n_j + n_k = 0, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \quad \Rightarrow \quad -i \mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i \tilde{n}_i}{\prod s_{\alpha_i}}. \]

Strategy

- Squaring relations trivial at 3-point:
  \[ -i \mathcal{M}_3 = A_3 \times \tilde{A}_3. \]
- Proceed inductively, using on-shell recursion relations

\[ \mathcal{A}_n = \sum_{\alpha} \hat{A}_L \frac{i}{s_{\alpha}} \hat{A}_R, \quad \mathcal{M}_n = \sum_{\alpha} \hat{M}_L \frac{i}{s_{\alpha}} \hat{M}_R, \]

\[ \begin{array}{c}
\hat{A}_L \\
\downarrow \quad s_{\alpha} \quad \uparrow \\
\hat{A}_R
\end{array} \]
Deriving the Squaring Relations

\[ n_i + n_j + n_k = 0, \quad \bar{n}_i + \bar{n}_j + \bar{n}_k = 0 \quad \Rightarrow \quad -i \mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i \bar{n}_i}{\prod s_{\alpha_i}}. \]

Assumptions

- A local choice of \( n_i \) exists such that

\[ \mathcal{A}_n = \sum_{\text{diags. } i} \frac{n_i c_i}{\prod s_{\alpha_i}}, \quad n_i + n_j + n_k = 0. \]
Deriving the Squaring Relations

\[ n_i + n_j + n_k = 0, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \implies -i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i \tilde{n}_i}{\prod s_{\alpha_i}}. \]

Assumptions

- A local choice of \( n_i \) exists such that
  \[ \mathcal{A}_n = \sum_{\text{diags. } i} \frac{n_i c_i}{\prod s_{\alpha_i}}, \quad n_i + n_j + n_k = 0. \]

- Complex on-shell deformations of momenta
  \[ p_a \to \hat{p}_a(z) = p_a + zq_a, \quad p_a \cdot q_a = q_a^2 = 0 \]
  exist such that
  \[ \hat{\mathcal{M}}_n(z) \to 0, \quad \hat{\mathcal{A}}_n(z) \to 0, \quad \hat{\mathcal{A}}_n(z) \to 0 \quad \text{as} \quad z \to \infty. \]

(BCFW particularly suitable)
Deriving the Squaring Relations

\[ n_i + n_j + n_k = 0, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \quad \Rightarrow \quad -i \mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i \tilde{n}_i}{\prod s_{\alpha_i}}. \]

Outline of derivation

- Express gravity and gauge-theory amplitudes using recursion relation
- Use lower-point squaring relations for subamplitudes
Deriving the Squaring Relations

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Crucial manipulation of gravity amplitude:

\[
\mathcal{M}_n = \sum_{\alpha} \frac{i}{s_{\alpha}} \sum_{\alpha\text{-diags. } i} \left[ \frac{\hat{n}_i(z_{\alpha})\tilde{h}_i(z_{\alpha})}{\prod \hat{s}_{\alpha_i}(z_{\alpha})} - \frac{\Delta_i^{\alpha} \hat{n}_i(z_{\alpha}) + \tilde{\Delta}_i^{\alpha} \tilde{n}_i(z_{\alpha})}{\prod \hat{s}_{\alpha_i}(z_{\alpha})} + \frac{\Delta_i^{\alpha} \tilde{\Delta}_i^{\alpha}}{\prod \hat{s}_{\alpha_i}(z_{\alpha})} \right].
\]
Deriving the Squaring Relations

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\]

\[ \Delta_i^\alpha, \tilde{\Delta}_i^\alpha \text{ are generalized gauge transformations:} \]

\[
\sum_{\alpha\text{-diags. } i} \frac{\Delta_i^\alpha c_i}{\prod \hat{s}_{\alpha_i}(z_{\alpha})} = 0
\]
Deriving the Squaring Relations

\[ n_i + n_j + n_k = 0, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \quad \Rightarrow \quad -iM_n = \sum_{\text{diags. } i} \frac{n_i \tilde{n}_i}{\prod s_{\alpha i}}. \]

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M_n = \sum_{\alpha} \frac{i}{s_{\alpha}} \sum_{\alpha\text{-diags. } i} \left[ \frac{\hat{n}_i(z_\alpha)\hat{n}_i(z_\alpha)}{\prod \hat{s}_{\alpha_i}(z_\alpha)} - \frac{\Delta_{\alpha} \hat{n}_i(z_\alpha) + \tilde{\Delta}_{\alpha} \hat{n}_i(z_\alpha)}{\prod \hat{s}_{\alpha_i}(z_\alpha)} + \frac{\Delta_{\alpha} \tilde{\Delta}_{\alpha}}{\prod \hat{s}_{\alpha_i}(z_\alpha)} \right].
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\[ \Delta_{\alpha} , \tilde{\Delta}_{\alpha} \] are generalized gauge transformations:

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\]

\[ \Delta_{\alpha} , \tilde{\Delta}_{\alpha} \]
Deriving the Squaring Relations

\[ n_i + n_j + n_k = 0, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \quad \Rightarrow \quad -i \mathcal{M}_n = \sum_{\text{diags.} i} \frac{n_i \tilde{n}_i}{\prod s_{\alpha_i}}. \]

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\[ \mathcal{M}_n = \sum_{\alpha} \frac{i}{s_{\alpha}} \sum_{\text{\textalpha-diags.} i} \frac{\hat{n}_i(z_{\alpha}) \hat{n}_i(z_{\alpha})}{\prod \hat{s}_{\alpha_i}(z_{\alpha})}. \]

\[ \Delta^\alpha_i, \tilde{\Delta}^\alpha_i \text{ are generalized gauge transformations:} \]

\[ \sum_{\text{\textalpha-diags.} i} \frac{\Delta^\alpha_i c_i}{\prod \hat{s}_{\alpha_i}(z_{\alpha})} = 0 \quad \Rightarrow \quad \sum_{\text{\textalpha-diags.} i} \frac{\Delta^\alpha_i \hat{n}_i(z_{\alpha})}{\prod \hat{s}_{\alpha_i}(z_{\alpha})} = 0 \]
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\[ \Rightarrow \text{consistency identity implies squaring relations!} \]
1. Generalized Gauged Invariance

2. Field Theory Derivation of the Squaring Relations

3. The Squaring Relations from a Lagrangian Viewpoint

4. BCJ at loop level
Motivation

- Amplitudes computed from the ordinary YM Lagrangian do not satisfy Jacobi-like relations!
- Can Jacobi-like relations arise from a Lagrangian?
- In what sense is $\mathcal{L}_{\text{gravity}} = (\mathcal{L}_{\text{gauge}})^2$? 

$$n_i + n_j + n_k = 0$$
Ordinary $\mathcal{L}_{YM}$ does not lead to BCJ-compatible amplitudes
Ordinary $\mathcal{L}_{YM}$ does not lead to BCJ-compatible amplitudes

Strategy

- Expand gauge theory Lagrangian as
  \[ \mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_5 + \mathcal{L}_6 + \ldots \]
- Determine $\mathcal{L}_n$, $n \geq 5$ to make Jacobi-like relations manifest
Ordinary $\mathcal{L}_{YM}$ does not lead to BCJ-compatible amplitudes

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- $\mathcal{L}_n$, $n \geq 5$ must not alter amplitudes!
Ordinary $\mathcal{L}_{YM}$ does not lead to BCJ-compatible amplitudes

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  \]
- Determine $\mathcal{L}_n$, $n \geq 5$ to make Jacobi-like relations manifest
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- Use auxiliary fields to turn Lagrangian cubic
Ordinary $\mathcal{L}_{YM}$ does not lead to BCJ-compatible amplitudes

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- Determine $\mathcal{L}_n, \ n \geq 5$ to make Jacobi-like relations manifest
- $\mathcal{L}_n, \ n \geq 5$ must not alter amplitudes!
- Use auxiliary fields to turn Lagrangian cubic
- Square cubic interactions in momentum space $\Rightarrow \mathcal{L}_{\text{gravity}}$
The Squaring Relations from a Lagrangian Viewpoint

5-point

- No covariant, local $\mathcal{L}_5$ can ensure Jacobi
5-point

- No covariant, local $\mathcal{L}_5$ can ensure Jacobi
- Instead:

$$\mathcal{L}_5 = -\frac{1}{2} g^{-3} \left( f^{a_1 a_2 b} f^{b a_3 c} + f^{a_2 a_3 b} f^{b a_1 c} + f^{a_3 a_1 b} f^{b a_2 c} \right) f^{c a_4 a_5}$$

$$\times \partial_{[\mu} A^{a_1}_{\nu]} A^{a_2}_{\rho} A^{a_3\mu} \frac{1}{\Box} (A^{a_4\nu} A^{a_5\rho}).$$

- non-local and vanishing by Jacobi-identity
No covariant, local $L_5$ can ensure Jacobi

Instead:

$$L_5 = - \frac{1}{2} g^3 (f_{a_1 a_2 b} f_{ba_3 c} + f_{a_2 a_3 b} f_{ba_1 c} + f_{a_3 a_1 b} f_{ba_2 c}) f_{ca_4 a_5}$$

$$\times \partial_{[\mu} A_{a_1}^{a_1} A_{a_2}^{a_2} A_{a_3}^{a_3} A_{\mu}^{1} (A_{a_4}^{a_4} A_{a_5}^{\rho}) .$$

- non-local and vanishing by Jacobi-identity
- can add one “self-BCJ” term:

$$\Delta L_5 \propto g^3 f_{a_1 a_2 b} f_{ba_3 c} f_{ca_4 a_5} \left( \partial_{(\mu} A_{a_1}^{a_1} A_{a_2}^{a_2} A_{a_3}^{a_3} A_{\mu)}^{1} + \partial_{(\mu} A_{a_2}^{a_2} A_{a_3}^{a_3} A_{a_4}^{a_4} A_{a_5}^{\rho}) .

\times \partial_{(\mu} A_{a_3}^{a_3} A_{a_4}^{a_4} A_{a_5}^{a_5} A_{\mu)}^{1} (A_{a_4}^{a_4} A_{a_5}^{\rho}) .$$
The Squaring Relations from a Lagrangian Viewpoint

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- No covariant, local $L_5$ can ensure Jacobi
- Instead:

$$L_5 = -\frac{1}{2} g^3 \left( f^{a_1 a_2 b} f^{b a_3 c} + f^{a_2 a_3 b} f^{b a_1 c} + f^{a_3 a_1 b} f^{b a_2 c} \right) f^{c a_4 a_5} \times \partial_{[\mu} A^{a_1}_\nu A^{a_2}_\rho A^{a_3}_\mu \frac{1}{\Box} (A^{a_4 \nu} A^{a_5 \rho}).$$

- non-local and vanishing by Jacobi-identity
- can add one “self-BCJ” term:

$$\Delta L_5 \propto g^3 f^{a_1 a_2 b} f^{b a_3 c} f^{c a_4 a_5} \left( \partial_{(\mu} A^{a_1}_{\nu}) A^{a_2}_\rho A^{a_3}_\mu + \partial_{(\mu} A^{a_2}_{\nu}) A^{a_3}_\rho A^{a_1}_\mu \right.$$  
$$+ \partial_{(\mu} A^{a_3}_{\nu}) A^{a_1}_\rho A^{a_2}_\mu \right) \frac{1}{\Box} (A^{a_4 \nu} A^{a_5 \rho}).$$

- introducing auxiliary fields $\Rightarrow$ local and cubic
The Squaring Relations from a Lagrangian Viewpoint

6-point

- $\mathcal{L}_5$ not sufficient to ensure Jacobi at 6-point
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- $\mathcal{L}_6$ contains terms of the form
  
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  \quad \frac{1}{\Box} \left( A^{a_1} A^{a_2} \right) \partial A^{a_3} \frac{1}{\Box} \left( \partial A^{a_4} A^{a_5} \right) A^{a_6}, \ldots
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- full local cubic Lagrangian ⇒ infinitely many auxiliary fields
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- To find general $\mathcal{L}_n$: systematic approach? symmetry principle?
1. Generalized Gauged Invariance

2. Field Theory Derivation of the Squaring Relations

3. The Squaring Relations from a Lagrangian Viewpoint

4. BCJ at loop level
KLT at loop level

- KLT used in unitarity cuts for tree subamplitudes:

- Only applicable on the cut, and different for each cut
- Of practical importance, but no loop-level KLT relation
The Squaring Relations at Loop Level

Squaring relations at loop level

see also: Bern, Carrasco, Johansson [arXiv:1004.0476]

- through the unitarity method, tree derivation generalizes to loop level
- large \( z \) behavior \( \leftrightarrow \) cut-constructability
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(-i)^L A_n^{L\text{-loop}} = \sum_{\text{diags. } i} \int \prod_{a=1}^L d^D l_a \frac{n_i(l_1, \ldots, l_L) c_i}{(2\pi)^D \prod s_{\alpha_i}(l_1, \ldots, l_L)}, \quad n_i + n_j + n_k = 0.
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Then:

$$(-i)^{L+1} M_n^{L-\text{loop}} = \sum_{\text{diags. } i} \int \prod_{a=1}^L \frac{d^D l_a}{(2\pi)^D} \frac{n_i(l_1, \ldots, l_L) \tilde{n}_i(l_1, \ldots, l_L)}{\prod s_{\alpha_i}(l_1, \ldots, l_L)},$$

- holds for arbitrary loop momenta (with internal lines off-shell)
- A universal relation, not a different one for each cut
Summary and Outlook

Summary of results

- Origin of Squaring Relations understood from a QFT perspective
- Squaring implemented at a Lagrangian level
- Better understanding of “gravity=(gauge)^2” (for trees and loops)

Various useful new expressions for gauge and gravity amplitudes

Open problems

- Simple, explicit expression for local, Jacobi-satisfying numerators
- For recent progress: see Stieberger, Mafra, Schlotterer [1104.5224]
- Better understanding of BCJ at loop level
- Can we see BCJ in the Grassmannian for planar \( N = 4 \) SYM?
  (reconcile manifest locality with manifest planarity)
- Implications for the UV properties of \( N = 8 \) supergravity
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