Holography, entanglement and thermalization

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Based on:


One-point functions provide important information about the nature of a system:

\[ \langle \mathcal{O}(\vec{x}, t) \rangle = f(\vec{x}, t) \]

For example, the expectation value of components of the stress tensor contains information about the energy density etc.

In suitable circumstances, time evolution of one-point functions can be described by an effective “low-energy theory”. For the stress-tensor, hydrodynamics is such a low-energy theory.
Of course, much more information is contained in non-local quantities such as

Correlation functions, in particular Green’s functions

Expectation values of Wilson loops

Entanglement entropy

This is especially relevant for systems with translational or rotational invariance: the metric outside a planet is the same as that outside a black hole.

**Interesting question**: is there an analogue of a low-energy effective theory like hydrodynamics for non-local observables?
Here, focus on **entanglement entropy**:

Has a simple definition, though quite difficult to compute in general, even for a free field theory.

Contains universal information, not related to specific operators.

Often used in many contexts, especially in condensed matter physics and quantum computing (e.g. as order parameter for topological phase transitions).

I will only consider entanglement entropy in translational invariant systems.
Definition of entanglement entropy: Separate a system in two subsystems A and B. Decompose the Hilbert space accordingly

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Given a pure or mixed state $\rho$, define

$$\rho_A = \text{Tr}_{\mathcal{H}_B} \rho$$

and the entanglement entropy as

$$S = -\text{Tr}_{\mathcal{H}_A} (\rho_A \log \rho_A)$$
Entanglement entropy is UV divergent. If $\epsilon$ denotes a UV cutoff and the subsystem $A$ refers to a region $V$ then the leading divergence is famously holographic

$$S(V) \sim \frac{\text{area}(\partial V)}{\epsilon^{d-2}}$$

For a conformal field theory in $d$ dimensions

$$S(V) \sim \frac{g_{d-2}(\partial V)}{\epsilon^{d-2}} + \ldots + \frac{g_1(\partial V)}{\epsilon} + g_0(\partial V) \log \epsilon + s(V)$$

All divergent terms are non-universal, except for the logarithmic term (cf stress tensor in ads/cft).

E.g. for a 2d CFT for an interval: $$S(\ell) = \frac{c}{3} \log \frac{\ell}{\epsilon}$$
To compute entanglement entropy in field theory, one typically uses the replica trick:

$$S(A) = - \frac{\partial}{\partial n} \text{Tr}_{\mathcal{H}_A} (\rho^n_A) \big|_{n=1}$$

The right hand side can be computed for integer $n$ by gluing together $n$ copies of the original theory along the boundary of $V$. This result needs to be analytically continued to arbitrary $n$.

Gluing together $n$ copies of the original theory introduces a branch cut along the boundary of $V$ with deficit angle $2\pi(1-n)$. 
To compute this using AdS/CFT, one needs to find a bulk solution whose boundary is a branched cover.

Fursaev proposed that the bulk solution would itself also be a branched cover. This has been shown to be an incorrect assumption by Headrick.

If the bulk solution is a branched cover there will be a delta function contribution to the curvature at the codimension two branching locus

\[ R \sim 2\pi(n - 1)\delta(\text{branch locus}) \]

The Einstein Hilbert action picks up an extra contribution proportional to the area of the branching locus which then leads to the proposal of Ryu-Takayanagi:

\[ S(V) = \frac{1}{4G} \int_{BL} \sqrt{g} \]
Though a basic assumption of the derivation of Fursaev was incorrect, the final result still stands and has passed many non-trivial checks.

In the case of 2d CFT’s, the entanglement entropy gave information about the central charge. For a non-conformal field theory it gives us a \( c \)-function

\[
C(\ell) = \ell \frac{\partial S(\ell)}{\partial \ell}
\]

which for short and long distances approaches the UV and IR values of the central charge and which is **monotically decreasing**. Can prove this directly using strong subadditivity (Casini, Huerta) and holographically.

\[
S(A) + S(B) \geq S(A \cap B) + S(A \cup B)
\]
Can we also get interesting functions of distance scales in higher dimensions? Does the relation between central charges and entanglement entropy hold in $d=4$?

Statement (Solodukhin):

$$g_0(\partial V) = \frac{c}{720\pi} g_{0c}(\partial V) - \frac{\alpha}{720\pi} g_{0a}(\partial V)$$

with

$$g_{0c}[\partial V] = \int_{\partial V} R_{\mu\nu\sigma\tau} (n_i^\mu n_i^\sigma) (n_j^\nu n_j^\tau) - R_{\mu\nu} n_i^\mu n_i^\nu + \frac{1}{3} R +$$

$$+ \int_{\partial V} \left[ \frac{1}{2} k^i k^i - (k^i_{\mu\nu})^2 \right]$$

$$g_{0a}[\partial V] = \int_{\partial V} R(\partial V)$$
In particular, for a ball $B$ of radius $R$

$$g_0(B) = \frac{a}{90}$$

And for a very long cylinder of length $L$ and radius $R$

$$g_0(C) = \frac{c}{720} \frac{L}{R}$$

We verified that these results are indeed reproduced in Gauss-Bonnet gravity where now the method of Fursaev also yields a curvature contribution on the branching locus

$$R \wedge R \rightarrow R \delta(\text{branching locus})$$

(see also Hung, Myers, Smolkin).
What about the analogue of a c-function? There are indications for an a-theorem in four dimensions.

Previous results suggest:

\[ A(R) = -90 R \frac{\partial S_{\text{sphere, regular}}(R)}{\partial R} \]

Which contains the regular part of the entanglement entropy of a sphere of radius R. Takes the right values at fixed points.

Cannot prove monotonicity: subadditivity does not seem to work, and in holographic RG flows cannot relate A(R) directly to null energy condition.

(Myers, Paulos, Sinha define a c-function in the bulk which is ok for RG flows but it is not clear whether it has a field theory definition.)
This was an application of entanglement entropy in static systems.

Next look at a time-dependent system: **infalling shells in AdS**.

Take translationally invariant shells.

Described by Vaidya metric:

$$ds^2 = \frac{1}{z^2}[-(1 - m(v) z^d)dv^2 - 2dzdv + d\vec{x}^2]$$

mass function

$$m(v) = \frac{M}{2} \left(1 + \tanh \frac{v}{v_0}\right)$$

Bhattacharyya, Minwalla
Lin, Shuryak
Abajo-Arrastia, Aparício, López
Albash, Johnson
5. Individual hadrons freeze out

4. Hadron gas cooling with expansion

3. Quark Gluon Plasma (QGP) thermalization, expansion

2. Non-equilibrium state (Glasma) collision

1. High energy nuclei (CGC)

Big unsolved question in heavy-ion physics

Q: How is thermal equilibrium (QGP) achieved after the collision? What is the dominant mechanism for thermalization?
Formation of the quark gluon plasma from the glasma is a far-from equilibrium process – not described by e.g. hydrodynamics.

Infalling translationally invariant shell is, at best, a very crude model for this.

Typical configuration of a single event just after the collision

\[ E^z = ig[\alpha^i_1, \alpha^i_2] \]
\[ B^z = ig\epsilon^{ij}[\alpha^i_1, \alpha^j_2]. \]
Color fluxtubes in AdS/CFT will look like fundamental strings extending into AdS.

For many strings, middle part resembles an infalling shell.
One point functions are instantaneously thermalized.
Look at entanglement entropy instead.
Results: difference between thermal answer and entanglement entropy as a function of time for a spherical region of diameter $D=1,2,3,4$ in $d=2$ (analytical), $d=3,4$ (numerical).
Features:

Thermalization is complete after $t=D/2$.

Behavior is non-analytic at $t=D/2$ (due to thin shell)

Thermalization proceeds from UV to IR and propagates with the speed of light (cf causality bound of Cardy-Calabrese)

Growth rate of entropy density linear (as seen in other situations as well) and nearly independent of volume (no longer true for large volumes)
Here, thermalization is very fast and proceeds from the UV to the IR (top-down)

This is very different from other descriptions of thermalization which are bottom-up: hard quanta radiate many soft gluons which then thermalize the system: IR to UV.

Still not clear what ultimately the right description will be.

Another interesting question is to determine the entropy production in heavy ion collisions: entanglement entropy is not a good notion for this.
Towards an effective description of the dynamics?

In thermal equilibrium, occupation numbers are given by Bose distributions. Typical equations that could describe time evolution are e.g. the Boltzmann equation or the Fokker Planck equation

\[ \frac{\partial n(k)}{\partial t} = \gamma \nabla_k \cdot \left( \frac{k}{\omega} n(k) + T \nabla_k n(k) \right) \]

Can such equations be translated into effective equations that govern the time evolution of e.g. the entanglement entropy with just a few adjustable parameters?

Analogue of hydrodynamics for non-local quantities?
Conclusions:

Entanglement entropy is a useful tool with many interesting applications.

Extend to non-translationally invariant backgrounds. Is thermalization still captured by non-local observables?

We also looked at other quantities such as Green’s functions and Wilson loops whose behavior is qualitatively similar to that of entanglement entropy in the collapsing shell background.

It would be very interesting to find some effective equations governing the time evolution: should know about the speed of light and the stationary solution should be the thermal one. New microscopic parameters?