A Large N Dual
For 2d CFTs
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Based on

Outline of the talk

✦ Introductory Remarks
  ✦ Statement of the Duality

✦ The Minimal Model CFTs
  ✦ $SU(N)$ Coset WZW theories and their large $N$ limit

✦ Higher Spin Symmetries
  ✦ In the Bulk and in the Boundary

✦ Further Checks of the Proposal
  ✦ Matching of Partition Functions and RG Flow Pattern

✦ Towards a Derivation
  ✦ Simplest Realisation of Holography?

✦ Where To?
1 Motivation and Introductory Remarks
**Motivation**

- Common Complaint: AdS/CFT works wonderfully in all these SUSY theories.
- But what about regular CFTs/QFTs?
- How do we know that holography is not something special to SUSY theories?
- To what extent do we need to have an embedding in a complete string theory?
- 2d CFTs are archetypes of well understood non-trivial QFTs. Can we study a large N family and find a gravity dual?
- **Yes.** For a class of $SU(N)$ WZW coset models (which includes the usual Virasoro minimal models as a special case).
- Exploit a *different* symmetry - higher spin algebra. (Fradkin-Vasiliev).
- In many ways analogous to the Klebanov-Polyakov conjecture for the $O(N)$ model - with additional ’tHooft coupling $\lambda$. 

The Duality

- **The CFT**: a coset WZW model.

\[
SU(N)_k \times SU(N)_1 \overline{\quad \quad SU(N)_{k+1}}
\]

- Take the 't Hooft large \( N \) limit, keeping \( 0 \leq \lambda = \frac{N}{N+k} \leq 1 \) fixed.

- A family of theories with central charge \( c_N(\lambda) = N(1 - \lambda^2) \) - vector like model.

- **The Bulk Dual**: A Vasiliev type higher spin theory (including spins 2, 3 \( \ldots \infty \)) in \( AdS_3 \) coupled to two equally massive complex scalar fields.

- \( M^2 = -(1 - \lambda^2) \). But the two scalars have opposite quantisation.

- \( \lambda \) also plays the role of a deformation parameter of the higher spin algebra ("like \( \alpha'' \)"). Also \( G_N \propto \frac{1}{N} \)
2 The Minimal Model CFTs
Coset Models

- A $G/H$ coset theory is defined in terms of a $G$ WZW theory in which a subgroup $H$ is gauged (without kinetic term).

- Therefore $T_{G/H}(z) = T_G(z) - T_H(z)$ and $c_{G/H} = c_G - c_H$.

- Our coset: $G = SU(N)_k \times SU(N)_1$ and $H = SU(N)_{k+1}$ (diagonal).

- For this family,

$$c_N(k) = (N - 1)[1 - \frac{N(N + 1)}{p(p + 1)}] \leq (N - 1)$$

where $p = k + N$. i.e. ($p = N + 1, N + 2, \ldots$)

- So-called $\mathcal{W}_N$ minimal model series (since it possesses a $\mathcal{W}_N$ symmetry).

- In the case $N = 2$, this is the coset construction of the unitary Virasoro discrete series (GKO). $c_2(k) = 1 - \frac{6}{p(p+1)} \leq 1$ with $p = 3, 4, \ldots$
Coset Models  

Spectrum

✧ Spectrum of Primaries are labelled by two integrable representations \((\Lambda^+, \Lambda^-)\) of \(SU(N)_k\) and \(SU(N)_{k+1}\) respectively.

✧ Dimensions given by:

\[
h(\Lambda^+; \Lambda^-) = \frac{1}{2p(p+1)} \left( (p+1)(\Lambda^+ + \rho) - p(\Lambda^- + \rho) \right)^2 - \rho^2
\]

\(\rho\) is the Weyl vector for \(SU(N)\).

✧ In the case, \(N = 2\) reduces to usual Feigin-Fuchs expression

\[
h(r, s) = \frac{(r(p + 1) - sp)^2 - 1}{4p(p + 1)} = h(p - r, p + 1 - s)
\]

✧ Particular cases:

\(h(0; f) = \frac{(N-1)}{2N} \left( 1 - \frac{N+1}{N+k+1} \right)\); \(h(f; 0) = \frac{(N-1)}{2N} \left( 1 + \frac{N+1}{N+k} \right)\)

\(h(0; \text{adj}) = 1 - \frac{N}{N+k+1}\); \(h(\text{adj}; 0) = 1 + \frac{N}{N+k}\)
Coset Models  Characters

✧ The partition function of the coset theory given in terms of contributions from each of these primaries.

✧

\[ b_{(\Lambda^+; \Lambda^-)}(q) = \frac{1}{\eta(q)^{N-1}} \sum_{w \in \hat{W}} \epsilon(w) q^{\frac{1}{2p(p+1)} ((p+1)w(\Lambda^+ + \rho) - p(\Lambda^- + \rho))^2}. \]

\( \hat{W} \) is the affine Weyl group (affine translations + usual Weyl reflections).

✧ (Diagonal) modular invariant partition function given by

\[ Z_{CFT} = \sum_{\Lambda^+, \Lambda^-} |b_{(\Lambda^+; \Lambda^-)}(q)|^2. \]
Coset Models  \textit{RG flows}

✧ One can flow between the minimal models with different \( k \) or \( p \) (for fixed \( N \)).

✧ The \textbf{relevant} operator of the \( p \text{th minimal model}, (0; \text{adj}), \) induces the RG flow. The \textbf{IR fixed point} is the \( p - 1 \)th model.

✧

\[
(0; \text{adj})_p \xrightarrow{\text{RG-flow by } (0; \text{adj})} (\text{adj}; 0)_{p-1}.
\]

✧ Analogue of \((1, 3)\) operator flowing to \((3, 1)\) operator for Virasoro minimal models.

✧ Similar analogues of \((1, 2)\) operator flowing to \((2, 1)\) operators

\[
(0; f)_p \xrightarrow{\text{RG-flow by } (0; \text{adj})} (f; 0)_{p-1}
\]

\[
(0; \bar{f})_p \xrightarrow{\text{RG-flow by } (0; \text{adj})} (\bar{f}; 0)_{p-1}.
\]
Now take the 'tHooft limit $N, k \to \infty$ with $\lambda = \frac{N}{k+N}$ fixed.

The central charge $c_N(\lambda) \simeq N(1 - \lambda^2) < N$

Dimensions of operators simplify remarkably:

$$h(0; f) = \frac{1}{2}(1 - \lambda), \quad h(f; 0) = \frac{1}{2}(1 + \lambda).$$

$$h(0; \text{adj}) = 1 - \lambda, \quad h(\text{adj}; 0) = 1 + \lambda.$$

In general, representations which are finite tensor products of fund./antifund. have finite dimension.

There is however, a very interesting degeneration phenomenon. Many closely spaced levels.
Coset Models \( 't \)Hooft limit

- Large degeneracy: e.g. \((R, R)\) operators

\[
h(R; R) = \frac{B(R)}{2} \times \frac{\lambda^2}{N},
\]

\((B(R)\) is the number of boxes in the young tableaux corresponding to \(R\).) 

- More generally leads to a band of states.

- Characters (branching functions) become reducible at \(N = \infty\).

- Need to understand the physical origin of these finely spaced states at large but finite \(N\).
3 Higher Spin Symmetries
Higher Spin Symmetries boundary

- The $SU(N)$ cosets have an extended $\mathcal{W}_N$ symmetry.

- In addition to $T(z)$, higher spin conserved currents $W^{(3)}(z), \ldots W^{(N)}(z)$.

- Constructed using higher order Casimir invariants.

- E.g $W^{(3)}(z) \propto d^{abc}[a_1(J^a_{(1)}J^b_{(1)}J^c_{(1)}) (z) + a_2(J^a_{(2)}J^b_{(1)}J^c_{(1)}) (z) + a_3(J^a_{(2)}J^b_{(2)}J^c_{(1)}) (z) + a_4(J^a_{(2)}J^b_{(2)}J^a_{(2)}) (z)]$.

- Similarly, $W^{(m)}(z)$ from $m$th order Casimir combinations of the currents $J^a_{(1,2)}(z)$ of $SU(N)_k$ and $SU(N)_1$ respectively.

- The $\mathcal{W}_N$ OPE gives rise to a non-linear symmetry algebra - rather than a Lie Algebra.
Unlike flat space, \(AdS\) admits consistent classical theories of interacting (massless) higher spin gauge fields. Typically, need an infinite tower of these fields.

Exception is \(AdS_3\) where one can truncate to a theory with spins \(2, 3\ldots N\). The gauge fields described by an \(SL(N, \mathbb{R}) \times SL(N, \mathbb{R})\) Chern-Simons theory.

A generalisation of the \(SL(2, \mathbb{R}) \times SL(2, \mathbb{R})\) description of pure \(AdS_3\) gravity.

Action is \(S = S_{CS}[A] - S_{CS}[\tilde{A}]\) with level \(k_{CS} = \frac{\ell}{4G_N}\) (Blencowe).

Brown-Henneaux type analysis of the asymptotic symmetry algebra (Henneaux-Rey, Campoleoni et.al.) shows that one gets precisely a boundary \(W_N\) algebra.

The central charge is exactly the same as Brown-Henneaux: \(c = \bar{c} = \frac{3\ell}{2G_N}\).
Higher Spin Symmetries \textit{bulk}

- For the bulk duals to the coset CFTs we have not just the higher spin fields.

- In $AdS_3$ can also have an \textit{additional} multiplet with scalars/fermions (consistent with the higher spin symmetry).

- Masses depend on a single parameter which also governs the structure of interactions.

- In our case, we need \textit{two complex scalars} with $M^2 = -(1 - \lambda^2)$.

- However, these can be quantised in two alternate ways.

- We need one of them in $(+)$ quantisation (corresponding to $h_+ = \frac{1}{2}(1 + \lambda)$) and the other in $(-)$ quantisation (corresponding to $h_- = \frac{1}{2}(1 - \lambda)$).

- Thus we will consider a bulk theory with a tower of massless fields together with the two complex scalars above.
4 Checks of the Proposal
Checks of the Proposal

- We have already seen the zeroth order check which is the matching of the $\mathcal{W}_N$ symmetries (More general picture of matching $\mathcal{W}_\infty(\lambda)$ symmetries: Gaberdiel-Hartman).

- Next match the partition functions of the large $N$ CFT with bulk answer.

- Generalised boundary gravitons captured by the MacMahon function:

$$Z_{HS} = |\tilde{M}(q)|^2 = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2} = \prod_{n=1}^{\infty} |1 - q^n|^2 \times \prod_{n=1}^{\infty} \frac{1}{|(1 - q^n)^n|^2}.$$

- This is accounted for in the CFT by the character of the vacuum representation $|b_{(0;0)}(q)|^2$ (the first null vectors are at level $k + 1$).

- We also have in the bulk fields corresponding to operators of dimension $h_\pm = \frac{1}{2}(1 \pm \lambda)$

- These correspond to the primaries $(0; f)$ and $(f; 0)$ as well as their complex conjugates.
Checks of the Proposal spectrum

Putting it all together, in the bulk we have the classical plus one loop answer

\[ Z_{\text{bulk}} = Z_{\text{class}} Z_{1-loop} = (q\bar{q})^{-c/24} Z_{\text{HS}} Z_{\text{scalar}}(h+)^2 Z_{\text{scalar}}(h-)^2. \]

\[ Z_{\text{scalar}}(h) = \prod_{l=0, l'=0}^{\infty} \frac{1}{(1 - q^{h+l}\bar{q}^{h+l'})}. \]

We can write the scalar determinant in a suggestive way as a sum over characters of representations of \( U(\infty) \).

\[ Z_{\text{scalar}}(h) = \sum_Y \chi^{u(\infty)}_Y(z_i) \chi^{u(\infty)}_Y(\bar{z}_i) \quad (z_i = q^{i+h-1}). \]

Therefore

\[ Z_{\text{bulk}} = (q\bar{q})^{-c/24} |\tilde{M}(q)|^2 \sum_{Y_1, Y_2, Y_3, Y_4} |\chi_{Y_1}(z^+_i) \chi_{Y_2}(z^+_i) \chi_{Y_3}(z^-_i) \chi_{Y_4}(z^-_i)|^2. \]
Checks of the Proposal spectrum

☆ Let us compare to the CFT partition function and take \( N = \infty \) 'tHooft limit. For any finite \( N, k \) we have

\[
Z_{\text{CFT}}(N, k) = \sum_{\Lambda^+, \Lambda^-} |b_{(\Lambda^+; \Lambda^-)}(q)|^2.
\]

☆ Recall that the CFT characters are given by:

\[
b_{(\Lambda^+; \Lambda^-)}(q) = \frac{1}{\eta(q)^{N-1}} \sum_{w \in \hat{W}} \epsilon(w) q^{\frac{1}{2p(p+1)}((p+1)w(\Lambda^+ + \rho) - p(\Lambda^- + \rho))^2}.
\]

where \( p = k + N \).

☆ However, it is somewhat subtle to take the large \( N, k \) limit.

☆ Reason: Irreducible Virasoro representations generically become reducible.

☆ However, they are often not completely reducible - rather they are indecomposable.
Thus a portion of the states in the original representation decouples in this limit - like null states. Need to compare bulk answer with the contribution of the remaining states in the CFT partition function in this limit.

The branching functions simplify remarkably in the $N = \infty$ 'tHooft limit. Affine Weyl sum drops out.

Using the Verlinde formula (at large $N$) we see the reducibility

$$b_{(\Lambda^+;\Lambda^-)}(q) \approx \sum_{\Lambda} N_{\Lambda_+\Lambda^*}^{\Lambda} q^{\frac{3}{2} \Delta} b_{(\Lambda;0)}(q)$$
Checks of the Proposal spectrum

☆ Again

$$b_{(0;\Lambda)}(q) \sim \frac{1}{\eta(q)^{N-1}} q^{C_2(\Lambda)} \sum_{w \in W} \epsilon(w) q^{-<w(\Lambda^+ + \rho),\rho>} \sim \tilde{M}(q) \times \text{dim}_q(\Lambda) q^{C_2(\Lambda)}.$$  

☆ Furthermore \(\lim_{N \to \infty} q^{C_2(\Lambda)} \text{dim}_q(\Lambda) = \chi_\Lambda(z_i)\) is a \(U(\infty)\) character.

☆ Correction for decoupled states means keeping states with \(B(\Lambda) = B(\Lambda^+) + B(\Lambda^-)\).

☆ Using factorisation of representations at large \(N\) into fundamentals and anti-fund. the remaining contributions exactly give the bulk answer!

☆

$$Z_{\text{bulk}} = (q\bar{q})^{-c/24} |\tilde{M}(q)|^2 \sum_{Y_1,Y_2,Y_3,Y_4} |\chi_{Y_1}(z_i^+) \chi_{Y_2}(z_i^+) \chi_{Y_3}(z_i^-) \chi_{Y_4}(z_i^-)|^2.$$
There is an RG flow between the model labelled by $p$ to $(p - 1) \Rightarrow \delta \lambda = \frac{\lambda^2}{N}$ and $\delta c = -2\lambda^2$.

The perturbing operator involving $(0; adj)$ (with $h = \bar{h} = 1 - \lambda$) can be written as

$$S_{\text{pert}} = g \int d^2 z \ O \ O^\dagger$$

where $O = (0; f)$ has $h = \bar{h} = \frac{1}{2} (1 - \lambda)$.

Thus a "double trace" perturbation in the boundary theory.

Bulk interpretation in such cases: the scalar field corresponding to $O$ is in $(\neg\neg)$-quantisation in the UV with dimension $\Delta_\neg\neg$.

Flows to a theory with the $(\neg\neg)$-quantisation in the IR where it corresponds to an irrelevant operator $O'$ with dimension $\Delta_\neg\neg = 2 - \Delta_\neg\neg$.

Matches exactly what we see in the boundary theory where $O' = (f; 0)$. 
5 Towards a Derivation
Towards a Derivation

☆ Can we derive this duality from "first principles"?

☆ Will give a sketch: key is the bulk $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ Chern-Simons description for the higher spin fields.

☆ No bulk propagating degrees of freedom. Boundary conditions crucial.

☆ The usual boundary conditions are $A_{\tilde{w}} = 0$, $\tilde{A}_w = 0$.

☆ Gauge fields are effectively in $SU(N)$. Might seem to give a $SU(N) \times SU(N)$ WZW theory.

☆ However, additional fall off conditions necessary for asymptotically $AdS_3$ geometry.
☆ Essentially, sets upper (lower) triangular components of $A$, $(\tilde{A})$ to zero.

☆ Amounts to a gauging of the WZW model - Drinfeld-Sokolov reduction. (Campoleoni et.al.)

☆ This was classical. At the quantum level, the DS reduction is more involved (Balog et.al.).

☆ In fact, the quantum version is *exactly* equivalent to our coset CFT with a specific relation between $k_{CS}$ and $k$.

☆ This gives the link between the higher spin theory and the coset.

☆ But need to understand better the interpretation of the scalar fields in the Chern-Simons picture - Wilson loops?
6 Where to?
Where to?

- A large family of non-SUSY examples of AdS/CFT with a fairly explicit description in both bulk and boundary.

- Intermediate complexity between pure gravity and full fledged string theory.

- Applications to real systems: $\mathbb{Z}_N$ Ising model and other stat. mech. models. (Jimbo, Miwa et al.)

- Black holes: Tangible context for microscopics, dynamics. New solutions with hair? (Gutperle-Kraus).

- Proving the duality: Nuts and bolts of Holography.

- Generalisations: Other coset models (Kiritsis) and supersymmetric versions.

- Embedding in String theories: A solvable sub-sector within theories such as D1-D5?

- Connection to Topological Strings: MacMahon function; $U(N)$ Chern-Simons Theory.
The end