What's quantum field theory?

- Quantum theory. Share the same basic postulates as quantum mechanics.
  - state, superposition,
  - unitary evolution: \( \text{id}_\psi |\psi\rangle = H |\psi\rangle \)
  - operator \( \leftrightarrow \) observables
  - measurement, probability

- Relativistic quantum theory.
  - In non-relativistic quantum mechanics we have finite number of degrees of freedom.
    - e.g. wave-function of a particle,
    - position operator of a particle.
  - With special relativity + quantum theory we must consider infinite number of degrees of freedom.

We cannot just make Schrödinger equation covariant!
$\Rightarrow$ analogy in classical physics:

- single particle: $(t, \vec{x})$ coordinates
- $\infty$ # of d.o.f.: $\phi(t, \vec{x})$ field
  
  (Fluid, E&M)

In a quantum theory, consider field operator $\hat{\phi}(t, \vec{x})$.

$\Rightarrow$ # of d.o.f. can also be seen from # of boundary conditions we need to specify.

- single particle: initial condition $x(0), \dot{x}(0)$
- field: $\phi(0, \vec{x})$ for all $\vec{x}$
Why quantum field theory?

1) Since classical mechanics → quantum mechanics, therefore, classical field theory → quantum field theory.

Not a good argument. Classical field is not fundamental at microscopic level, only particles here.

2) Relativity $E = mc^2$. More E, more particles.

True. But not compelling!

It's possible to have more particles with higher E, but is it necessary?
Before we start to have equations.

Unit: \( \hbar = c = 1 \). (Both quantum & relativistic effects important.)

\[ [m][L][E][T^{-1}] = [L^{-1}] \]

example: \( \frac{1}{(fm)^{-1}} \approx 0.2 \text{ GeV} \).

use:

\[ g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

\( \chi^\mu = (x^0, x^i) = (x^0, \vec{x}) \).

\( p^\mu = (E, -\vec{p}) \), \( p^\mu = (E, \vec{p}) \).

\( p^\mu p_\mu = E^2 - p^2 = m^2 \leftarrow \text{rest mass} = m \)
Free particles.

- Fock space: \( |0\rangle, |\vec{p}_1\rangle, \ldots, |\vec{p}_N, \vec{p}_{N-1}, \ldots \vec{p}\rangle \)
  
  real scalar, no other labels.

- Creation & annihilation operators
  
  \[ a_{\vec{p}}^+, a_{\vec{p}} \] with \( [a_{\vec{p}}^+, a_{\vec{p}'}^+] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}') \).

  \[ a_{\vec{p}}^+ |0\rangle = 0 \quad \text{for all } \vec{p} \]

- Single particle state \( |\vec{p}\rangle = \sqrt{2E_{\vec{p}}^+} a_{\vec{p}}^+ |0\rangle \)

- Multiple particle state: \( \prod_{i=1}^{N} \sqrt{2E_{\vec{p}_i}^+} a_{\vec{p}_i}^+ |0\rangle \)

- Inner product:
  
  \[ \langle \vec{p}' | \vec{p} \rangle = 2E_{\vec{p}}^+ (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}') \]

- Lorentz transformation (L.T.).

  Require \( \langle \vec{p}' | \vec{p} \rangle \) invariant under L.T.

\[ U(\Lambda) |\vec{p}\rangle \propto |\Lambda \vec{p}\rangle \]

\( \vec{p} \rightarrow \Lambda \vec{p} \) denotes \( p^m \rightarrow \Lambda^m{}_{\nu} p^\nu \)

\( U(\Lambda) \) is a unitary representation (inner product preserving) of Lorentz group.
note that
\[ \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} = \int \frac{d^4\vec{p}}{(2\pi)^4} \frac{2\pi}{2E_p} \delta(\vec{p}^2 - m^2) \bigg|_{p^0 > 0} \]

\[ \rightarrow \text{ Lorentz invariant.} \]

then
\[ \int d^3\vec{p} \delta^2(\vec{p} - \vec{p}') = 1 \]

\[ \Rightarrow \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \delta^3(\vec{p} - \vec{p}') = 1 \\
\text{Lorentz invariant.} \]

Therefore, we have.

\[ \mathbf{U} (\vec{A}) \big| \vec{p} > = \mathbf{A} \big| \vec{p} >. \]
$$H_0 = \int \frac{d^3 p}{(2\pi)^3} \, \frac{N_p}{E_p} \, (a^+_p a_p) + \text{const}(\propto VT)$$

$$[H_0, a^+_p] = E_p \, a^+_p \quad \text{and} \quad [H_0, a_p] = -E_p \, a_p$$
Step 1: describing local interaction

- No specially deep reason, except it seems to be the case.

- However, a general function $H(a^+_p, a^-_p)$ wouldn't be local.

- Use Fourier transforms of $a^+_p, a^-_p$ as building blocks.

$$
\Psi^+_\vec{x}(\vec{x}) = \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_p^3}} a^+_\vec{p} e^{i\vec{p} \cdot \vec{x}}
$$

$$
\Psi^-\vec{x}(\vec{x}) = \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_p^3}} a^-\vec{p} e^{-i\vec{p} \cdot \vec{x}}
$$

$H(\Psi^+ , \Psi^-)$ will be automatically local. Note: This already implies that field theory is a very useful tool for considering local interactions.
meaning of $\psi_-(\vec{x})$.

$$\psi_-(\vec{x})|\text{0}\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\vec{p}}} \ a_\vec{p}^+ e^{-i\vec{p} \cdot \vec{x}} |\text{0}\rangle$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\vec{p}}} e^{-i\vec{p} \cdot \vec{x}} \frac{1}{\sqrt{2E_\vec{p}}} |\vec{p}\rangle.$$ 

$$\langle \vec{p}' | \psi_-(\vec{x}) |\text{0}\rangle = e^{-i\vec{p}' \cdot \vec{x}}$$

Therefore:

$$\psi_-(\vec{x}) |\text{0}\rangle \propto |\vec{x}\rangle$$

"creating a particle at $\vec{x}$."
Step 2: Imposing Lorentz invariance. This task will be easier in a formulation in which $t \& \mathbf{x}$ are on the same footing.

- "Picture"
  - Schrodinger: $\psi^+(\mathbf{x})$, $\psi^-(\mathbf{x})$.

- "Interaction picture" or "Dirac picture"
  \[
  \phi^\pm(t, \mathbf{x}) = \phi^\pm(\mathbf{x}) = e^{iH_0 t} \psi^\pm(\mathbf{x}) e^{-iH_0 t}.
  \]
  \[
  \phi^+(\mathbf{x}) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3 (2E_p)^{1/2}} a_{\mathbf{p}} e^{-i\mathbf{p} \cdot \mathbf{x}}
  \]
  \[
  \phi^-(\mathbf{x}) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3 (2E_p)^{1/2}} a^+_{\mathbf{p}} e^{+i\mathbf{p} \cdot \mathbf{x}}
  \]

- Dynamical evolution
  - Schrodinger picture.
  \[
  H = H_0 + H_{\text{int}}
  \]
  \[
  |\psi_t\rangle = U_S(t, t') |\psi\rangle.
  \]
  \[
  i \frac{d}{dt} U_S = H U_S,
  \]
Interaction picture - evolution

\[ \langle f \mid e^{-iH_0 t} e^{iH_0 t'} e^{-iH(t-t')} e^{iH_0 t'} e^{iH_0 t} \mid l \rangle \]

\[ \langle f \mid U_I \mid l \rangle \]

\[ i \partial_t U_I(t, t') = H_I(t) U_I \quad \text{where} \quad H_I(t) = e^{iH_0 t} H_{int} e^{-iH_0 t} \]

(\( H_{int} \) constructed out of \( \phi_{\pm}(t, x) \))

\[ \rightarrow \text{Solution of } U_I(t, t'). \quad \text{by} \quad \text{perturbation expansion:} \]

\[ U_I(t, t') = 1 + (-i) \int_{t'}^t dt_1 H_I(t_1) + (i)^2 \int_{t'}^t dt_1 \int_{t'}^{t_1} dt_2 \]

\[ \times H_I(t_1) H_I(t_2) + \ldots \]

Time order

\[ T(\theta(t_1) H_I(t_2)) = \theta(t_1 - t_2) H_I(t_1) H_I(t_2) \]

\[ + \theta(t_2 - t_1) H_I(t_2) H_I(t_1) \]

\[ \rightarrow U_I(t, t') = T S \exp \left[ -i \int_{t'}^t dt' H_I(t') \right] S \]

In particular, the second term is

\[ -\frac{1}{2!} \int_{t'}^t dt_1 \int_{t'}^{t_1} dt_2 T(\theta(t_1) H_I(t_2)) \]

\[ \times T(\theta(t_2) H_I(t_1)) \]
Consider transition amplitude

\[ T_{fi} = \langle f_i | U(t_1, t_2) | i_i \rangle. \]

- Consider the case in which
  \[ \langle f_i | \text{ wave-packet localized at } (t_f, \overline{x}_f) = x_1 \]
  \[ | i_i \rangle: \text{ wave-packet localized at } (t_i, \overline{x}_i) = x_2 \]
  
  & assume first \( t_1 > t_2 \).

- 2nd order of expansion in \( H_i \)

\[ T_{fi} \supset \langle f_i | H_i(t_1) H_i(t_2) | i_i \rangle. \]

≡ writing \( H_i(t) \) in terms of local density \( \rho_i(x) \)

\[ \int dt \ H_i(t) = \int d^4 x \ \rho_i(x). \]

\[ \langle f_i | \chi_i(x_1) \chi_i(x_2) | i_i \rangle \]

\[ = \langle f_i | \chi_i(x_2) \chi_i(x_1) + [\chi_i(x_1), \chi_i(x_2)] | i_i \rangle \]

\[ = \langle f_i | [\chi_i(x_2), \chi_i(x_2)] | i_i \rangle. \]
Define $S$-matrix \( \langle t_f \rightarrow \infty, \ t_i \rightarrow -\infty \rangle \).

\[
\langle f | S | i \rangle = T_{fi}
\]

and put back time-order:

\[
S = 1 + i \int d^4 x_1 H_1(x_1) - \frac{1}{2} \int 
\]

\[
+ \frac{1}{2i} \int d^4 x_1 \ d^4 x_2 T \left[ H_1(x_1), H_1(x_2) \right] + \ldots
\]

Require $S$ to be Lorentz invariant.
b) How about $T [ \mathcal{H}_I (x_1), \mathcal{H}_I (x_2) ]$?

1. $(x_1 - x_2)^2 > 0$, time-like.
   $\rightarrow$ Time ordering is L.I.

2. $(x_1 - x_2)^2 < 0$, space-like.
   $\rightarrow$ Time ordering is NOT L.I.

Therefore, we must require

$$[ \mathcal{H}_I (x_1), \mathcal{H}_I (x_2) ] = 0 \quad \text{for} \quad (x_1 - x_2)^2 < 0 \quad \text{(A)}$$

$\Rightarrow$ Strong constraints on $\mathcal{H}_I (x)$ as a function of $\phi^\pm$.

For example:

$$[ \phi^+ (x'_2), \phi^- (x'_3) ] = \int \frac{d^3 p}{(2 \pi)^3} \int \frac{d^3 p'}{(2 \pi)^3} \frac{1}{\sqrt{2 E_p} \sqrt{2 E_{p'}}} e^{i \langle p, x'_2 - p, \phi_2 \rangle} \langle \phi^+_p, \phi^-_{p'} \rangle \mathcal{H}_I (x'_2 - x'_3)$$

$\chi \frac{m}{4 \pi^3 r} \mathcal{K}_I (m r), = C(r) \oplus$

$$r = (x - x')^2, \quad C(r) \not\equiv 0 \quad \text{for} \quad r < 0$$
From weak coupling limit, we expect solutions to constraint (A) in terms of field variables which are linear in $\phi^\pm$

\[ \phi_\lambda(x) = \phi_+^\lambda(x) + \lambda \phi^-_\lambda(x) \]

\[ [\phi_\lambda(x), \phi^+_\lambda(x)] = (1 - \lambda^2) C(x) \]

1) Must have $|\lambda| = 1$, with addition phase absorbed.

\[ \phi(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p^3}} \left( a_{\vec{p}} e^{-i\vec{p} \cdot \vec{r}} + a^{+}_{\vec{p}} e^{i\vec{p} \cdot \vec{r}} \right) \]

2) Also can show, we must have

\[ [a_{\vec{p}}, a^{+}_{\vec{p}'}] \propto \delta^3(\vec{p} - \vec{p}') \]

Commutator. \rightarrow Bose statistics.

3) For fermions.

must have (b, $b^+$ satisfy $\delta^3(\vec{p} - \vec{p}')$)

\[ \uparrow \]

Fermi - statistics.

2) & 3) Spin - statistic theorem.
4) If particle have change, then there are two kinds of particles, particle & anti-particle with opposite charges. ±.

\[ a_{p}^{+} \text{ create particle.} \]

\[ b_{p}^{+} \text{ create anti-particle.} \]

\[ \phi(x) = \int \frac{d^{3}p}{(2\pi)^{3} \sqrt{2E_{p}}} a e^{i p \cdot x} + \alpha \int \frac{d^{3}p}{(2\pi)^{3} \sqrt{2E_{p}}} b^{+} e^{-i p \cdot x} \]

Again, we can show \( a, b \) satisfy commutation relations.

- \( M_{+} = M_{-} \).

- \( \alpha \) is at most a pure phase, which can be absorbed in \( b^{+} \).

\[ \rightarrow \phi(x) = \int \frac{d^{3}p}{(2\pi)^{3} \sqrt{2E_{p}}} \left( a e^{i p \cdot x} + b^{+} e^{-i p \cdot x} \right) \]
- particles + quantum mechanics + local interaction

→ field operator \( \psi^\pm(x) \)

- Lorentz invariance →

  - only specific combinations of \( \psi^\pm(x) \) are allowed, for example
    \[
    \Phi(x) = \int dp \left( a e^{i px} + b e^{-i px} \right).
    \]
  - Spin statistics

- More formally, a QFT can be defined as L. I. quantum theory, with

\[
[\Phi^a(x), \Phi^b(y)] = 0 \text{ for } (x-y)^2 < 0.
\]

+ a few assumptions about energy spectrum (asymptotic).
We have glossed over a # of things such as

- A careful definition of $S$-Matrix.

- A careful treatment of $\Phi(x_1) \Phi(x_2)$ as $x_1 \to x_2$

"Renormalization"

These will be discussed in detail later. But the conclusion here is unchanged.