Spin, just like mass, is a property of a particle. We have observed particles with different spins in nature.

Spin = 0: the Higgs boson,
spin = 1/2: electron, quark, neutrino, ... most of the "matter"
Spin = 1: photon, gluon, W/Z
Spin = 2: graviton
Spin = 3/2?: gravitino? Supersymmetry, not found yet.

In general, have to be multiples of 1/2. Theoretically, seems to be difficult to have an elementary particle with spin greater than 2.

We will be focusing on spin 1/2 here.

Why do we call something spin? Because it tells us how particle transform under rotation. We will use this to construct the spin operator and the state of spin 1/2 particles.
We begin with a particle polarized in z-direction. As demonstrated by the Stern-Gerlach experiment, there can only be two possible polarizations which are $\pm \hbar \frac{1}{2}$. We will represent these two states of the system by

$$|z^+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |z^-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and "spin-z" operator would be

$$S_z = \frac{\hbar}{2} |z^+\rangle \langle z^+| + \left(-\frac{\hbar}{2}\right) |z^-\rangle \langle z^-| = \frac{\hbar}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Next, we consider a particle with positive polarization in the x direction, $x^+$. We then perform an S-G experiment in the z direction. As we discussed earlier, the result is + and - each half of the time. Therefore, Born's rule tells us

$$|z^+ x^+\rangle = |z^- x^+\rangle = \frac{1}{\sqrt{2}}$$

Therefore, we can write

$$|x^+\rangle = \frac{1}{\sqrt{2}} \left( |z^+\rangle + e^{i\delta_1} |z^-\rangle \right)$$

$\delta_1$: arbitrary phase to be fixed later.

From $\langle x^- x^+\rangle = 0$

$$|x^-\rangle = \frac{1}{\sqrt{2}} \left( |z^+\rangle - e^{i\delta_1} |z^-\rangle \right)$$

An identical argument leads to

$$|y^\pm\rangle = \frac{1}{\sqrt{2}} \left( |z^+\rangle \pm e^{i\delta_2} |z^-\rangle \right)$$

$\delta_2$: phase
Spin – x operator

\[ S_x = \frac{\hbar}{2} \left( |x+1 \rangle \langle x+1| + (-1)^x |x-1 \rangle \langle x-1| \right) \]

\[ = \frac{\hbar}{2} \left[ e^{-i\delta_1} |z+1 \rangle \langle z-1| + e^{i\delta_1} |z-1 \rangle \langle z+1| \right] \]

Similarly, spin – y operator

\[ S_y = \frac{\hbar}{2} \left[ e^{-i\delta_2} |z+1 \rangle \langle z-1| + e^{i\delta_2} |z-1 \rangle \langle z+1| \right] \]

Now, consider the following sequence of S-G experiments

\[ \xrightarrow{X^+} \text{60\% Y± each} \]

\[ \rightarrow |Y+1 \rangle x+ \rightarrow |Y-1 \rangle x+ \]

\[ \Rightarrow |Y+1 \rangle x+ = |Y-1 \rangle x+ = \frac{1}{\sqrt{2}} \]

\[ \Rightarrow \frac{1}{2} \left| 1 \pm e^{i(S_1 - S_2)} \right| = \frac{1}{\sqrt{2}} \]

\[ \Rightarrow S_2 - S_1 = \pm \frac{\pi}{2} \]

\[ \Rightarrow S_x, S_y \text{ can NOT be all real!} \]
We can choose $\delta_1 = 0$ and $\delta_2 = \frac{\pi}{2}$

Therefore

$$S_x = \frac{\hbar}{2} \left[ |z+\rangle\langle z-1 + |z-\rangle\langle z+1 \right] = \frac{\hbar}{2} \sigma_1$$

$$S_y = \frac{\hbar}{2} \left[ -i |z+\rangle\langle z-1 + i |z-\rangle\langle z+1 \right] = \frac{\hbar}{2} \sigma_2$$

and

$$|x\pm\rangle = \frac{1}{\sqrt{2}} \left( |z+\rangle \pm |z-\rangle \right) = \frac{1}{\sqrt{2}} \left( \pm 1 \right)$$

$$|y\pm\rangle = \frac{1}{\sqrt{2}} \left( |z+\rangle \pm i |z-\rangle \right) = \frac{1}{\sqrt{2}} \left( \pm i \right)$$

We have been using these already in our discussion on SG experiments. Here is the justification of doing so.
Angular momentum are generators of rotation. We should show these spin operators do rotate physical quantities.

In particular, we will show operator
\[
\exp \left[ -\frac{i}{\hbar} \mathbf{\hat{S}} \cdot \mathbf{\hat{e}} \alpha \right]
\]
generate rotation around axis \( \mathbf{\hat{e}} \) by an angle \( \alpha \).

- "rotate the state"

- starting with a \( \mathbb{Z}^+ \) state \((\mathbf{0})\)
- rotate around \( y \) axis by \( \theta \)
\[
\exp \left[ -\frac{i}{\hbar} S_y \theta \right] \cdot (\mathbf{0})
\]
Then rotate around $z$-axis by $\phi$

$$\exp \left[ -i \frac{\hat{S}_z}{\hbar} \phi \right] \exp \left[ -i \frac{\hat{S}_y}{\hbar} \theta \right] | 0 \rangle$$

use $e^{\frac{1}{2} i \sigma_i x} = \cos(\frac{x}{2}) - i \sigma_i \sin(\frac{x}{2})$

$$| 1 \rangle \rightarrow \left( \begin{array}{c} \cos \frac{\varphi}{2} e^{-i\frac{\phi}{2}} \\ \sin \frac{\varphi}{2} e^{i\frac{\phi}{2}} \end{array} \right) = | \chi_n \rangle$$

Note that $| \chi_n \rangle$ is the eigenstate of $\hat{\vec{S}} \cdot \hat{n}$

with $\hat{n}$ the unit vector pointing at $(\theta, \phi)$ direction.

$$\hat{\vec{S}} \cdot \hat{n} | \chi_n \rangle = \frac{\hat{n}}{2} | \chi_n \rangle$$

We see that we have rotated state $| \varphi \rangle$ to $| \chi_n \rangle$ by using spin operators.

Let's now consider a rotation of $2\pi$.

$$\exp \left[ -i \frac{\hat{S}_z}{\hbar} 2\pi \right] | \chi \rangle = \exp \left[ -i \frac{\hat{S}_z}{\hbar} \cdot 2\pi \right] | \chi \rangle$$

$$= - | \chi \rangle \quad \text{(using eigenvalue of } \hat{S}_z = \pm 1)$$

The negative sign is the mark of spin-$\frac{1}{2}$.

There is no contradiction, $| \chi \rangle$ by itself is not an observable. All physical observable is periodic under rotation of $2\pi$.  

We now demonstrate the rotation with a different example, the rotation of a vector observable.

Angular momentum is a vector. In particular, \( \vec{S} = S_x \hat{i} + S_y \hat{j} + S_z \hat{k} \)

For simplicity, let’s consider the \( x \)-component of vector \( \vec{S} \).

We will consider a physical quantity, the expectation value

\[ \langle \Psi | S_x | \Psi \rangle \]

Again for simplicity, we consider a rotation along \( z \)-axis

\[ | \Psi \rangle \rightarrow | \Psi' \rangle = \exp \left( -\frac{i}{\hbar} S_z \phi \right) | \Psi \rangle \]

Under this rotation \( U_\phi \)

\[ \langle \Psi' | S_x | \Psi' \rangle = \langle \Psi | U_\phi^\dagger S_x U_\phi | \Psi \rangle \]

\[ U_\phi^\dagger S_x U_\phi = \exp \left( \frac{i}{\hbar} S_z \phi \right) S_x \exp \left( -\frac{i}{\hbar} S_z \phi \right) \]

\[ = \exp \left( \frac{i}{\hbar} S_z \phi \right) \frac{1}{2} (1 \uparrow \uparrow \downarrow \downarrow \downarrow + 1 \uparrow \downarrow \downarrow \downarrow \downarrow + 1 \uparrow \downarrow \downarrow \downarrow \downarrow - 1 \uparrow \downarrow \downarrow \downarrow \downarrow) \exp \left( -\frac{i}{\hbar} S_z \phi \right) \]

\[ \times S_x | \uparrow \rangle = \frac{1}{2} | \uparrow \rangle \rightarrow \exp \left( \frac{i}{\hbar} S_z \phi \right) | \uparrow \rangle = e^{i\phi/2} | \uparrow \rangle \]

Therefore

\[ U_\phi^\dagger S_x U_\phi = \frac{1}{2} \left[ (1 \uparrow \uparrow \downarrow \downarrow \downarrow + 1 \uparrow \downarrow \downarrow \downarrow \downarrow + 1 \uparrow \downarrow \downarrow \downarrow \downarrow - 1 \uparrow \downarrow \downarrow \downarrow \downarrow) \cos \phi + i(1 \uparrow \downarrow \downarrow \downarrow \downarrow - 1 \uparrow \downarrow \downarrow \downarrow \downarrow) \sin \phi \right] \]

\[ \begin{align*}
S_x^2 = S_x \cos \phi - S_y \sin \phi
\end{align*} \]

Therefore, under rotation of \( \phi \) along \( z \)-axis

\[ \langle \Psi' | S_x | \Psi' \rangle = \langle \Psi | U_\phi^\dagger S_x U_\phi | \Psi \rangle = \cos \phi - \sin \phi \langle \Psi | S_y | \Psi \rangle \]

Exactly as the rotation of \( x \)-component of a vector, periodic for \( \phi = 2\pi \). Since \( \langle \Psi | S_x | \Psi \rangle \) is a physical observable.
More algebraic properties

Spin is angular momentum, which is a vector operator
\[ \vec{S} = S_x \hat{i} + S_y \hat{j} + S_z \hat{k} \]
For spin \( \frac{1}{2} \), we have \( \vec{S} = \frac{1}{2} \vec{\sigma} \)

We have the following commutation relations
\[
\begin{align*}
[S_x, S_y] &= i \hbar S_z \\
[S_y, S_z] &= i \hbar S_x \\
[S_z, S_x] &= i \hbar S_y
\end{align*}
\]
\[ [S_i, S_j] = i \hbar \epsilon_{ijk} S_k \]
\( i, j, k \in \{1, 2, 3\} \quad \epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1 \)
\( \epsilon_{132} = \epsilon_{321} = \epsilon_{123} = -1 \)

We can also define "Size of spin" operator
\[ S^2 = \vec{S} \cdot \vec{S} = S_x^2 + S_y^2 + S_z^2 \]

We have
\[ [S^2, S_x] = [S^2, S_y] = [S^2, S_z] = 0 \]

We can use the common eigenstate of \( S^2 \) and any one of \( S_i \) to characterize spin-\( \frac{1}{2} \) states.

Typically, we choose \((S^2, S_z)\)
We have
\[ S^2 = \frac{3}{4} \hbar^2 \]
It's customary to define \( S = \frac{1}{2} \), and write
\[ S^2 = S(S+1)\hbar^2 \]
There are two eigenstates of \( S_z \), with eigenvalues \( m\hbar \), \( m = \pm \frac{1}{2} \)

The eigenstates are
\[ |s, m\rangle \Rightarrow |\frac{1}{2}, \frac{1}{2}\rangle \text{ and } |\frac{1}{2}, -\frac{1}{2}\rangle \]

Shorthand notation: \( |+\rangle \) ↓ \( |\downarrow\rangle \) ↓ \( |1\rangle \) ↓
\( |\uparrow\rangle \) ↓ \( |\rightarrow\rangle \)

or \( |\uparrow\rangle \) ↓ \( |\rightarrow\rangle \)

More generally, a state with angular momentum \( j \) and \( z \)-component \( m \) is written as

\[ |j, m\rangle \]

\[ J_z |j, m\rangle = m\hbar \]

\[ \hat{J}_z |j, m\rangle = m\hbar \]

\( J \) is a general angular momentum operator.
Define ladder operator as

\[ S_+ = S_x + i S_y, \quad S_- = S_x - i S_y \]

We have

\[ [S_z, S_+] = [S_z, S_x] + i [S_z, S_y] = \frac{\hbar}{2} S_+ \]

and

\[ [S_z, S_-] = -\frac{\hbar}{2} S_- \]

Suppose

\[ S_z |s m\rangle = m \hbar |s m\rangle \]

Consider

\[ |\psi_\pm\rangle = S_\pm |s m\rangle \]

\[ S_z |\psi_\pm\rangle = S_z S_\pm |s m\rangle \]

\[ = ( [S_z, S_\pm] + S_\pm S_z ) |s m\rangle \]

\[ = ( \pm \hbar S_\pm + m \hbar S_z ) |s m\rangle \]

\[ = (m \pm 1) \hbar |\psi_\pm\rangle \]

Therefore

\[ S_\pm |s m\rangle \propto |s m \pm 1\rangle \]

Since

\[ [S^2, S_\pm] = 0 \rightarrow [S^2, S_\pm] = 0 \]

Therefore, state

\[ |\psi_\pm\rangle = S_\pm |s m\rangle \]

has the same \( S \).

\[ S_\pm |s m\rangle \propto |s m \pm 1\rangle \]
In matrix form,

\[ S_+ = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_- = \frac{-\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

We have

\[ S_+ |+\rangle = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} |+\rangle \]

\[ S_- |+\rangle = \frac{-\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{-\hbar}{2} |+\rangle \]

Moreover,

\[ S_+ |+\rangle = 0, \quad S_- |+\rangle = 0 \]

In principle, we can start from the "highest" state, the |+\rangle, then use the "lowering" operator S- to obtain |->. The |-> would also be the "lowest" state since S- |-> = 0. Therefore, we have obtained all possible states of a spin 1/2 particle.

Of course, this is probably the most complicated way of enumerate all the possible states of a spin 1/2 particle. However, the ladder operator introduced here will be more useful in more complicated cases.
Here, we elaborate on the physical observables for spin-1/2 particle.

The most commonly used and instructive model concerns spin-1/2 particle in constant magnetic field. We have met this case already in earlier examples and homework problems. Here, we describe the physical picture of the so called spin precession. For concreteness, we consider electron with electric charge $e$.

\[ \mathbf{H} = -\left( \frac{e}{m_e c} \right) \mathbf{s} \cdot \mathbf{B} \]

or, with $\mathbf{\mu} = \frac{e}{m_e c} \mathbf{s}$

\[ \mathbf{H} = -\mathbf{\mu} \cdot \mathbf{B} \]

- Take $\mathbf{B}$ along $z$-direction $\mathbf{B} = B \mathbf{k}$

\[ \mathbf{H} = \omega \mathbf{S}_z \quad \text{with} \quad \omega = \frac{|e| B}{m_e c} \]

Eigenvalues: $E_\pm = \pm \frac{1}{2} \hbar \omega$; eigenstates: $|\pm\rangle$

- Evolution of state

$t=0$, state is $|\alpha_0\rangle$

\[ |\alpha, t\rangle = e^{-\frac{i}{\hbar} E_+ t} |+\rangle \langle +| \alpha_0 \rangle + e^{-\frac{i}{\hbar} E_- t} |-\rangle \langle -| \alpha_0 \rangle \]

\[ = e^{-\frac{i \omega t}{2}} |+\rangle \langle +| \alpha_0 \rangle + e^{i \frac{\omega t}{2}} |-\rangle \langle -| \alpha_0 \rangle \]
Evolution of expectation value \[ \langle S_i \rangle = \langle \psi(t) \mid S_i \mid \psi(t) \rangle \]

\[ \langle S_x \rangle = \langle S_x \rangle_{t=0} \cos \omega t - \langle S_y \rangle_{t=0} \sin \omega t \]

\[ \langle S_y \rangle = \langle S_y \rangle_{t=0} \cos \omega t + \langle S_x \rangle_{t=0} \sin \omega t \]

\[ \langle S_z \rangle = \langle S_z \rangle_{t=0} \]

This is called spin precession. As expected:

\[ \text{torque} \quad \vec{\tau} = \mu \times \vec{B} \]

\[ \alpha = \vec{S} \times \vec{B} \]

Analogous to spinning top.

We also note that

period for spin precession \[ T_{pre} = \frac{2\pi}{\omega} \]

“back to itself after 2π precession.”

However,

period for state evolution \[ T_{state} = \frac{4\pi}{\omega} \]

“back to negative itself after 2π”

trademark for spin \( \frac{1}{2} \).