Physics 235 Homework 4 Solutions

Problem 1

(a)

The potential

\[ V(x) = \begin{cases} \infty & \text{if } x < 0 \\ 0 & \text{if } 0 < x < a \\ V_0 & \text{if } x > a. \end{cases} \]

has the value \( V_0 \) as \( x \to +\infty \). Because of this asymptotic behavior, \( E < V_0 \) condition we impose ensures that the wave function can be exponentially damped as \( x \to +\infty \) and be a normalizable solution to the energy eigenvalue problem. More precisely, in the \( x > a \) region, the time independent Schrödinger equation gives

\[ -\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V_0 \psi = E \psi. \]

For \( E < V_0 \), the solution to this equation is of the form \( \psi(x) = B_1 e^{\kappa x} + B_2 e^{-\kappa x} \) where we define the positive real variable \( \kappa \equiv \sqrt{\frac{2m(V_0-E)}{\hbar^2}} \). Moreover, for a normalizable wavefunction, the \( e^{\kappa x} \) solution should not appear.

Since the potential is infinite when \( x < 0 \), the wave-function should vanish for \( x < 0 \). In the region \( 0 < x < a \), on the other hand, the eigenvalue problem gives

\[ -\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E \psi. \]

Note that \( \psi \) has to be concave down in this region so that the solution \( e^{-\kappa x} \) as \( x \to +\infty \) can be joined to a solution which vanishes at \( x = 0 \). This tells us that \( E > 0 \). Therefore, defining the positive real variable \( q \equiv \sqrt{\frac{2mE}{\hbar^2}} \) the equation above reduces to \( \frac{d^2 \psi}{dx^2} = -q^2 \psi \), which has the general solution, \( \psi(x) = A_1 \sin qx + A_2 \cos qx \). Among these two we should pick the solution \( \sin qx \) in \( 0 < x < a \) region since by the continuity of the wave-function we should have \( \psi(0^+) = 0 \).

Putting everything together, we have the solution

\[ \psi(x) = \begin{cases} 0 & \text{if } x < 0 \\ A \sin qx & \text{if } 0 < x < a \\ B e^{-\kappa x} & \text{if } x > a. \end{cases} \]

The continuity of the wave function and its first derivative at \( x = a \) gives the following set of equations.

\[ A \sin qa = B e^{-\kappa a} \quad \text{by } \psi(a^-) = \psi(a^+), \]

\[ q A \cos qa = -\kappa B e^{-\kappa a} \quad \text{by } \frac{d\psi}{dx} \bigg|_{a^-} = \frac{d\psi}{dx} \bigg|_{a^+}. \]
The first equation gives the relation

\[ B = Ae^{\kappa a} \sin qa. \]

Plugging this in the second one we obtain

\[ qA \cos qa = -\kappa A \sin qa. \]

Since, \( A \neq 0 \) for a nontrivial solution, this equation gives a condition on the energy eigenvalue as

\[ \cot qa = -\frac{\kappa}{q}. \]

In terms of the variables given in the problem:

\[ \cot \left( \frac{a\sqrt{2mE}}{\hbar} \right) = -\sqrt{\frac{V_0}{E} - 1} \]

In other words, for a given value of \( V_0 \), bound states exist if and only if the equation above has solutions for some \( 0 < E < V_0 \). This is a transcendental equation, for which we can find the solutions numerically. However, discussing the existence of solutions is a much easier problem.

(b)

Using the relation

\[ B = Ae^{\kappa a} \sin qa. \]

we can put the energy eigenfunctions to the form

\[ \psi(x) = C \begin{cases} 0 & \text{if } x < 0 \\ \frac{\sin qx}{\sin qa} & \text{if } 0 < x < a \\ e^{-\kappa(x-a)} & \text{if } x > a \end{cases} \]

where \( C \) is an overall normalization factor.

**Problem 2**

(a)

For the harmonic oscillator, we will label \( E_n \) by \( n \), the eigenvalue with respect to the number operator, \( \hat{N} = a^\dagger a \). Remember that \( \hat{H} = \hbar \omega (\hat{N} + 1/2) \), so the eigenstates of \( \hat{H} \) and \( \hat{N} \) are common, in particular, \( E_n = \hbar \omega (n + 1/2) \). The eigenvalues, \( n \), of \( \hat{N} \) run over \( 0, 1, 2, \ldots \).

Let us start by finding the time evolved state \( |\psi(t)\rangle \) for

\[ |\psi(0)\rangle = \frac{1}{\sqrt{2}} (|\psi_0\rangle + |\psi_1\rangle). \]
Since the Hamiltonian does not have any explicit time dependence, the solution to the Schrödinger’s equation
\[
\frac{i\hbar}{\text{d}t}\ket{\psi(t)} = \hat{H}\ket{\psi(t)}
\]
is given by
\[
\ket{\psi(t)} = e^{-i\hat{H}t/\hbar}\ket{\psi(0)}. 
\]
Since \(\ket{\psi_0}\) and \(\ket{\psi_1}\) are energy eigenstates of simple harmonic oscillator with energy eigenvalues \(\hbar\omega/2\) and \(3\hbar\omega/2\), respectively, we simply get
\[
\ket{\psi(t)} = \frac{1}{\sqrt{2}} e^{-i\omega t/2}\ket{\psi_0} + e^{-i\omega t/2}\ket{\psi_1}. 
\]
Of course, that also means
\[
\langle \psi(t) \rangle = \frac{e^{i\omega t/2}}{\sqrt{2}} (\langle \psi_0 | + e^{i\omega t} \langle \psi_1 |). 
\]
In order to find the expectation value \(\hat{x}, \hat{p}, \hat{x}^2\) and \(\hat{p}^2\), we can use the operators \(a\) and \(a^\dagger\). Since \(a = \sqrt{m\omega/2\hbar}(\hat{x} + i\hat{p}/m\omega)\) and \(a^\dagger = \sqrt{m\omega/2\hbar}(\hat{x} - i\hat{p}/m\omega)\) we can write \(\hat{x}\) and \(\hat{p}\) as
\[
\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \quad \text{and} \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a). 
\]
The operators \(a^\dagger\) and \(a\) acts on an energy eigenstate as
\[
a^\dagger \ket{\psi_n} = \sqrt{n + 1} \ket{\psi_{n+1}}, \quad \text{and} \quad a \ket{\psi_n} = \sqrt{n} \ket{\psi_{n-1}}. 
\]
We can finally evaluate the expectation values \(\langle \hat{x} \rangle_t, \langle \hat{p} \rangle_t, \langle \hat{x}^2 \rangle_t, \langle \hat{p}^2 \rangle_t\) as follows.

\[
\langle \hat{x} \rangle_t = \sqrt{\frac{\hbar}{2m\omega}} \langle \psi(t) | a^\dagger + a | \psi(t) \rangle 
= \sqrt{\frac{\hbar}{8m\omega}} (\langle \psi_0 | + e^{i\omega t} \langle \psi_1 |) \left( a^\dagger + a \right) (\ket{\psi_0} + e^{-i\omega t} \ket{\psi_1}) 
= \sqrt{\frac{\hbar}{8m\omega}} (\langle \psi_0 | + e^{i\omega t} \langle \psi_1 |) \left( \ket{\psi_1} + e^{-i\omega t} \sqrt{2} \ket{\psi_2} + e^{-i\omega t} \ket{\psi_0}) 
= \sqrt{\frac{\hbar}{8m\omega}} (e^{i\omega t} + e^{-i\omega t}),
\]
which gives
\[
\langle \hat{x} \rangle_t = \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t). 
\]
\[ \langle \hat{p} \rangle_t = i \frac{\hbar}{2} \langle \psi(t)|a^d - a|\psi(t) \rangle \]
\[ = i \frac{\hbar}{8} \left( (|\psi_0\rangle + e^{i\omega t} |\psi_1\rangle) \left( (a^d - a) \left( |\psi_0\rangle + e^{-i\omega t} |\psi_1\rangle \right) \right) \]
\[ = i \frac{\hbar}{8} \left( (|\psi_0\rangle + e^{i\omega t} |\psi_1\rangle) \left( |\psi_1\rangle + e^{-i\omega t} \sqrt{2} |\psi_2\rangle - e^{-i\omega t} |\psi_0\rangle \right) \right) \]
\[ = i \frac{\hbar}{8} \left( e^{i\omega t} - e^{-i\omega t} \right), \]
which gives
\[ \langle \hat{p} \rangle_t = -\frac{\hbar}{2} \sin(\omega t). \]

- We will compute \( \langle \hat{x}^2 \rangle_t \) by noting that
\[ \langle \psi(t)|(a^d + a) = \left[(a^d + a)|\psi(t)\right]^\dagger \]
and that we computed \((a^d + a)|\psi(t)\) already for \( \langle \hat{x} \rangle_t \):
\[ \langle \hat{x}^2 \rangle_t = \frac{\hbar}{2m\omega} \langle \psi(t)|(a^d + a)^2|\psi(t) \rangle \]
\[ = \frac{\hbar}{4m\omega} \left( |\psi_1\rangle + e^{i\omega t} \sqrt{2} |\psi_2\rangle + e^{i\omega t} (|\psi_0\rangle + e^{-i\omega t} \sqrt{2} |\psi_2\rangle + e^{-i\omega t} |\psi_0\rangle \right), \]
which gives
\[ \langle \hat{x}^2 \rangle_t = \frac{\hbar}{m\omega}. \]
- We will similarly compute \( \langle \hat{p}^2 \rangle_t \) by noting that
\[ \langle \psi(t)|(a^d - a) = -\left[(a^d - a)|\psi(t)\right]^\dagger \]
as:
\[ \langle \hat{p}^2 \rangle_t = -\frac{\hbar}{2} \langle \psi(t)|(a^d - a)^2|\psi(t) \rangle \]
\[ = \frac{\hbar}{4} \left( |\psi_1\rangle + e^{i\omega t} \sqrt{2} |\psi_2\rangle - e^{i\omega t} (|\psi_1\rangle + e^{-i\omega t} \sqrt{2} |\psi_2\rangle - e^{-i\omega t} |\psi_0\rangle \right), \]
which gives
\[ \langle \hat{p}^2 \rangle_t = m\omega \hbar. \]

As a final check, we note that
\[ \left[ \langle \hat{x}^2 \rangle_t - \langle \hat{x} \rangle_t^2 \right] \left[ \langle \hat{p}^2 \rangle_t - \langle \hat{p} \rangle_t^2 \right] = \frac{\hbar^2}{4} \left( 1 + \sin^2 \omega t \right) \left( 1 + \cos^2 \omega t \right) \]
which is manifestly greater than or equal to \( \frac{\hbar^2}{4} \) for all times, in line with the uncertainty bound.
(b) Right after the measurement we start with a wavefunction

\[ \langle x | \psi(t_0) \rangle = \psi(x, t_0) = \delta(x) \]

at \( t = t_0 \). Of course, strictly speaking, this is not in the Hilbert space of square integrable functions. Nonetheless, we will find that we can compute a time evolved version of this wavefunction. However, the time evolved wavefunction will also fail to be a square integrable function.

To perform the time evolution we will simply expand the initial state in terms of energy eigenstates of harmonic oscillator:

\[
\langle x | \psi(t) \rangle = \langle x | e^{-i\hat{H}(t-t_0)/\hbar} | \psi(t_0) \rangle \\
= \sum_{n=0}^{\infty} \langle x | e^{-i\hat{H}(t-t_0)/\hbar} | \psi_n \rangle \langle \psi_n | \psi(t_0) \rangle \\
= e^{-i\omega(t-t_0)/2} \sum_{n=0}^{\infty} e^{-i\omega n(t-t_0)} \langle x | \psi_n \rangle \langle \psi_n | \psi(t_0) \rangle
\]

To obtain the last line we have used the fact that \( \hat{H} | \psi_n \rangle = \hbar \omega (n + \frac{1}{2}) | \psi_n \rangle \) for the harmonic oscillator. The wavefunctions, \( \langle x | \psi_n \rangle \), for the energy eigenstates are

\[
\langle x | \psi_n \rangle \equiv \psi_n(x) = \frac{(m\omega/\pi\hbar)^{1/4}}{\sqrt{2^nn!}} e^{-m\omega x^2/2\hbar} H_n \left[ x(m\omega/\hbar)^{1/2} \right].
\]

The inner products, \( \langle \psi_n | \psi(t_0) \rangle \), on the other hand are given by

\[
\langle \psi_n | \psi(t_0) \rangle = \int dx' \langle \psi_n | x' \rangle \langle x' | \psi(t_0) \rangle \\
= \int dx' \psi_n^\dagger(x') \delta(x') \\
= \psi_n^\dagger(0).
\]

Combining these results we get

\[
\langle x | \psi(t) \rangle = e^{-i\omega(t-t_0)/2} e^{-m\omega x^2/2\hbar} \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \sum_{n=0}^{\infty} \frac{1}{2^n n!} e^{-i\omega n(t-t_0)} H_n \left[ x(m\omega/\hbar)^{1/2} \right] H_n(0).
\]

We can simplify this result further by noting that \( H_n(0) \) is zero for odd \( n \) and that for even ones we have \( H_{2n}(0) = (-1)^n \frac{(2n)!}{n!} \) which finally gives us

\[
\langle x | \psi(t) \rangle = e^{-i\omega(t-t_0)/2} e^{-m\omega x^2/2\hbar} \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \sum_{n=0}^{\infty} \frac{1}{n!} \left( -e^{-2i\omega(t-t_0)/4} \right)^n H_{2n} \left[ x(m\omega/\hbar)^{1/2} \right].
\]

An even simpler result can be obtained using the expression for \( \langle x | e^{-i\hat{H}(t-t')/\hbar} | x' \rangle \) given in equation 7.3.28 of Shankar. However, its simplest derivation comes from path integral formalism which is beyond our scope here.

5