Spontaneous Symmetry Breaking (SSB)

- Ground state vs symmetry.

The ground state of a quantum mechanical system does not necessarily respect the full set of symmetries of the system. For a simple example, consider a magnet modeled by nearest neighbor interactions

\[ H = -\lambda \sum_{i,j} \hat{s}_i \cdot \hat{s}_j \]

The Hamiltonian has 3D rotational symmetry, SO(3). At the same time, the ground state would be a configuration in which all the spins \( \hat{s}_i \) are aligned along a certain direction \( \vec{n} \). Therefore, the symmetry of the ground state is only the rotation around axis \( \vec{n} \). In other words, the ground state only preserves an SO(2) subgroup of the full symmetry group SO(3).

This is called spontaneous symmetry breaking (as opposed to explicit symmetry breaking).

- Spontaneous Symmetry breaking with real scalar.

\[ \mathcal{L} = \frac{1}{2} (\partial \phi)^2 - V(\phi), \quad V(\phi) = \frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4} \phi^4 \]

This system has a symmetry \( \phi \rightarrow -\phi \) \( (\mathbb{Z}_2) \). The ground state of this system can be found by minimizing potential energy \( V(\phi) \), since the minimal kinetic energy is zero.

\[ \mu^2 > 0 \quad \phi = 0 \rightarrow V_{\text{min}}. \]

ground state preserves symmetry: \( \phi \leftrightarrow -\phi \).

\[ \mu^2 < 0 \]

\[ V(\phi) = \pm \frac{1}{\sqrt{\lambda}} |\phi| \]

Either \( \phi = 0 \) or \( \phi = -\phi \) breaks \( \phi \leftrightarrow -\phi \) symm.

Since \( \phi = \pm \phi \) labels the ground states, also known as the "vacuum", \( \pm \phi \) is called the vacuum expectation value, \( (\text{V E V}) \), or \( \phi \text{ v.e.v.} \).

The arguments so far are too quick. Let's be more careful.
- SSB in quantum mechanics.

Consider double well potential

\[ \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle) \]

$|+\rangle$ and $|-\rangle$ center on the two centers of the double well.

However, it's well known that neither $|+\rangle$ nor $|-\rangle$ is the ground state. Quantum tunnelling mixes $|+\rangle$ and $|-\rangle$. The true ground state is

\[ |0\rangle = \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle) \]
Back to field theory.

The key difference is that in field theory, we have infinite # of degrees of freedom. Or, in other words, the volume is infinite.

In particular, the tunelling rate is proportional to

\[ e^{-V} \]

where \( V \) is the spatial volume. \( V \to \infty \) in field theory.

The Hamiltonian can be written as

\[
\begin{pmatrix}
E_0 & \delta \\
\delta^* & E_0
\end{pmatrix}
\]

\[ E_0 = \langle +1 | H | +1 \rangle = \langle -1 | H | -1 \rangle \]
\[ \delta = \langle +1 | H | -1 \rangle \]

where \( | +1 \rangle \) and \( | -1 \rangle \) are ground states centered on \( \phi = \pm \phi_0 \).

In field theory, \( J \propto e^{-V} \to 0 \)

Therefore, \( \phi = \pm \phi_0 \) are legitimate ground states, we can denote them as \( | + \rangle \) and \( | - \rangle \).

Now, due to the degeneracy of ground state, we can as use any combination of them as the ground state.

In particular, we could use

\[ | S \rangle = \frac{1}{\sqrt{2}} (| + \rangle + | - \rangle) \]
\[ | A \rangle = \frac{1}{\sqrt{2}} (| + \rangle - | - \rangle) \]

as ground states.
Note that $|s\rangle$ and $|A\rangle$ preserves the $\phi \leftrightarrow -\phi$ symmetry. Why do we usually work in the $|t\rangle$ and $|\rightarrow\rangle$ basis and talking about symmetry breaking?

It's important to recognize that

$$\langle -1 \phi | 1 + \rangle = 0$$

for any local operator $O(x)$. It's easy to argue that any such matrix element must vanish as $e^{-\text{Volume}}$, since in order to turn $|t\rangle$ into $|\rightarrow\rangle$, change must happen over the full spatial volume. For any local operator $O(x)$, this must be vanishing.

Therefore, if we start with a particular vacuum state $|t\rangle$ (same argument applies to $|\rightarrow\rangle$), we can build Hilbert space $\mathcal{H}_t$, which is completely factorized from the Hilbert space on the other vac. state $|\rightarrow\rangle$. If we consider local operators acting on $\mathcal{H}_t$, we never need to worry about the other ground state.

Take for example the field operator $\phi(x)$. We have argued that $\langle +1 \phi(x) | \rightarrow\rangle = 0$. On the other hand,

$$\langle A \mid \phi(x) \mid s\rangle = \frac{1}{2} \langle +1 \phi(x) | t\rangle - \frac{1}{2} \langle -1 \phi(x) | \rightarrow\rangle$$

$$= \frac{1}{2} v - \frac{1}{2} (-v) = v$$

So in $|A\rangle$ and $|s\rangle$ basis, we have to worry about both of them at the same time.
Another to illustrate the same inconvenience is to consider the so-called “cluster decomposition principle”. For field theory on $\mathcal{H}_+$, we have

$$\langle +1 | \phi(x) \phi(y) | + \rangle \rightarrow \langle +1 | \phi(x) | + \rangle \langle -1 | \phi(y) | - \rangle$$

$$| x - y | \rightarrow \infty$$

Which is formally saying $\phi(x)$ is “local”.

On the other hand,

$$\langle s | \phi(x) \phi(y) | s \rangle = \frac{1}{2} \left( \langle +1 | \phi(x) \phi(y) | + \rangle + \langle -1 | \phi(x) \phi(y) | - \rangle \right)$$

$$| x - y | \rightarrow \infty \quad \frac{1}{2} \left( \langle +1 | \phi(x) | + \rangle \langle +1 | \phi(y) | + \rangle + \langle -1 | \phi(x) | - \rangle \langle -1 | \phi(y) | - \rangle \right)$$

$$= v^2$$

and

$$\langle s | \phi(x) | s \rangle = \frac{1}{2} \left( \langle +1 | \phi(x) | + \rangle + \langle -1 | \phi(x) | - \rangle \right) = 0$$

Therefore, in the $| + \rangle$ and $| s \rangle$ basis, cluster decomp is broken for field operator $\phi(x)$.

At the same time, let’s remember that spontaneous symmetry breaking (the choice of $| + \rangle$ or $| - \rangle$) is fundamentally a choice of basis. The real physical statement is that there is a degeneracy of vacua.
Spontaneous breaking of continuous symmetry

- Consider complex scalar field

\[ L = \partial_{\mu} \phi^* \partial^{\mu} \phi - V(\phi), \quad V(\phi) = \mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2 \]

There is a global $U(1)$ symmetry

\[ \phi(x) \rightarrow e^{i\nu} \phi(x) \]

Or, equivalently, with $\phi(x) = \phi_1(x) + i \phi_2(x)$, the symmetry is an $SO(2)$ rotation in the space of 2 real fields $\phi_1$ and $\phi_2$.

- "Phases" of this theory

1) $\mu^2 \gtrless 0$ groundstate or vacuun has $\phi(x) = 0$

This ground state is invariant under $\phi(x) \rightarrow e^{i\nu} \phi(x)$. We call this symmetry "manifest".

Physically, this implies that the particles, or oscillations, in $\phi_1$ and $\phi_2$ have same masses, $\mu^2$. Therefore, without SSB, the manifest $U(1)$ symmetry implies a degeneracy in the particle spectrum.
We have the so-called "minimum bias" potential,
\[ V(x) = \lambda (2p^2 - m^2) + \text{constant} \]
with \[ p^2 = \frac{\text{\textit{g}}}{\text{\textit{t}}} \].

A ground state solution is \( 1/2 |x| = \frac{1}{2} \). There are inﬁnitely many degenerate ground states, which can be parameterized by
\[ \phi(x) = \frac{i}{\sqrt{2}} \exp(i \pi \text{\textit{g}} / \text{\textit{t}}), \quad \text{\textit{g}} \in [0, \pi]. \]

Similar to the case of discrete symmetry, we can choose any one of them as starting point. For invariance, we consider states with \( \text{\textit{g}} = 0 \).

The original O(3) symmetry acts on the wave state as a shift
\[ \phi \rightarrow e^{i \phi}; \quad \varphi \rightarrow \varphi + \varphi \]

This shift, called a \textit{shift symmetry}, leaves the scalar potential invariant.
We then study the particle spectrum by considering small fluctuations around the vac., parameterized as

\[ \phi(x) = \frac{1}{\sqrt{2}} (\psi + h(x)) e^{i \frac{\pi(x)}{\nu}} \]

We obtain

\[ L = \frac{1}{2} (\partial \psi)^2 - \frac{1}{2} m^2 \psi^2 - \lambda \psi^3 - \frac{\lambda}{4} \psi^4 + \frac{1}{2} (\partial \pi)^2 + \frac{1}{\sqrt{2} \nu} (\partial \pi)^2 h + \frac{\hbar^2}{4 \nu^2} h^2 (\partial \pi)^2 \]

where \( m^2_h = 2 \lambda^2 \nu^2 = -\mu^2 \)

At the same time, we observe that \( \Pi \) only has derivative couplings, or equivalently, it is massless.

\( \Pi \) is called a Goldstone boson, or Nambu–Goldstone boson. The existence of a massless mode for each broken continuous global symmetry is called Goldstone theorem.

The fact \( M_{\Pi} = 0 \) can be understood from the shift symmetry, acting on \( \Pi \) as

\[ \Pi \rightarrow \Pi + 5 \alpha \]

A would-be non-zero mass term \( m^2_{\Pi} \Pi^2 \) clearly breaks this symmetry.

A shift symmetry is also called a non-linear realization
A shift symmetry is also called a non-linear realization.
Pattern of general SSB.

Consider global symmetry group $G$, under which a set of scalar fields $\phi_a$ furnish an irreducible representation.

Under $G$, the scalar fields transform as

$$\phi^a \rightarrow \phi^a + \alpha \Delta^a(\phi)$$

where $\Delta^a(\phi)$ is a general linear function in $\phi$.

After SSB, a subset of scalar fields acquire VEVs $\langle \phi^a \rangle = \phi^a_0$. As a result, the global symmetry is broken down to its subgroup

$$G \xrightarrow{\langle \phi \rangle} H.$$  

Note that $\phi^a_0$'s do not transform under the unbroken subgroup:

$$\kappa \Delta^a(\phi) = 0$$

$$\in H.$$

To study the spectrum after SSB, Taylor expand the scalar potential around the ground state

$$V(\phi) = V(\phi_0) + \frac{1}{2} \delta \phi^a \delta \phi^b \left. \frac{\partial^2 V}{\partial \phi^a \partial \phi^b} \right|_{\phi^a = \phi^a_0} \phi^a_0 \delta \phi^b + \ldots$$

where $\delta \phi^a = \phi^a - \phi^a_0$.

Note that there is no linear term, since the ground state satisfies

$$\left. \frac{\partial V(\phi)}{\partial \phi^a} \right|_{\phi^a = \phi^a_0} = 0.$$
Since transformation $\phi \rightarrow \phi + \omega \Delta^a(\phi)$ is a symmetry, it leaves $V(\phi)$ unchanged

$$V(\phi^a) = V(\phi^a + \omega \Delta^a(\phi))$$

or equivalently

$$\Delta^a(\phi) \frac{\partial}{\partial \phi^a} V(\phi) = 0$$

Taking another derivative w.r.t. $\phi^b$, and setting all scalar fields to their VEVs, we obtain

$$\frac{\partial \Delta^a(\phi)}{\partial \phi^b} \left( \frac{\partial V}{\partial \phi^a} \right) \phi^a, b = \phi^o, b + \Delta^a(\phi) \left( \frac{\partial^2 V}{\partial \phi^a \partial \phi^b} \right) \phi^a, b = \phi^o, b = 0$$

$$0 \text{ since } \frac{\partial V}{\partial \phi^a} \bigg|_{\phi} = 0$$

minimization condition

$$\Rightarrow \Delta^a(\phi) \left( \frac{\partial^2 V}{\partial \phi^a \partial \phi^b} \right) \phi^a, b = \phi^o, b = 0$$

Two solutions

i) $\Delta^a(\phi) = 0$ VEV of $\phi^a$ does not break symmetry.

i.e., symmetry transformation belongs to the unbroken subgroup H.

ii) $\Delta^a \frac{\partial^2 V}{\partial \phi^a \partial \phi^b} = 0$

$\Delta^a$ is an eigenvector $M^2_{ab} = \frac{\partial^2 V}{\partial \phi^a \partial \phi^b}$ with zero eigenvalue. From equation 1, we see that $M^2_{ab}$ is just the mass matrix. The existence of zero eigenvalues $\rightarrow$ zero mass states (Goldstones), identified with the corresponding eigenvectors.
From this discussion, it's clear that for symmetry breaking \( G \rightarrow H \), the number of Goldstones is the same as the number of symmetry transformation generators which is not in \( H \), or in other words, in the coset \( G/H \).

\* A simple example

Consider 2 real scalar fields \( \phi_1, \phi_2 \) with an \( SO(2) \) symmetry (This is equivalent to one complex scalar \( \phi = \phi_1 + i \phi_2 \) with a \( U(1) \) symmetry)

The \( SO(2) \) transformation is

\[
\begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}
\]

\( \alpha \rightarrow 0 = \phi + \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \)

Or in the notation we used earlier

\[
\Delta(\phi) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -\phi_2 \\ \phi_1 \end{pmatrix}
\]

In the symmetry breaking phase, we can choose VEVs

\[
\langle \phi_1 \rangle = \phi_{10}, \quad \langle \phi_2 \rangle = 0
\]

Hence, the Goldstone is

\[
\Delta(\phi) \big|_{\phi = \phi_0} = \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix}
\]

along the \( \phi_2 \) direction, as expected.
Consider U(1) gauge symmetry, with vector gauge boson $A_\mu$. The gauge coupling constant is $g$. We also introduce complex scalar $\phi$ with charge $Q$.

Covariant derivative is defined as

$$ D_\mu \phi = (\partial_\mu + igQ A_\mu(x)) \phi(x) $$

Under gauge transformation

$$ \phi \to e^{-iQ \alpha(x)} \phi, \quad A_\mu \to A_\mu - \frac{1}{g} \partial_\mu \alpha(x) $$

we have

$$ D_\mu \phi \to e^{-iQ \alpha(x)} D_\mu \phi $$

The Lagrangian of this system is

$$ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi|^2 - V(\phi) $$

where

$$ V(\phi) = + \mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2 $$

$$ \mu^2 < 0 \Rightarrow \langle \phi \rangle = \frac{\mu}{\sqrt{2}} = \frac{|m|}{\sqrt{2}} $$

Parameterize excitation as

$$ \phi(x) = \frac{1}{\sqrt{2}} (\sin + h \alpha) e^{i \frac{\pi \alpha}{1}} $$

For simplicity, we work with the example of $Q = +1$. 

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We have

\[ |D_\mu \phi|^2 = (D^\mu \phi)^* (D^\mu \phi) \]

\[ = \frac{1}{2} (\partial h)^2 \]

\[ + \frac{1}{2} \left( (\partial \pi)^2 + \frac{1}{2} u^2 (\partial \pi)^2 h^2 + \frac{2}{\sqrt{2}} \frac{1}{\sqrt{2}} (\partial \pi^0 h) \right) \]

\[ + \frac{1}{2} \left( g^2 (u + h)^2 A^2 + 2 g (\partial \pi^0) (u + h) A \right) \]

\[ + \frac{3}{4} \frac{g}{\sqrt{2}} (\partial \pi) h (u + h) A \]

To interpret this result, we need to remember the gauge equivalence

\[ A \rightarrow A - \frac{1}{9} \partial \omega(x), \quad \pi \rightarrow \pi - \nu \omega(x) \]

In particular, we can completely remove \( \pi(x) \) using this equivalence. This particular gauge is called unitary gauge. In unitary gauge, we have

\[ \tilde{L} = \frac{1}{2} (\partial h)^2 + \frac{g^2}{2} (u + h)^2 A^2 - \frac{1}{4} F_{\mu \nu}^2 \]

\[ = \frac{1}{2} (\partial h)^2 + \frac{1}{2} m_A^2 A^2 - \frac{1}{4} F_{\mu \nu}^2 \]

\[ + g^2 \partial h A^2 + \frac{g^2}{2} h A^2 \]

with \( m_A^2 = g^2 u^2 \)

Therefore, gauge boson acquired a mass.

This is called the Higgs mechanism.
Essentially, what “going to unitary gauge” doing is to work with

$$\hat{A} = A - \frac{1}{3} \pi \Delta$$

and relabeling $$\hat{A} \rightarrow A$$

Note that $$\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$ is invariant under this redefinition.
This fact may seem to be a little bit mysterious. After all,

1) Massive vector boson has 3 degrees of freedom (called 2 transverse and 1 longitudinal). On the other hand, massless vector boson only has 2 degrees of freedom. They seem to be very different.

2) The Goldstone boson, $\pi(x)$, seems to have disappeared.

In fact, everything fits well together. In gauge theory with SSB, $\pi(x)$ (now called the "would-be Goldstone") becomes the 3rd degree of freedom. Together with the other 2 d.o.f. already there in the vector before SSB, the form a massive vector.

Before SSB:

\begin{align*}
A_\mu (\pm \text{polarization}) & \quad \phi(0) \quad 2 \text{ d.o.f.} \\
\text{down} & \quad \Rightarrow \\
\text{after SSB: massive } A_\mu, 3 \text{ d.o.f.}
\end{align*}

The procedure we followed earlier, using gauge transformation to remove $\pi(x)$, is sometimes called "gauging away the Goldstone".
The "Higgsing" of a gauge theory is sometimes called SSB of gauge symmetry. Just like the SSB of global symmetry, this terminology is somewhat misleading. After SSB, there is still a full gauge invariance in the Lagrangian. To see this explicitly, it is useful not to go to the unitary gauge. In general, the gauge boson + goldstone terms can be written as

\[ \frac{1}{2} g^2 v^2 \left( A^\mu - \frac{1}{g} \partial^\mu \frac{\Pi(x)}{v} \right)^2 \]

which is manifestly invariant under gauge transformation

\[ A \rightarrow A - \frac{1}{g} \partial \nu(x), \quad \Pi \rightarrow \Pi - \nabla \times (x) \]

Again similar to the case of SSB in global symmetry, this is called a non-linear realization.
More complicated symmetry breaking.

1) Consider 2 complex scalar field, $\phi_a, \phi_b$, with charge $Q_a, Q_b$. After SSB, we have VEVs $\langle \phi_a \rangle = \phi_a, \langle \phi_b \rangle = \phi_b$.

The covariant derivative is

$$|D_\mu \phi_a|^2 + |D_\mu \phi_b|^2, \quad D_\mu \phi_{a,b} = (\partial_\mu - i g Q_{a,b} A_\mu) \phi_{a,b}.$$ 

We know after SSB, we will have 1 massive vector. In addition, out of the 4 d.o.f. in the original 2 complex scalars, 1 is the would-be Goldstone and will be "eaten". There are 3 additional scalars.

2) 2 Gauge symmetries $U(1)_1, U(1)_2$ with coupling $g_1$ and $g_2$. 1 complex scalar $\phi$ charged under both with charges $Q_1$ and $Q_2$.

$$D_\mu \phi = \partial_\mu - i g_1 Q_1 A_1 - i g_2 Q_2 A_2.$$ 

After SSB, there will only be a massive vector, since there is only 1 Goldstone to be eaten.
3) Mass of chiral fermion.

Consider left handed fermion \( \psi_L \) charged under \( U(1) \) with charge \( Q_L \), and \( \psi_R \) with \( Q_R \). And \( Q_L \neq Q_R \) (called chiral fermion).

Kinetic term can be fully gauge invariant

\[
\psi^+_L (\partial^\mu - i g Q_L A^\mu) \overline{\psi}^-_L + \psi^+_R (\partial^\mu - i g Q_R A^\mu) \overline{\psi}^-_R
\]

However, mass term \( m (\psi^+_L \psi^-_R + \psi^+_R \psi^-_L) \) is not gauge invariant. Therefore, in this case, chiral fermion is massless.

However, chiral fermion could acquire mass after SSB, if the Higgs field \( \phi(x) \) has charge \( Q_{\phi} = Q_L - Q_R \). In this case

\[
\phi \psi^+_L \psi^-_R + \phi^* \psi^+_R \psi^-_L
\]

is gauge invariant and allowed. After SSB, \( \langle \phi \rangle = \frac{\mu}{\sqrt{2}} \). These terms becomes fermion mass terms with \( m = \frac{\mu}{\sqrt{2}} \).

This is exactly analogous to the case of fermion mass in the standard model.