3 S-Matrix

Consider scattering process in which
n particles with momenta $k_1, \ldots, k_n$
(isolated wave-packets at $t = -\infty$)

\[ \downarrow \text{scattering} \]

m particles with momenta $p_1, \ldots, p_m$
(isolated wave-packets at $t = +\infty$)

We would like to calculate amplitude

\[ \left< p_1, \ldots, p_m \mid k_1, \ldots, k_n \right> \]

\[ = \left< p_1, \ldots, p_m \mid S \mid k_1, \ldots, k_n \right> \sim S\text{-matrix} \]

\[ = \left< p_1, \ldots, p_m \mid e^{-iHT} \mid k_1, \ldots, k_n \right> \bigg|_{T \to \infty} \]

We have learned how to calculate $\left< e^{iHT} \phi(x_1) \ldots \phi(x_n) \right>$
in terms of free field correlator $\left< 0 \mid T \phi(x_1) \phi(x_2) \mid 0 \right>$
and perturbative expansion.

If $\left| k_1, \ldots, k_n \right> \sim \lim_{T \to \infty (1-i\epsilon)} e^{-iHT} \left| k_1, \ldots, k_2 \right> \bigg|_{T \to \infty (1-i\epsilon)}$
eigenstate of
interactiy theory.
eigen state of free field $\phi$. 
Then, since

\[ |k_1 \ldots k_n \rangle \propto a_{k_1}^+ a_{k_2}^+ \cdots a_{k_n}^+ \langle 0 | \]

and similarly for our state

\[ \langle p_1 \ldots p_m | \propto \langle 0 | a_{p_1} a_{p_2} \cdots a_{p_m} \], \quad \text{and} \]

We can write \( a, a^\dagger \) in terms of free field (\& and its time derivative).

We can rewrite S-matrix in terms of correlators

\[ \langle 0 | \mathcal{T} \phi(x_1) \cdots \langle 0 | \]

(with a perturbative expansion in interaction Hint or Hint)

However, the assumption in \( \bigstar \) is a big "if".

We will begin with a discussion which can cast it into a more rigorous form.
Interacting QFT

1) Particles interacting with each other.

2) Particle interacting with itself. (\[ \text{interaction} \])

\[ \text{Interaction connected diagram which cannot be separate further into single propagation} \]

\[ \text{"Propagation on its own"} \]

\[ \text{~"Free particle"} \]
1. Particle Irreducible diagrams

(1PI).

Any diagram than cannot be split in two by cutting open a single line.

Examples:

1PI:

\[ \begin{align*}
&0 \\
&0
\end{align*} \]

NOT 1PI:

\[ \begin{align*}
&0 \quad 0 \\
&X
\end{align*} \]

- Schematically: Scattering process looks like

\[ \text{general 2-point function} \]

All "blob"s can be expressed in terms \(1\text{PI} \). (discussed later).
Note: We will write

\[ S = 1 + i\tau \]

1. Trivial "forward" processes such as

\[ \begin{array}{c}
   & & \\
   & 0 & \\
   & & \\
\end{array} \]

\[ i\tau: \text{non-trivial scattering} \]

\[ \begin{array}{c}
   \times & \times \\
\end{array} \]

We will focus only on the nontrivial part of the S-matrix: \( i\tau \).

Therefore, we only include connected diagrams.
3.1 Källén–Lehmann Spectral Representation

- Consider general 2-point function

\[ \langle \Omega | T \Phi(x) \Phi(y) | \Omega^\prime \rangle \]

and assume \( \kappa^0 \geq \gamma^0 \).

Insert a complete set of eigenstate of full theory \( \{ | n \rangle \} \),
and choose \( | n \rangle \) to be also the momentum eigenstate.
(possible since \( [\hat{P}, \hat{H}] = 0 \)).
\( \hat{P} | n \rangle = p_n | n \rangle \).

- Spectrum of \( | n \rangle \)

Considering the theory with single particle states.
[isolated particles, very relevant for the real world, has meaningful scattering process]

\[ E = \sqrt{m^2 + p^2} \]
\( 1 \)-particle state.
- Assume \( \phi(x) \) can create 1 particle state out of the ground state.

(Reasonable assumption at least for weakly coupled theory).

\[
\langle \Omega | \phi(x) | \Omega \rangle (\neq 0)
\]

\[
\frac{\langle \Omega | \phi(x) | \Omega \rangle}{\sqrt{2}} e^{-i \mathbf{p} \cdot \mathbf{x}} = \langle \Omega | \phi(0) | \Omega \rangle e^{-i \mathbf{p} \cdot \mathbf{x}}
\]

Lorentz Inv.  A particle at rest.

\[
\mathbf{p}^2 + m^2 = E_p^2
\]

- We also have

\[
\langle \Omega | \phi(x) | \Omega \rangle = \langle \Omega | \phi(0) | \Omega \rangle e^{-i \mathbf{p} \cdot \mathbf{x}}.
\]

+ for a-particle state \( \phi(0) \) (at rest) \( \rightarrow \Omega \).

- We work with \( \langle \Omega | \phi(x) | \Omega \rangle = 0 \). It is always possible to make a change of field variable (by a constant shift) so that this is true.
Note:

a) It's possible to have bound state between $m$ & $2m$.

b) This is called a theory with a mass gap.

Lightest stable particle $m^2 = p_1^2$, where $p_1^2$ is the lowest non-zero eigenvalue.

c) For theory with massless particles, we have similar results.

d) For general theory with no mass gap, there could be no particles, no scattering theory.

- The completeness relation is

\[ \frac{1}{S} \int_0^\infty \frac{dM^2}{2\pi} \int \frac{d^6\vec{p}^'}{(2\pi)^4} \theta(p^0\sqrt{\vec{p}'^2 + M^2}) |n>=<n' | \\
\]

Note that we denote

i) $n$: quantum numbers other than mass $M$ & $\vec{p}$.

ii) $|n'>$: eigenvector with mass $M$ & momentum $\vec{p}'$.

iii) among $|n'>$, we denote $1-$ particle states by $|1'>$.
Finally have:

\[ \langle \sigma | \Phi(\vec{x}) \cdot \Phi(\vec{y}) | \sigma \rangle \]

\[ \int_0^\infty \frac{d M^2}{2 \pi} \int d^4 p \ \Theta(p^0) \ \delta(p^2 - M^2)(2\pi)^3 e^{-ip \cdot (\vec{x} - \vec{y})} \cdot \]

\[ \times \left[ 2\pi \int \frac{d^2 q}{(2\pi)^2} | \Phi(0) | m > ^2 \delta(M^2 - m^2) + \sum_n | \langle \sigma | \Phi(0) | n_\sigma > | ^2 \right] \]

\[ \rho(M^2) \leq \text{spectral density function.} \]

\[ \int_0^\infty \frac{d M^2}{2 \pi} \rho(M^2) D_F(\vec{x} - \vec{y}, M^2). \]

\[ \rho(M^2) = 2\pi \int \delta(M^2 - m^2) \cdot Z + \rho_{\text{continuum, bound states.}} \]
Note on expression of \( D_F(x-y) \)

\[
D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-i p \cdot (x-y)}
\]

\[
p^2 - m^2 + i\epsilon = p_0^2 - (|p|^2 + m^2) + i\epsilon
\]

\[
= p_0^2 - (E_p - i\epsilon)^2.
\]

\[
\text{complex plane of } p_0.
\]

for \( x^0 > y^0 \), complete contour from below. Pick up the residue at \( E_p - i\epsilon \)

\[
\Rightarrow D_F = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2E_p} e^{-i p \cdot (x-y)}.
\]
On the other hand

\[ \int \frac{d^4 p}{(2\pi)^4} \Theta(p^0) (2\pi) \delta(p^2 - m^2) e^{-ip \cdot (x - y)}. \]

using \( \delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} f(x - x_0). \)

\[ = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x - y)}. \]

Therefore, we verify that \( \text{for } x^0 > y^0 \)

\[ \int \frac{d^4 p}{(2\pi)^4} \Theta(p^0) (2\pi) \delta(p^2 - m^2) e^{-ip \cdot (x - y)} \]

\[ = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x - y)} \]

\[ = \mathcal{D}_p (x - y). \]
- 2-point function in momentum space \( G^{(2)}(p) \).

\[
\int d^4x \ e^{i p \cdot x} \langle 0 | \prod T \phi(x) \phi(0) | 0 \rangle = \frac{i \pi}{\rho^2 - m^2 + i \epsilon} + \int \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{\rho^2 - M^2 + i \epsilon}
\]

\( m^2 \), \( 4m^2 \), branch cut

single particle

isolated pole
- 1PI expression for $G^{(2)}$

\[ -i M^2(p^2) = \frac{\Box}{\lambda} + \frac{\Box}{\lambda^2} + \ldots \]

\[ = -1\text{PI} \]

\[ G^{(2)} \text{, full propagator.} \]

\[ = \frac{1}{p^2 - m_0^2} + \frac{1}{p^2 - m^2} \left( -\frac{M^2}{p^2 - m^2} \right) \frac{1}{p^2 - m^2} + \ldots \]

\[ = \frac{1}{p^2 - m_0^2 - M(p^2)} \]

Pole: $p^2 - m_0^2 - M(p_0^2) \bigg|_{p^2 = m^2} = 0$

$m^2_0$: physical "pole" mass.

\[ \frac{1}{p^2 - m_0^2 - M(p^2)} \xrightarrow{p^2 \to m^2} \frac{i \not{Z}}{p^2 - m^2} + \text{regular}. \]
3.2. **LSZ - reduction theorem (formula).**

Lehmann - Symanzik - Zimmermann.

- Scattering states, or, "in" and "out" states.

  a) "in" \( t = -\infty \) and "out" \( t = \infty \) states are multi-particle states in which particles are well separated wave packets.

  This requires interactions fall fast enough, at large space-like separation. ("official name: cluster decomposition principle")

  b) Very well motivated. We experimentally see such "almost free" particles.

  c) It's possible, however, to construct examples where there is no scattering theory.

    For example, the requirement that interaction falls off is easily satisfied in theory with a mass gap. Theory without mass gap generally gives more complicated behavior.
"in" state (straightforward generalization to "out" state)

→ An "in" state particle are prepared as wave packet

\[ |\phi_{\text{in}}\rangle = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi^*(k) \, k \rangle \]

\( k \rangle \) is a single particle state with definite momentum \( k \).

→ "in" state wave function is

\[ \langle \chi | \phi_{\text{in}} \rangle = \int \frac{d^3 k}{(2\pi)^3} \phi^*(k) e^{-ik \cdot \chi} = \phi_{\text{in}}^*(\chi). \]

\( k^2 = (k^2 + m^2) \).

\( \phi_{\text{in}}(\chi) \) satisfies Klein-Gordon equation.

→ "plane wave" limit:

\[ \phi^*(k) \propto \delta^{(3)}(k^2 - k_0^2). \]

"creation" operator at \( t \to -\infty \)

\[ A^+_{\text{in}}(t) = \frac{i}{\sqrt{2}} \int d^3 \chi \left( \partial_\chi \phi_{\text{in}} \Phi(\chi) - \phi_{\text{in}} \partial_\chi \Phi(\chi) \right). \]

We assume non-zero matrix element for field operator

\[ \langle \Sigma | \Phi(\chi) | k \rangle = \sqrt{2} e^{-i k \cdot \chi}. \]

Under reasonable assumptions, it can be shown

\[ \lim_{t \to -\infty} A^+_{\text{in}}(t) |\Sigma\rangle = |\phi_{\text{in}}\rangle. \]

This statement is equivalent to "interaction falls off at large separation" (therefore, particles can be created by \( A(-\infty) \) independently.)
LSZ reduction

We can replace \( \lim_{t \to -\infty} A^t(t) \) with

\[
\lim_{t \to -\infty} A^t(t) \rightarrow \frac{-i}{\sqrt{2}} \int dt \, \varphi_0 \int d^3x \left( \varphi_{\text{in}} \varphi_{\text{in}}^{\dagger} - \varphi_{\text{in}}^{\dagger} \varphi_{\text{in}} \right).
\]

Note that this integral also gives \( A^t(+\infty) \) which creates an identical wave packet in the future. However, since we are only interested in the non-trivial part of the S-matrix: \( iT \), \( A^t(+\infty) \) does not contribute.

(Note of the note: even \( iT \) has a forward scattering part in which in state is the same as out state. But, as we will see later, this part can be obtained from the rest of the \( iT \), so we don't have to worry about calculating it here.)

\[
\rightarrow \frac{-i}{\sqrt{2}} \int dt \, \varphi_0 \int d^3x \left( \varphi_{\text{in}} \varphi_{\text{in}}^{\dagger} - \varphi_{\text{in}}^{\dagger} \varphi_{\text{in}} \right)
\]

\[
= \frac{-i}{\sqrt{2}} \int d^4x \left[ (\vec{p}^2 - m^2) \varphi_{\text{in}} \varphi_{\text{in}}^{\dagger} - \varphi_{\text{in}}^{\dagger} \varphi_{\text{in}} \right]
\]

\[
= \frac{-i}{\sqrt{2}} \int d^4x \varphi_{\text{in}} \left[ \vec{p}^2 + m^2 \right] \varphi_{\text{in}}(x).
\]

We used the equation \( (\vec{p}^2 + m^2) \varphi_{\text{in}} = 0 \)
Recall $|\phi_{\text{in}}\rangle = A^+(-\infty) \mid \Sigma \rangle$

and use an identical treatment for the out state, we obtain the LSZ reduction for S-matrix

$$
\langle \phi_{\text{out}}^1 \ldots \phi_{\text{out}}^j \ldots \phi_{\text{out}}^n | \phi_{\text{in}}^1 \ldots \phi_{\text{in}}^i \ldots \phi_{\text{in}}^m \rangle
$$

$$
= \int \frac{d y_j}{\sqrt{2}} \phi_{\text{out}}^j(y_j) \left( \frac{i}{\sqrt{2}} \left( \partial_y + m^2 \right) \right).
$$

$$
\times \int \frac{d x_i}{\sqrt{2}} \phi_{\text{in}}^i(x_i) \left( \frac{i}{\sqrt{2}} \left( \partial_x + m^2 \right) \right).
$$

$$
\times \langle \Omega | T \Phi(y_1) \ldots \Phi(y_j) \ldots \Phi(x_1) \ldots \Phi(x_m) \mid \Sigma \rangle.
$$

\text{Momentum Space}

In practice, such as in collider experiment, we often prepare "in" state very close to momentum eigenstate and measure the momentum of "out" state. Therefore, it's convenient to replace wave-packet with plane-wave by taking

$$
\Phi(k') = \delta^{(3)}(k' - k_0) (2\pi)^3
$$

$$
\phi_{\text{in}} = e^{-i \mathbf{k}_0 \cdot \mathbf{x}}
$$
\[ \langle \text{out} \mid P_1 \ldots P_j \ldots P_n \mid k_1 \ldots k_i \ldots k_m \rangle \]

\[
= \prod_j \frac{P_j^2 - m^2}{i \sqrt{Z}} \cdot \prod_i \frac{k_i^2 - m^2}{i \sqrt{Z}}.
\]

\[
\times G^{(m+n)} \left( P_1 \ldots P_j, k_1 \ldots k_m \right).
\]

Note that for external particles far away from each other, \( P_j^2 - m^2 = 0 \) and \( k_i^2 - m^2 = 0 \) (on-shell).

Therefore, S-matrix for on-shell in and out states vanishes unless \( G^{(m+n)} \left( P_1 \ldots P_n, k_1 \ldots k_m \right) \) contains poles at \( P_j^2 = m^2 \) and \( k_i^2 = m^2 \).

And, this is precisely what \( G^{(m+n)} \left( P_1 \ldots P_n, k_1 \ldots k_m \right) \) has when external states are separated particles.
Källen–Lehmann representation tells us

\[ G^{(2)}(p) = \int d^4x e^{ip\cdot x} \langle S2 | T\phi(x)\phi(x) | S2 \rangle. \]

\[ = \frac{i\pi}{p^2 - m^2 + i\epsilon} + \int \frac{dM^2}{2\pi} \frac{\rho(M^2)}{4\pi^2} \frac{i}{p^2 - M^2 + i\epsilon} \]

- Single particle pole.
- Cancels \( \frac{p^2 - m^2}{2\sqrt{Z}} \) from LSZ (except a factor of \( \sqrt{Z} \)).

So, LSZ gives us the following prescription for calculation S-Matrix

\[ \langle p_1, ..., p_n | k_1, ..., k_m \rangle_{\text{out}} \]

\[ = (\sqrt{Z})^{m+n} \quad \text{connected} \]

\[ Z = 1 \quad \text{for free theory, and it can be computed perturbatively for interacting theory.} \]
- Crossing relation.

From LSZ.

\[ \langle \psi_0 \cdots \psi_n | \mathcal{S} | k_1 \cdots k_m \rangle = \left( \prod_{i=1}^{n} \int \frac{d\psi_i}{\sqrt{Z}} e^{i\int \psi_i \partial \bar{\psi}_i - \frac{\bar{\psi}_i^2 - m^2 + i\epsilon}{\sqrt{Z}}} \right) \left( \prod_{j=1}^{m} \int \frac{d\bar{\phi}_j}{\sqrt{Z}} e^{-i\int \bar{\phi}_j \partial \phi_j - \frac{\bar{\phi}_j^2 - m^2 + i\epsilon}{\sqrt{Z}}} \right) \]

\[ \langle \mathcal{D} | T \phi(x_1) \cdots \phi(x_n) \bar{\phi}(y_1) \cdots \bar{\phi}(y_m) | \mathcal{S} \rangle. \]

One can make an out-state with \( p \) into our in-state with momentum \( k \) simply by taking \( p \rightarrow -k \).

So we have

\[ \langle \psi_0 \cdots \psi_n | p \mathcal{S} | k_1 \cdots k_m \rangle \bigg|_{p = -k} = \langle \psi_0 \cdots \psi_{n-1} | \bar{\mathcal{S}} | k k_1 \cdots k_m \rangle. \]

"Crossing symmetry" of \( S \)-matrix.

It's a general property of QPT. It does not depend on perturbative expansion in terms Feynman diagrams.
a couple of loose ends.

1) Strictly speaking, plane-wave is not localized and an exact plane-wave cannot be separated from others. When we use plane-wave as an approximation, we actually mean:

a) In and out states are made of long wave packets with length \( L \), with \( \frac{1}{L} \approx \Delta p \). \( \Delta p \) is the precision we have in preparing and measuring those states as momentum eigenstates.

b) \( L \ll \) separation we use to define isolated in and out states.

2) What about the continuum?

a) In general, field operator can create multiparticle states

\[
\langle n^+ \mid \phi(x) \mid \Omega \rangle \neq 0 = e^{i \cdot \vec{p} \cdot \vec{x}} A_n
\]

b) Consider multi-particle wave-packet

\[
| \psi \rangle = \frac{1}{\sqrt{n}} \int \frac{d^3 \vec{p}}{(2\pi)^3} \Psi_n(\vec{p}) \mid n^+ \rangle
\]

\[
\langle \psi \mid A^+(\vec{p}) \mid \Omega \rangle = \frac{-i}{\sqrt{2}} \frac{1}{n} \int \frac{d^3 \vec{p}}{(2\pi)^3} \Psi_n(\vec{p}) \int \frac{d^3 \vec{k}}{(2\pi)^3} \Phi^*(\vec{k}) A_n
\]

\[
\times \int d^3 \vec{x} (e^{-i\vec{k} \cdot \vec{x}} \psi_0 e^{i\vec{p} \cdot \vec{x}} - \bar{\psi}_0 e^{-i\vec{k} \cdot \vec{x}} e^{i\vec{p} \cdot \vec{x}}).
\]
\[ \langle \downarrow \downarrow | A^+(t) | \uparrow \uparrow \rangle = \sum_n \int \frac{d^3 p}{(2\pi)^3} \left( \frac{1}{2\pi} \right)^2 (\tau^0 + k^0) \Psi_n^{*}(\tau) \Phi(k) A_n \]

\[ \times e^{i(p_0 - k_0) t} \]

where \( p_0 = (\vec{p}^2 + m^2)^{1/2} \quad k_0 = (\vec{k}^2 + m^2)^{1/2} \)

Since \( M \gg 2m > m, \quad p_0 > k_0 \).

As \( t \to \pm \infty \),

\[ \langle \downarrow \downarrow | A^+(t) | \uparrow \uparrow \rangle = 0 \quad \text{by Riemann-Lebesgue Lemma}. \]

In words, multi-particle and single particle state spread out differently, their overlap \( \to 0 \) as \( t \to \pm \infty \). Therefore \( A^+(\pm \infty) \) does not create multiple particle states.
3.3 S-Matrix element calculation.

We begin with \( LS Z \)

\[
\langle p_1 \cdots p_n | \mathcal{S} | k_1 \cdots k_m \rangle = 
\left( \prod_{i=1}^{n} \int d^4 x_i \frac{e^{i \mathbf{p}_i \cdot \mathbf{x}_i}}{i \sqrt{2}} \frac{\mathbf{p}_i^2 - m^2 + i \epsilon}{i \sqrt{2}} \right) \left( \prod_{j=1}^{m} \int d^4 y_j \frac{e^{i \mathbf{k}_j \cdot \mathbf{y}_j}}{i \sqrt{2}} \frac{\mathbf{k}_j^2 - m^2 + i \epsilon}{i \sqrt{2}} \right)
\]

\(\times \langle 52 | T \Phi(x_1) \cdots \Phi(x_n) \Phi(y_1) \cdots \Phi(y_m) | 52 \rangle \).

Consider the following diagram. (and setting \( Z = 1 \) for now).

Consider the following diagram. (and setting \( Z = 1 \) for now).

\[
\langle 52 | T \Phi(x_1) \cdots \Phi(x_n) \Phi(y_1) \cdots \Phi(y_m) | 52 \rangle = \int d^4 y \mathcal{D} \mathcal{P} (x_i - y) x F(x_1, y, \ldots) \rightarrow \text{rest of the diagram. (No } x_i \text{)}
\]

\[
= \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-i \mathbf{p} \cdot \mathbf{x}_i}}{\mathbf{p}^2 - m^2 + i \epsilon} \int d^4 y e^{i \mathbf{k} \cdot \mathbf{y}} F(y, x_i, \ldots).
\]

\[
= \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-i \mathbf{p} \cdot \mathbf{x}_i}}{\mathbf{p}^2 - m^2 + i \epsilon} F(p, x_i, \ldots).
\]
\[ \int d^4 \xi \ e^{i \cdot \xi} \ \frac{\partial^2}{\partial \xi^2 - m^2 + i \epsilon} \ \langle 52 | T \phi(x_1) \ldots \phi(x_n) \phi(y) \ldots | 0 \rangle \]

\[ = \ \frac{\partial^2}{\partial \xi^2 - m^2 + i \epsilon} \int d^4 p \ \delta(p - p_i) \ \frac{1}{p^2 - m^2 + i \epsilon} \ \tilde{F}(p, \xi_1 \ldots) \]

\[ = \ \tilde{F}(p, \xi_1 \ldots) \cdot \langle \text{`amputate propagator } \frac{1}{p^2 - m^2 + i \epsilon} \rangle \]

This procedure obviously carries through all external legs.

\[ \langle p_1 \ldots p_n | S' | k_1 \ldots k_n \rangle \]

\[ = \ \tilde{F}(p_1 \ldots p_n, k_1 \ldots k_n), \ e \ \text{fully amputated diagram.} \]

Note that \( \tilde{F}(p_1 \ldots p_n, k_1 \ldots k_n) \propto \delta^{(4)}(0, \sum_{i=1}^{n} p_i - \sum_{j=1}^{m} k_j) \)

(global 4-momentum conservation).

Again, we focus on nontrivial part of \( S' (S = 4 + i T) \)

\[ i T = (2\pi)^4 \delta^{(4)} \left( \sum_{i=1}^{n} p_i - \sum_{j=1}^{m} k_j \right) \cdot M \]

\[ i M: \ \text{fully connected, fully amputated, d-dimensional diagrams.} \]

With \( g \neq 1 \), there is an overall factor of \( (\sqrt{g})^{n+m} \).

We will treat this more carefully later.
Rules for calculating $iM_i$ in $\frac{1}{4}f^\alpha$ theory:

1) propagator: $\frac{i}{p^2 - m^2 + i\epsilon}$
2) vertex: $-i\lambda$
3) external line: 1
4) energy momentum conservation at each vertex.
5) integrate over loop momentum (other all fixed).
6) divide by symm. factor.

Example: 2 → 2 scattering.

Leading order:

Next leading order:
Next leading order.

\[ (-i\lambda)^2 \int \frac{d^4l}{(2\pi)^4} \frac{i}{\left(l^2 - m^2 + i\epsilon\right)^2} \]

"s-channel"

\[ \times \frac{i}{(k_1 + k_2 - l)^2 - m^2 + i\epsilon}. \]

\[ \lambda \quad (k_1 - p_1 - l) \]

"t-channel"

\[ \times \frac{i}{(k_1 - p_1 - l)^2 - m^2 + i\epsilon}. \]

\[ \lambda \quad (k_1 - p_1 - l) \]

"u-channel"

\[ \times \frac{i}{(k_1 - p_1 - l)^2 - m^2 + i\epsilon}. \]

The final result can be expressed in terms of a set of Lorentz invariants made of final and initial state momenta. Mandelstam variables,

\[ s = (k_1 + k_2)^2 = (p_1 + p_2)^2 = \sqrt{E_{cm}^2} \]

\[ t = (k_1 - p_1)^2 \]

\[ u = (k_1 - p_2)^2. \]

\[ s + t + u = 4 \left(\frac{m}{2}\right)^2. \]
3.4 Unitarity

$S$-matrix evolves in coming state a $t = -\infty$ to outgoing state at $t = +\infty$. Therefore, it must be unitary to preserve probability, i.e.

$$S^+ S = 1$$

Using $S = 1 + iT$, we have

$$-i(T - T^+) = T^+ T$$

Consider final state $\langle f |$, and initial state $\langle i |$, and matrix element

$$\langle f | T^+ T | i \rangle$$

Insert a complete set of states

$$\{ q_1, q_2, \ldots \}, \ldots, \{ q_1, i \}$$

1-particle 2-particle $n$-particle.

$$1 = \frac{1}{n} \left[ \left( \prod_{i=1}^{n} \int \frac{d^3q_i}{(2\pi)^3} \frac{1}{2E_i} \right) \langle \{ q_i \} | \{ q_i \} \rangle \right]$$

Using eq. (6), we have

$$-i \left[ \langle f | T | i \rangle - \langle f | T^+ | i \rangle \right]$$

$$= -i \left[ M(?i \rightarrow f) - M^+(f \rightarrow i) \right]$$

$$= \sum \left( \prod_{i=1}^{n} \int \frac{d^3q_i}{(2\pi)^3} \frac{1}{2E_i} \right) \left[ M\* (f \rightarrow \{ q_i \}) \cdot M (\emptyset i \rightarrow \{ q_i \}) \right].$$
In particular, when in-coming state is identical to the out-going state ("forward scattering"), we have

\[ 2 \text{Im} \sum_{11'} = \sum_{X} \int d^4 \Pi_X \left( \begin{array}{c} 11' \\ \leftrightarrow \\ \left\langle 1I \right| 1X \rangle \langle X1 | \left\langle i1 \right\rangle \end{array} \right) \text{phase-space}. \]

\[ X \text{ includes all possible intermediate state (on-shell)}. \]

\[ \leftrightarrow \]

Optical theorem.

Example: Verifying optical theorem in $\lambda \phi^4$.

We will consider scattering amplitude for $k_1 k_2 \rightarrow k_1 k_2$.

At leading order in $\lambda$: \[ i M^{(1)} = -i \lambda \]

\[ M^{(1)} = \lambda \quad (\text{real}) \]

Therefore, \[ \text{Im} M \text{ starts at least at order } \lambda^2. \]
Order $\lambda^2$:

Consider s-channel loop diagram

\[ \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2. \]

(t and u channel do not contribute to imaginary part of $\mathbf{M}$)

\[ i \mathcal{M}^{(s)} = \frac{\lambda^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{(\mathbf{k}_2 - \mathbf{q})^2 - m^2 + i\epsilon} \frac{1}{(\mathbf{k}_1 + \mathbf{q})^2 - m^2 + i\epsilon}. \]

Poles (in c.o.m. frame $\mathbf{k} = (\mathbf{k}^0, \mathbf{0})$).

\[ E_k = \sqrt{|\mathbf{p}|^2 + m^2} \]

\[ -\frac{1}{2} k^0 - E_q \quad \frac{1}{2} k^0 - E_q \quad -\frac{1}{2} k^0 + E_q \quad \frac{1}{2} k^0 + E_q \]

We choose to close the contour from below.

Consider first the pole at $-\frac{1}{2} k^0 + E_q$.

Residue contribution from the pole $\frac{2\pi i}{-2E_q}$.

Therefore, equivalently, we can replace

\[ \frac{1}{(q + \frac{1}{2})^2 - m^2 + i\epsilon} \rightarrow -2\pi i \delta((\frac{1}{2} + q)^2 - m^2), \]

"putting the propagator on-shell"
\[ i \mathcal{M}^{(2)} = -2\pi i \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{2E_q} \frac{1}{k^0(k^0-2E_q)+i\epsilon} \]

As \( \epsilon \to 0 \), \( \mathcal{M}^{(2)} \) is only imaginary when \( k^0 = 2E_q \). Using \( q_0 = -\frac{1}{2}k^0 + E_q \) as fixed by the pole at \( -\frac{1}{2}k^0 + E_q \), we have

\[
\left( \frac{k - q}{2} \right)^2 - m^2 \phi = 0
\]

Therefore, the propagator \( \frac{1}{(k - q)^2 - m^2 + i\epsilon} \) is on-shell as well.

Using \( \frac{1}{k^0 - 2E_q + i\epsilon} = \left\{ \frac{1}{k^0 - 2E_q} - i\pi \int \frac{d(k^0 - 2E_q)}{k^0 - 2E_q} \right\} \) principle value

we find in calculating \( 2 \text{ Im}(\mathcal{M}^{(2)}) \), we just need to replace

\[
\frac{1}{(k - q)^2 - m^2 + i\epsilon} \rightarrow -2\pi i \int \frac{d(k - q)^2 - m^2}{(k - q)^2 - m^2 + i\epsilon}
\]

Note, similarly, we can verify the pole at \( \frac{1}{2}k^0 + E_q \) does not contribute to the imaginary part.
To summarize, relabel the loop momenta as $p_1$, $p_2$ with constraint $p_1 + p_2 = k$.

Therefore, $\int \frac{d^4 p_1}{(2\pi)^4} = \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + p_2 - k)$.

To calculate $2\text{ Im} M$, we replace the 2 propagators with $\frac{1}{p^2 - m^2 + i\epsilon} \rightarrow -2\pi i \delta(p_i - m^2)$.

and carrying out $p_1^0, p_2^0$ integral. We have

$$2\text{ Im}(M) = 2\text{ Im}(M^{(2)}) \Rightarrow 2\text{ Im} \left( \begin{array}{c} \chi \end{array} \right)$$

$$= \frac{1}{2} \int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2E_1} \int \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2E_2} |M^{(3)}|^2 (2\pi)^4 \delta(p_1 + p_2 - k).$$

Bose-statistics

$$\int d^{4} \mathbf{p} \left| \prod \mathbf{p} \right|^{2}.$$  

This verifies optical theorem at the order of $\lambda^2$.

General cutting rules (Cutkosky). For calculating $2\text{ Im}(M)$.

1) Cut through diagram in all possible ways in which all the intermediate particles can be put on-shell.

2) For each cut, replace $\frac{1}{p^2 - m^2 + i\epsilon} \rightarrow -2\pi i \delta(p^2 - m^2)$ and carry out $p$ integral.

3) sum contribution from all possible cuts.
- Unstable particle.

- Consider $1 \rightarrow 1$ scattering, the amplitude is

$$M(p \rightarrow p) = -Z M^2(p^2).$$

Where $M^2(p^2)$'s sum of $1\Phi$ 2-point functions, and full propagator is

$$\frac{i}{p^2 - m_0^2 - M^2(p^2)}.$$

$$M^2(p^2) = \frac{\Delta 1\Phi}{\text{external particle}}.$$

If we can cut through $1\Phi$ and make intermediate particles on-shell, the following process is allowed

In other words, the external particle can decay and it is not stable.

$$\rightarrow$$

- On the other hand, Cutkosky rules tell us this is also when $-Z M^2(p^2) = M$ develops an imaginary part.

- Working in the limit where the imaginary part is small (true in weakly coupled theories since its proportion to some small couplings).

Define the physical mass by using the real part of $M^2(p^2)$:

$$m^2 - m_0^2 = \text{Re}[M^2(p^2 = m^2)] = 0.$$
The full propagator is

\[ \sim \frac{i Z}{p^2 - m^2 - i Z \text{Im}[M^2(p^2)]} \]

imaginary part of propagator.

For small \( \text{Im}[M^2(p^2)] \), its contribution is only important when \( p^2 \approx m^2 \). So, we can approximate.

\[ \sim \frac{i Z}{p^2 - m^2 + i \gamma} \] (Breit - Wigner)

Width: \( \Gamma = \frac{Z}{m} \text{Im}[M^2(m^2)] \)

From optical theorem,

\[ Z \text{Im}[M^2(p^2)] = - \text{Im}(M(p \to \phi)) = -\frac{1}{2} \frac{Z}{\gamma} \int d^4x |M(p \to X)|^2 \]

Therefore, decay width can be computed by

\[ \Gamma = \frac{1}{2m} \frac{Z}{\gamma} \int d^4x |M(p \to X)|^2 \]

all possible states (kinematically allowed).

- Note: If the imaginary part is large, i.e., \( \text{Im}[M^2(p^2)] \approx m^2 \), the notion of particle with well defined mass is not valid. This is usually the case for strongly coupled theories.
3.5 Cross Sections.

- Consider the collision of two bunches of particles. Note that modern collider experiments typically use two beams, each is consist of many bunches of particles with well defined momenta. There are also experiments in which a beam of particles are impinged on on a fixed target.

The likelihood of any scattering process in such experiments can be expressed in terms of cross sections.

- **Definition.**

![Diagram]

\[ n_A \rightarrow n_A' \]

\[ n_B \rightarrow n_B' \]

\[ l_A' \]

\[ l_B' \]

\[ n_{A/B} \text{ density of particles, } \]

Assume \( n_A, n_B \) are constant.

\[ \sigma = \frac{N_S}{n_B n_A \cdot A} \]

\[ n_{A/B} = n_{A/B} \cdot l_{A/B} ; \, \text{density of particles per unit area.} \]

A common area between two bunches.

Note that cross section has the dimension of area. It's NOT Lorentz invariant except with respect to boost along the direction of \( v_B - v_A \).
Consider incoming wave packets.

\[ |\psi_A\rangle = \int \frac{d^3k_A'}{(2\pi)^3} \frac{1}{\sqrt{2E_{k_A}}} \phi_A(k_A') |k_A'\rangle \]

and

\[ |\psi_B\rangle = \int \frac{dk_B'}{(2\pi)^3} \frac{1}{\sqrt{2E_{k_B}}} \phi_B(k_B') |k_B'\rangle. \]

where \( \phi_{A,B}(k_{A,B}) \) are sharply peaked at \( k_A \) and \( k_B \).

Consider a specific case (to simplify our steps without losing generality)

\[ \begin{array}{c}
\uparrow \\
( \text{spread out in transverse direction} ) \\
\downarrow
\end{array} \]

\[ \begin{array}{c}
\text{multiple (B) particles} \\
\text{1 particle (A)}
\end{array} \]

\[ |\psi_A \psi_B\rangle_{\text{in}} = \int \frac{d^3k_A'}{(2\pi)^3} \int \frac{d^3k_B'}{(2\pi)^3} \frac{\phi_A(k_A') \phi_B(k_B')}{\sqrt{2E_{k_A}} \sqrt{2E_{k_B}}} e^{-i\mathbf{b} \cdot \mathbf{k}_B} |k_A' k_B'\rangle \]

We will also approximate outgoing states as momentum eigenstates. Note that in practice, particle detectors measure momenta (rather than positions) of the particles, and typically with a resolution worse than the natural spread of particle momenta. Therefore, this is a good approximation.
\[ d_{out} \leq 1 = < p_1 \cdots p_n > \]

Therefore, we have differential probability

\[ dP(AB \rightarrow p_1 \cdots p_n) = \left( \frac{\pi}{f} \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |< p_1 \cdots p_n | S | \Phi_A \Phi_B >|^2 \]

\[ dN = \sum \text{all incoming particles} \text{dN} = \int d^2 b \ n_B \cdot dP. \]

- differential cross section

\[ \sigma = \frac{dN}{n_B n_A} = \int d^2 b \ dP. \]

\[ = \left( \frac{\pi}{f} \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \int d^2 b \ e^{i \cdot \bar{b} \cdot (k_b - k_b')} . \]

\[ x \left[ \frac{d^3 k_a}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi(k_a) \right] \left[ \frac{d^3 k_a''}{(2\pi)^3} \frac{1}{\sqrt{2E_k'}} \phi(k_a'') \right] . \]

\[ x \left[ \frac{d^3 k_b}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi(k_b) \right] \left[ \frac{d^3 k_b''}{(2\pi)^3} \frac{1}{\sqrt{2E_k'}} \phi(k_b'') \right] . \]

\[ x (\langle p_1 \cdots p_n | k_a' k_b' >) (\langle p_1 \cdots p_n | k_a'' k_b'' >^*) . \]

where we used notation \( \langle p_1 \cdots P | k \cdots > \).
Consider only the non-trivial scattering:

\[
\langle \psi_1 \cdots \psi_n | i T | \psi'_A, \psi'_B \rangle = i \int \frac{d^4 k''}{(2\pi)^4} \phi_{k''}^{(a)}(k'' + k'_A - 2p') \left( \psi_1 \psi_n \right) \phi_{k''}^{(b)}(k'' + k'_B - 2p')
\]

Integration over \( d^2 b \) produces \( \delta^{(a)}(k'_{b} - k'_{b}) \).

(\( k' \) only includes components perpendicular to \( U_B, U_A \)).

Note also

\[
\int d k'' \, d k'' \, \delta \left( k'' + k'' + \frac{\Sigma \mathbf{p}_F^2}{F} \right) \left( \mathbf{E}_A + \mathbf{E}_B - \Sigma \mathbf{E}_F \right) = \frac{1}{|U_A - U_B|}.
\]

We will use the fact (or the assumption) that \( \phi_A(k'_A) \) and \( \phi_B(k'_B) \) are sharply peaked around central value \( k'_A \) and \( k'_B \).

Carrying out \( S \)-function integrals (eliminating integration over \( d^3 k'' \) and \( d^3 k'' \)), replacing all functions (except \( \phi_{k'_A} \) and \( \phi_{k'_B} \)) of \( k'_A \) and \( k'_B \) with their central values \( k'_A \) and \( k'_B \), and using

\[
\int \frac{d^3 k'_{A,B}}{(2\pi)^3} \left| \phi_{k'_{A,B}} \right|^2 = 1,
\]

we finally have
\[ d\Sigma = \frac{1}{2 E_A 2 E_B \sqrt{\Delta - \mu_B}} \left( \frac{\prod d^3p_i}{(2\pi)^3} \frac{1}{2E_i} \right) (2\pi)^3 \delta^{(3)}(k_A + k_B - \sum p_i) \times |\mathcal{M}(k_A, k_B \rightarrow p_1 \ldots p_n)|. \]

**Example:**

2-body. We choose to work in the center of mass frame.

\[ \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{2E_1} \left[ \int \frac{d^3p_2}{(2\pi)^3} \frac{1}{2E_2} \delta^{(3)}(p_1 + p_2) \right] (2\pi)^3 \delta(E_{cm} - E_1 - E_2) \]

\[ = \int d\Omega \frac{p_1^2}{16\pi^2 E_1 E_2} \left( \frac{p_1}{E_1} + \frac{p_1}{E_2} \right)^{-1} \quad p_1 = |\vec{p}_1| = |\vec{p}_2|. \]

\[ = \int d\Omega \frac{1}{16\pi^2} \frac{p_1}{E_{cm}} \]

If reaction is symmetric with respect to rotation along the collision axis, we can further perform the integration over azimuth angle \( \phi \).

\[ \Phi d\Pi_2 = \int d\cos \theta \frac{1}{8\pi} \frac{p_1}{E_{cm}} \]
Implication of Unitarity:

\[ 2 \text{ Im} = \sum_{f'} \int \frac{d \Pi_f}{\mathcal{N}} \left[ \text{all possible final states} \right] \times \left[ f' \right] \text{f} \left( k_1 \right) \text{f} \left( k_2 \right) \]

\[ = 2 E_{\text{cm}} \left( 2 P_{\text{cm}} \right) \Upsilon_{\text{tot}} \left( k_1, k_2 \rightarrow \text{anything} \right) \]

\( P_{\text{cm}} \): momentum of particle A or B in c.o.m. frame.