4. Renormalization.

4.1 Introduction.

The universe as we understand it

\[ M_{\text{Planck}} = 10^{19} \text{ GeV}, \quad \text{Quantum gravity} \]

something else? unification, strings

New physics?

\[ 100 \text{ GeV}, \quad \text{weak interaction} \]
\[ W^\pm, Z \text{ boson.} \]

\[ \text{GeV}, \quad \text{strong interaction, } q, g \text{-bound states} \]

\[ \sim \text{MeV} \quad \text{nuclear force, } p, n \rightarrow \text{nucleus} \]

\[ \lesssim \text{eV} \quad \text{atoms} \]
\[ \text{molecule} \]

\[ \rightarrow \quad \text{Macroscopic, Newton, Maxwell} \]

Very different physics at very different length (energy) scales.
A remarkable fact:

At each energy scale, we are able to understand the physics while being very ignorant about the physics at higher energy or shorter distances.

Understanding renormalization in QFT will allow us to systematically understand this fact.

Note that if the physics cannot be understood at separate scales, it would be much more difficult (if possible at all) to understand anything, since it would require understanding physics at all possible scales.

We will discuss renormalization in 2 steps.

1) How do we do it in practice?

2) What does it mean?
4.2. Renormalization in practice.

4.2.1. Divergences

Consider $L = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{4!} \phi^4$.

$$-\imath M^2(p^2) = \frac{\mathcal{O}}{\lambda_0} + \frac{\mathcal{O}}{\lambda_0^2} + \cdots$$

$$\mathcal{O} = -\imath \lambda_0 \int \frac{\imath \delta^4(k)}{(2\pi)^4} \frac{i}{k^2 - m^2 + \imath \epsilon}.$$

Wick rotation:

$$\int_{-\infty}^{+\infty} \imath k_0^0 \delta(k^0) = i \int_{-\infty}^{+\infty} \imath k_E^0 \delta(k^0)$$

$$k^2 = k_0^2 - |k|^2 = -k_E^2$$

$$= -(k_E^2 + |k|^2).$$
\[ \langle \sigma \rangle = i \frac{\lambda_0}{2} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2} \]

This is divergent as \( |k_E| \to \infty \). To see how it diverges, we introduce momentum cutoff \( \Lambda \). This means the loop integral will only be performed for \( |k_E| < \Lambda \).

\[ \langle \sigma \rangle = \frac{\lambda_0}{32\pi^2} \Lambda^2 \quad \text{divergent as } \Lambda^2 \text{ as } \Lambda \to \infty \]

Introducing such a cutoff is called regularization. (more precisely, one of the regularization schemes).

a) The root of divergence is UV or short distance physics.

b) Cut off regularization scheme introduces a minimal distance, breaks translational invariance, etc. It could be used in practice but it's awkward.

We would like to have our prediction for experiments performed not depending on \( \Lambda \). This is not just because we cannot handle divergent result. Much more importantly, our experimental results at low energies should not depend on the precise knowledge of physics at much higher energy scales.

We will see, indeed, physical quantities do not depend on \( \Lambda \). The process of understand what precisely this
statement means is called renormalization.

A simple analogy (to show you that this is not crazy)

Vibration on the string (classical).

\[ \mu \frac{\partial^2 \psi}{\partial t^2} - T \frac{\partial^2 \psi}{\partial x^2} = 0. \]

mass density \( \mu \)
tension \( T \)

Let's try to make a microscopic model of string

\[ \frac{1}{k} \frac{\partial}{\partial x} \psi \]

\( T = k \Delta x \)
\( \mu = \frac{m}{\Delta x} \)
\( \Delta x: "UV\ cut-off" \)

However, as \( \Delta x \to 0 \), infinities!

The proper way of doing this is to write the theory in term of low energy observable \( T, \mu \) only.

(rather than "fundamental" parameters \( m, k \).) \( \therefore \) The results will be finite and independent of the choice of cutoff \( \Delta x \).
Superficial degree of divergence in $\frac{1}{4} \phi^4$.

$N_E$: # of external lines
$N_I$: # of internal lines
$N_V$: # of vertices

$4N_V = 2N_I + N_E$

$N_L$: # of loop momenta integral

$N_L = N_I - N_V + 1$

Each internal line gives $\int \frac{d^4k}{(2\pi)^4}$

Each loop momentum integral gives $\int \frac{d^4k}{(2\pi)^4}$

Superficial degree of divergence of Feynman diagram

$D = 4N_L - 2N_I$

$= 4 - N_E$

It's not the precise measure of whether a Feynman diagram is divergent. But, it does show that divergent diagrams are there (not just at lowest order of perturbation theory).
4.2.2. Renormalization of $\lambda_4^4 \phi^4$ theory.

We will show, using the simple example of $\lambda \phi^4$, how systematically renormalization works.

Every renormalization procedure begins with a regularization procedure which separates out the divergent piece to be dealt with.

We could use the momentum cut-off. But it's awkward in practice since it breaks a lot of symmetries.

The most popular regularization scheme is dimensional regularization. (dim reg)

dim reg asks we do loop integral not in space-time dimension $d=4$, but $d=4-\epsilon$.

We will try to understand what this means better later. Right now, just think of it as a pure technical trick which allows us to separate out divergent parts. (We are by no means implying there is some physical meaning for $d=4-\epsilon$.)


Carrying out dim-reg.ed loop integral

a) Generally, a loop integral can contain multiple propagators with different momenta (and masses in a theory with multiple fields).

We can simplify this by introducing the Feynman parameter trick, which is just the following identities.

\[ \frac{1}{A \cdot B} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} = \int_0^1 dx \, dy \, \delta(x+y-1) \frac{1}{[xA + yB]^2} \]

or more generally

\[ \frac{1}{A_1 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \, \delta(\sum x_i - 1) \frac{(n-1)!}{[x_1A_1 + \cdots + x_nA_n]^n} \]

Therefore, we can always combine multiple propagators into the form of

\[ \int \frac{d[x]}{(p^2 + \Delta)^m} \]

where \( d[x] \) denotes the necessary integration over Feynman parameters, \( m \) is some integer power, \( \Delta \) is a function of external momenta and Feynman parameters.
b) After using Feynman parameters, the loop integral has the general form

\[ \int \frac{d^{d}p}{(2\pi)^{d}} \frac{\left(p^{2}\right)^{m}}{(p^{2}+\Delta)^{m}} = \int \frac{d^{d}R}{(2\pi)^{d}} \int_{0}^{\infty} \frac{d\rho}{\rho} \frac{\rho^{d+2n}}{(\rho^{2}+\Delta)^{m}} \]

\[ = \left( \int \frac{d^{d}R}{(2\pi)^{d}} \right) \times \frac{d+2n-2m}{2} \times \frac{1}{2} \int_{0}^{1} x^{n+\frac{d}{2}-1} (1-x)^{m-n-\frac{d}{2}-1} \, dx \]

\[ = \left( \int \frac{d^{d}R}{(2\pi)^{d}} \right) \times \frac{d+2n-2m}{2} \times \frac{1}{2} B \left( n+\frac{d}{2}, m-n-\frac{d}{2} \right) \]

where we have separated the integration over Euclidean solid angle \( \int d\Omega_4 \) and the radial momentum \( \int_{0}^{\infty} d\rho \).

\[ (\pi)^{d/2} = \left( \int_{-\infty}^{\infty} dp \, e^{-p^2} \right)^{d} = \int d^{d}p \, e^{-\frac{d}{2} \cdot p^{2}} \]

\[ = \int d^{d}R \int_{0}^{\infty} dp \, \rho^{m-1} e^{-\rho^{2}} \]

\[ = \int d^{d}R \frac{1}{2} \int d(p^{2}) \left( p^{2} \right)^{d/2-1} e^{-p^{2}} \]

\[ \Gamma \left( d/2 \right) \]

<table>
<thead>
<tr>
<th>( d )</th>
<th>( \Gamma \left( \frac{d}{2} \right) )</th>
<th>( \int d^{d}R )</th>
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<tbody>
<tr>
<td>1</td>
<td>( \sqrt{\pi} )</td>
<td>( 2 )</td>
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<td>2</td>
<td>( \frac{\Gamma(1)}{\sqrt{\pi}} )</td>
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<td>3</td>
<td>( \frac{\Gamma(1/2)}{\sqrt{\pi}} )</td>
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<td>( \frac{\Gamma(1/4)}{\sqrt{\pi}} )</td>
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\[ B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \]

\[ \Gamma(z) = (z-1) \Gamma(z-1). \]

The \( \Gamma \)-function has isolated poles at \( z = 0, -1, -2, \ldots, -k \ldots \) with residue \( \frac{(-1)^k}{k!} \) at \( -k \).

For example:

\[ \Gamma(z - \frac{d}{2}) = \Gamma \left( \frac{\xi}{2} \right) = \frac{2}{\xi} - \delta_E + \ldots \]

\( \delta_E = \text{Euler constant} \approx 0.5772 \ldots \)

\( \delta \) often cancels in physical quantities.
Renormalization with counter-terms (BPHZ).

a) Begin with Lagrangian

$$\mathcal{L} = -\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{4!} \phi^4$$

{\phi_0, \lambda_0, m_0} are called bare variable (\phi_0) and bare parameter (\lambda_0, m_0). Doing calculation with bare para. and variables we will encounter divergences.

b) We change variable and rewrite the same Lagrangian as

$$\mathcal{L} = -\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$

$$+ \frac{1}{2} \delta_\lambda (\partial \phi)^2 - \frac{1}{2} \delta m m^2 \phi^2 - \frac{\delta \lambda}{4!} \phi^4$$

with

$$\phi_0 = Z^{1/2} \phi, \quad \delta_\lambda = Z - 1, \quad \delta m = m_0^2 Z - m^2, \quad \delta \lambda = \lambda_0 Z - \lambda.$$ 

\phi, renormalized field variable, \lambda, m: renormalized parameter

Claim:

@ Doing computation with renormalized (\phi, \lambda, m), with counter terms \delta_\lambda, \delta m, and \delta \lambda fixed by a set of physical conditions (renormalization conditions), we obtain finite results.

(\phi, \lambda, m) "more physical". We will see exactly what this means when discussing Wilsonian EFT later
c) Renormalization conditions.

There can be many choices of renormalization condition. All of them are equivalent, i.e., physical predictions do not depend on the choice of renorm. condition.

In practice, some of them are more intuitive, and some of them are more convenient.

For example, in this example, we will use

1) Full propagator

\[
\begin{align*}
\Gamma(p^2) &= \frac{\alpha^2}{p^2 - m^2} + \text{regular terms}.
\end{align*}
\]

(2 conditions: the position of the pole and its residue)

2) \[\begin{align*}
\text{Diagram}
\end{align*}\]

\[\begin{align*}
&= -i\lambda \quad \text{at } s = 4m^2, \ t = u = 0
\end{align*}\]

\[\text{amputated.}\]
- 1-loop renormalization of $\lambda \phi^4$.

\begin{align*}
\rho &= \rho_1 + \rho_2, \quad \rho^2 = s \\
\frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} & \frac{i}{k^2 m^2} \frac{i}{(k+p)^2 - m^2} \\
= (-i\lambda)^2 & i V(p).
\end{align*}

\begin{align*}
V(p^2) &= \frac{i}{2} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[ k^2 + 2x(k \cdot p) + (1-x)p^2 - m^2 \right]^2} \quad \text{(Feynman parameter)} \\
&= \frac{1}{2} \int_0^1 dx \int \frac{d^4l}{(2\pi)^d} \frac{1}{\left[ l^2 + \alpha(1-x)p^2 - m^2 \right]^2} \quad \text{(Wick rotation)} \\
&= -\frac{1}{2} \int_0^1 dx \int \frac{d^4l}{(2\pi)^d} \frac{1}{\left[ l^2 - \alpha(1-x)p^2 + m^2 \right]^2} \\
&= -\frac{1}{2} \int_0^1 dx \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} \frac{1}{\left[ m^2 - \alpha(1-x)p^2 \right]^{2-d/2}} \\
&\Rightarrow d = 4 - \epsilon = -\frac{1}{32\pi^2} \int_0^1 dx \left( \frac{2}{\epsilon} - \frac{\epsilon}{6} + \log(4\pi) + \log(m^2 - \alpha(1-x)p^2) \right) \\
&= 1 + \epsilon \log A + \ldots
\end{align*}

Note the logs come from expansion

\begin{align*}
(A)^\epsilon &= e^{\epsilon \log A} = 1 + \epsilon \log A + \ldots
\end{align*}
Next, we use renormalization condition on $2 \to 2$ $S$-matrix

\[ i\mathcal{M}(p_1, p_2 \to p_1, p_2) = \]

\[ -i\lambda + (i\lambda)^2 \left( iV(s) + iV(t) + iV(u) \right) \]

\[ \text{counterterm} \quad (-i\delta\lambda), \]

\[ = -i\lambda + (i\lambda)^2 \left( iV(s) + iV(t) + iV(u) \right) + (-i\delta\lambda). \]

The renormalization condition used here

\[ i\mathcal{M} = -i\lambda \quad \text{at} \quad s=4m^2, \quad t=u=0 \]

\[ \Rightarrow \delta\lambda = -\lambda^2 \left[ V(s=4m^2) + 2V(0) \right] \]

\[ = \frac{\lambda^2}{2} \frac{\Gamma(2-d)}{(4\pi)^{d/2}} \int_0^1 dx \left( \frac{1}{[m^2 - \chi(1-x)^4 m^2]^{d/2}} + \frac{2}{(m^2)^{d/2}} \right) \]

\[ d=4-\epsilon \]

\[ \Rightarrow \lambda^2 \frac{\Delta}{(2\pi)^{d/2}} \int_0^1 dx \left( \frac{6}{\epsilon} - 3\delta_E + 3 \log(4\pi) - 2 \log(m^2) \right) \]

\[ - \log \left( m^2 - \chi(1-x)^4 m^2 \right) \]

\[ \uparrow \]

divergent as $\epsilon \to 0$
For any $S, t, u,$

4-loop renormalized $2\to2$ scattering amplitude is

(\text{using $8\pi$})

\[ i\lambda(s, t, u) = -i\lambda \]

\[ = -\frac{i\lambda}{32\pi^2} \int_0^1 dx \left[ \log \left( \frac{m^2 - x(1-x)s}{m^2 - x(1-x)t} \right) + \log \left( \frac{m^2 - x(1-x)t}{m^2} \right) \right] \]

Several comments on renormalized amplitude.

1) It's finite.
2) It cancels.
3) Renormalization gives important, finite, physical contributions to scattering amplitude.
4) Schematically, at \( \lambda^2 \) order.

\[ i\lambda \equiv \lambda^2 \left[ \text{loop (divergent)} + \text{counter term (divergent)} \right] \]

\[ = \lambda^2 \left[ \text{finite} \right] \]

The same pattern persists to all orders in perturbation theory.

Although we are seemingly dealing with divergent quantities when performing loop integrals, it is in this sense we have a well-defined perturbative expansion.
b) denote \(-\frac{d\Pi}{dp^2} = -iM^2(p^2)\)

Full propagator:

\[
\begin{align*}
\quad = & -i \frac{1}{p^2 - m^2 + M^2(p^2)} \\
\text{renormalization condition} & = \frac{i}{p^2 - m^2} + \text{regular terms}
\end{align*}
\]

implies

\[M^2(p^2 = m^2) = 0, \quad \text{and } \frac{d}{dp^2} M^2(p^2) = 0\]

At 1-loop order

\[-iM^2(p^2) = \quad \text{counter term: } i(p^2 \delta \bar{\Lambda} - \delta m)\]

\[
\begin{align*}
&= (-i\lambda) \frac{1}{2} \int \frac{dk}{(2\pi)^d} \frac{i}{k^2 - m^2} + i(p^2 \delta \bar{\Lambda} - \delta m) \\
&= -\frac{i\lambda}{2} \frac{1}{(4\pi)^d} \frac{\Gamma(1-d/2)}{(m^2)^{1-d/2}} + i(p^2 \delta \bar{\Lambda} - \delta m).
\end{align*}
\]

Using renormalization condition:

\[
\frac{d}{dp^2} M^2(p^2) = 0 \implies \delta \bar{\Lambda} = 0 \quad \text{(vanishes at 1-loop order, non-vanishing at higher order)}
\]

\[\delta m = -\frac{\lambda}{2(4\pi)^d} \frac{\Gamma(1-d/2)}{(m^2)^{1-d/2}} \]

Higher order, non-zero \(\delta \bar{\Lambda}\)

\[
\begin{align*}
\quad &+ \quad \text{2-loop counter term} \quad \delta \Lambda \quad \text{(computed earlier)}
\end{align*}
\]
2-loop diagrams for $2 \rightarrow 2$ scattering $O(\lambda^3)$.

$\begin{align*}
\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} &\rightarrow 1\text{-loop counter term } O(\lambda^2) \\
\text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} &\rightarrow 2\text{-loop counter term } O(\lambda^3).
\end{align*}$

(to be determined by using renormalization condition).

+ Other diagrams related to these by crossing symmetry (no need to calculate).
- The renormalization program.

a) rewrite Lagrangian in renormalized parameters and field variables, and introduce counter terms, as a result of first change of variable.

b) Fix counter terms, order-by-order in perturbation theory, by a set of renormalization conditions.

c) Compute physical predictions using the renormalized parameters and variables. We obtain finite and physically meaningful results with important contributions from loop diagrams.

This program can be carried out to higher orders successfully, produce renormalized results as a well defined perturbative expansion. (BPHZ theorem).

d) A theory is called renormalizable if all the effect of UV divergences can be absorbed in to finite number of parameters (or using finite number of counter terms fixed by finite number of renorm. conditions).

A renormalizable theory has predictive power since we only need finite number of conditions to fix the counter terms. We can use the resulting renormalized Lagrangian to compute in finite number of observables.