Follow equations (20.110-20.116) in section 20.2 of Peskin and Schroeder.

(a)

The vev of the scalar is found by minimizing the potential \( V(\phi) = -\mu^2|\phi|^2 + \lambda(|\phi|^2)^2 \), with \( \mu^2 > 0 \),

\[
\frac{\partial V}{\partial \phi} \bigg|_{\phi=v/\sqrt{2}} = 0 \quad \Rightarrow \quad v = \sqrt{\frac{\mu^2}{\lambda}}. \quad (1)
\]

From the charge assignments and using the Peskin and Schroeder convention, the covariant derivative is \( D_\mu \phi = \left( \partial_\mu - ig_1 B_1 - ig_2 B_2 \right) \phi \). We obtain the mass terms for the gauge bosons by inserting the vev into the kinetic term of the scalar:

\[
(D_\mu \phi)^\dagger (D^\mu \phi) \bigg|_{\phi=v/\sqrt{2}} \ni \frac{1}{2} (g_1 B_1 + g_2 B_2)^2 v^2. \quad (2)
\]

The mass eigenstates are

\[
Z = \frac{g_1 B_1 + g_2 B_2}{\sqrt{g_1^2 + g_2^2}}, \quad m_Z^2 = (g_1^2 + g_2^2) v^2, \quad (3)
\]

\[
A = \frac{-g_2 B_1 + g_2 B_2}{\sqrt{g_1^2 + g_2^2}}, \quad m_A^2 = 0. \quad (4)
\]

(b)

After using the gauge freedom to eliminate phases, we find the mass of \( h \) by inserting \( \phi = (v + h)/\sqrt{2} \) into the potential.

\[
V \left( \frac{v + h}{\sqrt{2}} \right) = (\text{constant and tadpole terms}) + \frac{1}{2} (-\mu^2 + 3\lambda v^2) h^2 + \lambda v h^3 + \frac{\lambda}{4} h^4, \quad (5)
\]

so that \( m_h^2 = 2\lambda v^2 \).
(c) With $\phi = (v + h)/\sqrt{2}$, the kinetic term becomes
\[
\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m_Z^2 Z_\mu Z^\mu \left(1 + \frac{h}{v}\right)^2.
\] (6)

So there are three- and four-point vertices with two gauge bosons and one or two Higgs bosons. Note that the Higgs couples only to the massive gauge boson.

(d)
We assume zero Dirac masses and parity conservation. The Lagrangian for the fermion-gauge sector is
\[
\mathcal{L}_{f-g} = \sum_{k=1}^{2} \bar{e}_k (i D_k - m_k) e_k,
\] (7)
\[
D_k = \partial + ig_k B_k.
\] (8)

Expressing this in terms of mass eigenstates, we have
\[
B_1 = \frac{g_1 Z - g_2 A}{\sqrt{g_1^2 + g_2^2}}, \quad B_2 = \frac{g_2 Z + g_1 A}{\sqrt{g_1^2 + g_2^2}},
\] (9)

so that the fermion-gauge couplings are
\[
\mathcal{L}_{f-g} \ni \frac{1}{\sqrt{g_1^2 + g_2^2}} \bar{e}_1 \left( - g_1^2 Z + g_1 g_2 A \right) e_1 - \frac{1}{\sqrt{g_1^2 + g_2^2}} \bar{e}_2 \left( g_2^2 Z + g_1 g_2 A \right) e_2,
\] (10)
i.e. $e_1$ ($e_2$) is charged under $U(1)_{em}$ as $+1$ ($-1$), with gauge coupling $\frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}}$.

Note that we cannot write renormalizable Yukawa terms that preserve both Lorentz and gauge invariance. At mass dimension 5 (non-renormalizable), we can write
\[
\frac{\lambda_k}{\Lambda} \phi^\dagger \phi \bar{e}_k e_k,
\] (11)
where $\Lambda$ is some mass scale.