What's quantum field theory?

- Quantum theory. Share the same basic postulates as quantum mechanics.
  - state, superposition.
  - unitary evolution: \( i\hbar \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle \)
  - operator \( \leftrightarrow \) observables
  - measurement \( \leftrightarrow \) probability

- Relativistic quantum theory.
  - In non-relativistic quantum mechanics, we have finite number of degrees of freedom.
    e.g. wave-function of a particle, position operator of a particle.
  - With special relativity + quantum theory, we must consider infinite number of degrees of freedom.

We cannot just make Schrödinger equation covariant!
An analogy in classical physics:

- single particle: $(t, \vec{x})$ coordinates
- \( \infty \) # of d.o.f.: \( \phi(t, \vec{x}) \) field. (fluid, E&M)

In a quantum theory, consider

field operator \( \phi(t, \vec{x}) \).

# of d.o.f. can also be seen from # of boundary conditions we need to specify

- single particle: initial condition \( x(0), \dot{x}(0) \)
- field: \( \phi(0, \vec{x}) \) for all \( \vec{x} \)
Why quantum field theory?

1) Since classical mechanics $\rightarrow$ quantum mechanics. Therefore, classical field theory $\rightarrow$ quantum field theory. Not a good argument. Classical field is not fundamental at microscopic level, only particles here. Classical field theory is derived from QFT!


It's possible to have more particles with higher E, but is it necessary?
Before we start to have equations.

Units: \[ \hbar = c = 1. \] (Both quantum & relativistic effects important.)

\[ [m] = [E] = [T^{-1}] = [L^{-1}] \]

Example: \( (fm)^{-1} \approx 0.2 \text{ GeV} \).

Use:

\[ g_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]  

Peskin & Schroeder

\( \chi^\mu = (\chi^0, \chi^i) = (\chi^0, \vec{\chi}) \).

\( p_\mu = (E, \vec{p}), \quad p^\mu = (E, \vec{p}) \).

\[ p_\mu p^\mu = E^2 - \vec{p}^2 = m^2 \]  \( \text{rest mass} = m \).
Free particles.

- Fock space: \( |0\rangle, |\mathbf{p}_1\rangle, \ldots, |\mathbf{p}_N, \mathbf{p}_{N-1}, \ldots, \mathbf{p}\rangle \)
  - real scalar, no other labels.

- Creation & annihilation operators

\[ a^+_{\mathbf{p}}, \ a_{\mathbf{p}}^\dagger \quad \text{with} \quad [a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \]

\[ a_{\mathbf{p}}^\dagger |0\rangle = 0 \quad \text{for all} \quad \mathbf{p} \]

- Single particle state

\[ |\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}} \ a_{\mathbf{p}}^\dagger |0\rangle \]

- Multiple particle state

\[ \frac{N}{!} \prod_{i=1}^{N} \sqrt{2E_{\mathbf{p}_i}} \ a_{\mathbf{p}_i}^\dagger |0\rangle \]

Inner product:

\[ \langle \mathbf{p}' | \mathbf{p} \rangle = 2E_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \]

- Lorentz transformation (L.T.).

Require \( \langle \mathbf{p}' | \mathbf{p} \rangle \) invariant under L.T.

\[ \text{Note that} \quad U(\Lambda) |\mathbf{p}\rangle \propto |\Lambda \mathbf{p}\rangle \]

\[ \mathbf{p} \rightarrow \Lambda \mathbf{p} \quad \text{denotes} \quad p^\mu \rightarrow \Lambda^\mu{}_{\nu} p^\nu \]

\[ U(\Lambda) \text{ is a unitary representation (inner product preserving) of Lorentz group.} \]
note that
\[
\int \frac{d^3 \mathbf{p}}{(2\pi)^3 2E_p} = \int \frac{d^4 \mathbf{p}}{(2\pi)^4} \frac{2\pi \delta(p^2 - m^2)}{p^0 > 0}
\]
\[\downarrow\]
\[\text{Lorentz invariant.}\]

then
\[
\int d^3 \mathbf{p} \delta^2 (\mathbf{p} - \mathbf{p}') = 1
\]
\[\Rightarrow\]
\[
\int \frac{d^3 \mathbf{p}}{(2\pi)^3 2E_p} \delta^3 (\mathbf{p} - \mathbf{p}') = 1
\]
\[\downarrow\]
\[\text{Lorentz invariant.}\]

Therefore, we have.

\[\cup (\Lambda) |\mathbf{p} > = |\Lambda \mathbf{p} > .\]
\[
H_0 = \int \frac{d^3 p}{(2\pi)^3} \frac{N_{\mathbf{p}}}{E_{\mathbf{p}}} \left( a_{\mathbf{p}}^+ a_{\mathbf{p}} \right) + \text{const}(\alpha V T).
\]

\[
[H_0, a_{\mathbf{p}}^+] = E_{\mathbf{p}} a_{\mathbf{p}}^+, \quad [H_0, a_{\mathbf{p}}] = -E_{\mathbf{p}} a_{\mathbf{p}}
\]
Introducing quantum Field

Step 1: describe local interaction

- Motivation: everything we know seems to be local.

- More formally, called "cluster decomposition principle", i.e., events far away don't affect each other.

- This is satisfied if Hamiltonian is of the form (see S. Weinberg's book)

\[ H = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \int dp_1 dp_2 \cdots dp_N dp_1 \cdots dp_M \]

\[ \times h_{NM} (p'_1 \cdots p'_N, p_1 \cdots p_M) \]

\[ \times \alpha^+, \alpha^+ \cdots \alpha^+, \alpha_1 \alpha_2 \cdots \alpha_M \]

Where \( h_{MN} \) only contains one \( S \)-function

\[ h_{NM} = \tilde{h} (p'_1 \cdots p'_N) \delta^{(3)} (p'_1 + \cdots + p'_N - p_1 - \cdots - p_M) \]
- However, this is not convenient. It's not manifestly local.

It's much more convenient to work with the Fourier transforms of $a_{\vec{p}}$, $a_{\vec{p}}^*$:

\[
\Psi_+ (\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \ a_{\vec{p}} \ e^{i \vec{p}.\vec{x}}
\]

\[
\Psi_- (\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \ a_{\vec{p}}^* \ e^{-i \vec{p}.\vec{x}}
\]

$H (\Psi_+(\vec{x}), \Psi_- (\vec{x}))$ will be manifestly local. Using field operators $\Psi_+(\vec{x}), \Psi_- (\vec{x})$ has advantage!
Step 1: describing local interaction

- No specially deep reason, except it seems to be the case.

- However, a general function $H(a, a^\dagger)$ won't be local.

- Use Fourier transforms of $a$, $a^\dagger$ as building blocks.

\[ \psi_+ (\vec{x}) = \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_p}} \; a^{\dagger}_{p} \; e^{i\vec{p} \cdot \vec{x}} \]

\[ \psi_- (\vec{x}) = \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_p}} \; a_{p} \; e^{-i\vec{p} \cdot \vec{x}} \]

$H(\psi_+, \psi_-)$ will be automatically local.

Note: This already implies that field theory is a very useful tool for considering local interactions.
meaning of $\psi_-(\vec{x})$.

$$\psi_-(\vec{x}|10> = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \ a_{\vec{p}}^+ e^{-i\vec{p}.\vec{x}} \ 10>$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \ e^{-i\vec{p}.\vec{x}} \frac{1}{\sqrt{2E_p}} \ 1\vec{p}>.$$  

$$\langle \vec{p}' | \psi_- (\vec{x}) | 10 > = e^{-i\vec{p}' \cdot \vec{x}}$$

Therefore: $\psi_-(\vec{x}|10> \propto 1\vec{x}>$.

"creating a particle at $\vec{x}$".
Step 2: Impose Lorentz invariance. This task will be easier in a formulation in which $t$ and $\vec{x}$ are on the same footing.

- "Picture"
  - Schrödinger: $\Psi^+(\vec{x}), \Psi^-(\vec{x})$

- "Interaction picture" or "Dirac picture"
  - $\phi^\pm(t, \vec{x}) = e^{i\,\mathcal{H}_0 \, t} \Psi^\pm(\vec{x}) \, e^{-i\,\mathcal{H}_0 \, t}$

  - $\phi^+ (\vec{x}) = \int \frac{d^3 \vec{p}}{(2\pi)^3 (2\,E_p)^{1/2}} \, a^\dagger_{\vec{p}} \, e^{-i\,\vec{p} \cdot \vec{x}}$
  - $\phi^- (\vec{x}) = \int \frac{d^3 \vec{p}}{(2\pi)^3 (2\,E_p)^{1/2}} \, a_{\vec{p}} \, e^{i\,\vec{p} \cdot \vec{x}}$

- Dynamical evolution
  - Schrödinger picture
  - $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}$
  - $|f>^t = U_S(t, t') \, |i>$

  - $i\partial_t \, U_S = \mathcal{H} \, U_S$
\[
[\mathcal{H}_0, a^+_p] = E^+_p a^+_p, \quad [\mathcal{H}_0, a^-_p] = -E^-_p a^-_p
\]

Using so called Baker–Campbell–Hausdorff formula

\[
e^{i\mathcal{H}_0 t} a^+_p e^{-i\mathcal{H}_0 t} = a^+_p + i [\mathcal{H}_0, a^+_p] + \frac{(i)^2}{2} [\mathcal{H}_0, [\mathcal{H}_0, a^+_p]] + \cdots
\]

\[
= a^+_p e^{-iE^+_p t}
\]

\[
\Phi^+(x) = e^{i\mathcal{H}_0 t} \psi^+(x) e^{-i\mathcal{H}_0 t} = \int \frac{d^3p}{(2\pi)^3(2E_p)^{1/2}} \ a^+_p \ e^{-i\mathbf{p}\cdot\mathbf{x}}
\]
- Dynamical evolution

i) Schrödinger picture

\[ H = H_0 + H_{\text{int}} \quad H_{\text{int}} \Rightarrow \text{interactions} \]

State evolves

\[ t = t_0 \quad \rightarrow \quad t \]

\[ |i\rangle \rightarrow |i_t\rangle \]

\[ |i_t\rangle = U_S(t, t_0) |i\rangle \]

Schrödinger equation

\[ i\partial_t U_S = H U_S \]

Transition amplitude

\[ A_{fi} = \langle f | i_t \rangle \]

ii) Interaction picture

\[ A_{fi} = \frac{\langle f | e^{-iH_0 t} e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t} e^{iH_0 t} | i \rangle}{\langle f_i | U_I(t, t_0) | i_i \rangle} \]

Interaction picture operator \( U_I \) satisfies

\[ i\partial_t U_I(t, t_0) = H_1(t) U_I \]

\[ H_1(t) = e^{iH_0} H_{\text{int}} e^{-iH_0} \]
Solution for $U_I(t,t_0)$

$$U_I(t,t_0) = 1 - i \int_{t_0}^{t} H_I(t') U_I(t',t_0) \, dt'$$

Using iteration

$$U_I(t,t_0) = 1 + (-i) \int_{t_0}^{t} dt_1 H_I(t_1)$$

$$+ (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) + \cdots$$

Time order

$$T (H_I(t_1) H_I(t_2)) = \Theta (t_1 - t_2) H_I(t_1) H_I(t_2)$$

$$+ \Theta (t_2 - t_1) H_I(t_2) H_I(t_1)$$

With this definition

$$U_I(t,t') = T \left\{ \exp (-i \int_{t_0}^{t} dt' H_I(t')) \right\}$$

$$= 1 + (-i) \int_{t_0}^{t} dt_1 H_I(t_1)$$

$$- \frac{1}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T (H_I(t_1) H_I(t_2))$$
Consequences of Lorentz invariance

Consider transition amplitude

\[ A_{pf} = \langle f | U_I(t, t_0) | i \rangle \]

S-Matrix limit: (with direct connection to physical observables)

\[ t_0 \to -\infty \quad t \to \infty \]

To make Lorentz invariance as manifest as possible, introduce Hamiltonian density

\[ \int_{-\infty}^{\infty} dt \: H_I(t) = \int d^4 x \: \mathcal{H}_I(x) \]

where

\[ H_I(t) = \int d^2 \vec{x} \: \mathcal{H}_I(t, \vec{x}) \]

\[ \mathcal{H}_I: \text{Hamiltonian density constructed out of field operators } \phi(x) \]

The transition amplitude to the 2nd order in \( \mathcal{H}_I \)

\[ A_{pf} = \langle f | \frac{1}{2} \int d^4 x_1 d^4 x_2 \: T(\mathcal{H}_I(x_1) \mathcal{H}_I(x_2)) | i \rangle \]
Only time order is not manifestly Lorentz invariant.

i) If \( (x_1 - x_2)^2 > 0 \) (time-like)
Lorentz transformation won't be able to change the time order (causality).

ii) If \( (x_1 - x_2)^2 < 0 \) (space-like)
\( x_1, x_2 \) causally disconnected.
Lorentz transformation can change time order.

To have Lorentz invariance for transition amplitude \( A_{fi} \), two different orders
\( H_I(x_1) H_I(x_2) \) and \( H_I(x_2) H_I(x_1) \)
must give the same answer for \( (x_1 - x_2)^2 < 0 \)

That is,

\[
[H_I(x_1), H_I(x_2)] = 0
\]
for \( (x_1 - x_2)^2 < 0 \)
iii) Condition \( \star \) puts strong constraint on \( \mathcal{H}_I \) as a function of field operators.

In general, generic functions of \( \phi^\pm(x) \) do not commute.

For example:

\[
[\phi^+(x_1), \phi^-(x_2)]
\]

\[
= \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{1}{\sqrt{2E_{p_1}}} \frac{1}{\sqrt{2E_{p_2}}} e^{i (p_2 \cdot x_2 - p_1 \cdot x_1)} [a_{p_1}, a_{p_2}^+] \\
= \frac{m}{4\pi^2} K_1 (mr) \equiv C(r) \\
r = (x_1 - x_2)^2 \\
C(r) \neq 0 \text{ for } r < 0 \]
From weak coupling limit, we expect solutions to constraint (A) is in terms of field variables which are linear in $\phi^\pm$

$$\phi_\lambda(x) = \phi_\lambda^+(x) + \lambda \phi_\lambda^-(x)$$

$$[\phi_\lambda(x), \phi_\lambda^+(x)] = (1 - \lambda |\lambda|^2) \phi_\lambda(x)$$

1) Must have $|\lambda| = 1$, with addition phase absorbed.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left( a^+_\tilde{p} e^{-i\tilde{p}x} + a^+_{\tilde{p}} e^{i\tilde{p}x} \right).$$

2) Also can show, we must have

$$[a_{\tilde{p}}, a^+_{\tilde{p}'}] \propto \delta^3(\tilde{p} - \tilde{p}').$$

Commutator $\rightarrow$ Bose statistics.

3) For fermions.

must have

$$\{ b, b^+ \} \propto \delta^3(\tilde{p} - \tilde{p}').$$

$\uparrow$

Fermi-statistics.

2) & 3) Spin-statistics theorem.
4. If particle have charge, then there are two kinds of particles, particle & anti-particle with opposite charges. ±.

\[ a^+_\frac{p}{q} \text{ create particle.} \]
\[ b^+_\frac{p}{q} \text{ create anti-particle.} \]

\[ \phi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{p^2 + m^2}} a e^{i p \cdot x} + \alpha \int \frac{d^3p}{(2\pi)^3 \sqrt{p^2 + m^2}} b^+ e^{-i p \cdot x} \]

Again: we can show \( a, b \) satisfy commutation relations.

- \( M^+ = \hat{M}^- \).

- \( \alpha \) is at most a pure phase, which can be absorbed in \( b^+ \).

\[ \rightarrow \phi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} (a e^{i p \cdot x} + b^+ e^{-i p \cdot x}) \]
Summary

- particles + quantum mechanics + local interaction

→ field operator $\phi^\pm(x)$

- Lorentz invariance →
  - only specific combinations of $\phi^\pm(x)$ are allowed, for example
    $\phi(x) = \int dq \left( a e^{ipx} + b^* e^{-ip\cdot x} \right)$.
  - spin statistics.

- More formally, a QFT can be defined as local Lorentz invariant quantum theory.

\[
[\phi^a(x), \phi^b(y)] = 0 \quad \text{for } (x-y)^2 < 0.
\]

+ a few assumptions about energy spectrum (asymptotic).
We have glossed over a # of things such as

- A careful definition of observable (S-Matrix)

- A careful treatment of

  \( \Phi(x_1) \Phi(x_2) \) as \( x_1 \rightarrow x_2 \)

  "Renormalization"

These will be discussed in detail later. But the conclusion here is unchanged!