1. Problem 1. Consider complex scalar field with

\[ L = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \frac{\lambda}{4} (\phi^\dagger \phi)^2 \]

(a) Add source terms \( J^\dagger \phi + J \phi^\dagger \). In the limit of \( \lambda = 0 \), compute generating functional \( Z_0[J] \).

(b) In the limit of \( \lambda = 0 \), compute correlation functions \( \langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle \), \( \langle 0 | T \phi^\dagger(x_1) \phi^\dagger(x_2) | 0 \rangle \), and \( \langle 0 | T \phi(x_1) \phi^\dagger(x_2) | 0 \rangle \).

(c) Derive Feynman rules for the full theory \( (\lambda \neq 0) \), in both momentum and position space.

(d) Write down the expression for two point function \( \langle \Omega | T \phi(x_1) \phi^\dagger(x_2) | \Omega \rangle \), both in position and in momentum space, up to order \( \lambda \). You don’t have to carry out the loop integral.

2. Particle production from source. Consider an external source \( J(x) \) which is turned on for a finite period of time, during \( t \in [-\tau, \tau] \). The source is coupled to a free real scalar field through \( L_{\text{int}} = \int d^4x J(x) \phi(x) \).

We will use \( Z_0[J = 0] = 1 \) for this problem.

(a) Show that the probability that the source creates no particle is given by

\[ P(0) = \left| \int D\phi e^{i \int d^4x L_0 + J \phi} \right|^2. \]

What is the appropriate boundary condition at \( t = \pm \infty \)?

(b) Expand \( P(0) \) in terms of source \( J \), and show that the first order term vanishes.

(c) Calculate \( P(0) \) to the second order in source \( J \). Write the answer as \( P(0) = 1 - \lambda \) and compute \( \lambda \).

(d) Show that the expansion of \( P(0) \) in terms of \( J \) can be summed exactly to \( P(0) = e^{-\lambda} \).

3. Energy momentum tensor. Consider spacetime translation symmetry \( x^\mu \rightarrow x^\mu + a^\mu \)

(a) Derive energy momentum tensor \( T^{\mu\nu} \) as the symmetry current.

(b) Write the corresponding Ward identity.

(c) Compute \( [T^{00}(x), T^{00}(y)] \), \( [T^{0i}(x), T^{00}(y)] \), and \( [T^{0i}(x), T^{0j}(y)] \), for \( x_0 = y_0 \).