We need to apply the commutators of the spin operators to solve this problem. We note that, for general spin operators, we get the commutation relation:

\[
[S_i, S_j] = i\hbar \epsilon_{ijk} S_k
\]  

(1)

We now require that \(\alpha = \phi/\hbar\).

This means that:

\[
C_1 = [S_z, S_x] = i\hbar S_y
\]  

(2)

and

\[
C_2 = [S_z, [S_z, S_x]] = i\hbar [S_z, S_y] = -(i\hbar)^2 S_x = \hbar^2 S_x
\]  

(3)

We notice a pattern forming. We can write this pattern in the following way:

\[
C_{2n} = (\hbar)^{2n} S_x, \quad C_{2n+1} = i(\hbar)^{2n+1} S_y
\]  

(4)

These can be plugged into the Baker-Hausdorff formula to find the answer:

\[
\exp \left\{\frac{i S_z \phi}{\hbar}\right\} S_z \exp \left\{-\frac{i S_z \phi}{\hbar}\right\}
\]

\[
= S_x + \frac{i \phi}{\hbar} (i\hbar) S_y + \frac{1}{2!} \left(\frac{i \phi}{\hbar}\right)^2 (\hbar^2 S_x) + \ldots
\]

\[
= S_x \left(1 - \frac{1}{2!} \phi^2 + \frac{1}{4!} \phi^4 - \ldots\right) - S_y \left(\phi - \frac{1}{3!} \phi^3 + \frac{1}{5!} \phi^5 - \ldots\right)
\]

\[
= S_x \cos \phi - S_y \sin \phi
\]  

(5)
This is exactly what you would expect if you rotated the $\hat{x}$ direction about the $z$-axis by an angle $\phi$.

2

Since $\vec{B}$ is along the $z$-direction, our Hamiltonian is proportional to $\sigma_3$. We have:

$$H = -\lambda \frac{\hbar}{2} B \sigma_3 \quad (6)$$

We next need the eigenstate of $\sigma_1$ with eigenvalue $+1$ as our initial state. We calculate this using the standard form for $\sigma_1$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (7)$$

This gives the eigenstate:

$$|\lambda_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (8)$$

We now decompose this into eigenstates of our Hamiltonian. These are the eigenstates of $\sigma_3$, which are given as:

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\mp\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (9)$$

This gives the decomposition:

$$|\lambda_+\rangle = \frac{1}{\sqrt{2}} (|+\rangle + |\mp\rangle) \quad (10)$$

We seek to evolve this state in time; this is done by exponentiating the eigenvalues according to the Schrödinger equation:

$$H |\pm\rangle \mp \lambda B \frac{\hbar}{2} |\pm\rangle = \mp e^{\pm i \lambda B \frac{\hbar}{2} t} |\pm\rangle \quad (11)$$

2
So we find the time evolution:

$$|\lambda_+(t)\rangle = \frac{1}{\sqrt{2}} \left( e^{+i\lambda B\frac{1}{2}t} |+\rangle + e^{-i\lambda B\frac{1}{2}t} |-\rangle \right)$$  \hspace{1cm} (12)$$

The expectation value of $S_y$ is just what we get when we sandwich $S_y$ between our state at time $t$:

$$\langle S_y \rangle = \langle \lambda_+(t) | S_y | \lambda_+(t) \rangle$$

$$= \frac{\hbar}{4} \left( e^{-i\lambda B\frac{1}{2}t} \ e^{+i\lambda B\frac{1}{2}t} \right) \left( \begin{array}{cc} 0 & -t \\ t & 0 \end{array} \right) \left( \begin{array}{c} e^{+i\lambda B\frac{1}{2}t} \\ e^{-i\lambda B\frac{1}{2}t} \end{array} \right)$$

$$= -\frac{\hbar}{2} \left( e^{+i\lambda Bt} - e^{-i\lambda Bt} \right)$$

$$= -\frac{\hbar}{2} \sin(\lambda Bt)$$  \hspace{1cm} (13)$$

We find that the expectation value of $S_y$ is an oscillatory function. We can think of this as a spin which is precessing about an axis in the direction of the magnetic field. This is known as Larmor precession.