5. Fermion

5.1 Representations of Lorentz Group.

We begin with a general discussion of representations of the Lorentz group. We have already used one of the simplest representation of the Lorentz group, the scalar. We will now add additional quantum numbers, in addition to the space-time coordinates of the fields, which can transform non-trivially under Lorentz transformation. We will see that such additional quantum numbers naturally label the irreducible representations of rotational group, they are therefore called the spins of the particles. Including particles with non-zero spin leads additional classes of field theories. What we have learned about the fundamentals of field theory, such as the principle of renormalization and effective field theory, are still valid here. At the same time, such theories do pose new technical problems, and more importantly, contain richer physics. Most (if not all) of elementary particles which we know indeed have non-zero spin.

Lorentz Group has 6 generators, 3 rotations + 3 boosts. These generators can be collectively denoted as $J^\mu\nu$, where $\mu$ and $\nu$ run from 0 to 3, and $J$s are antisymmetric under $\mu \leftrightarrow \nu$. The structure of the Lorentz group is defined by

$$[J^\mu\nu, J^\rho\sigma] = i (\{g^\nu\rho J^\mu\sigma - (\mu \leftrightarrow \nu)\} - [\rho \leftrightarrow \sigma])$$

where $g^{\mu\nu} = \text{Diag}(1, -1, -1, -1)$. Define

$$L^i = \frac{1}{2} \epsilon^{ijk} J^j k, \quad K^i = J^{0i}$$

where $i, j, k = 1, 2, 3$. $L^i$ are generators of rotations, and $K^i$ are generators of boosts. Indeed, we can verify

$$[L^i, L^j] = i \epsilon^{ijk} L^k \text{ (rotation)}, \quad [K^i, K^j] = -\epsilon^{ijk} L^k, \quad [L^i, K^j] = i \epsilon^{ijk} K^k.$$  

In the familiar representation in which Lorentz transformations are acting on space-time 4-vectors, we have each $J^\mu\nu$ as a $4 \times 4$ matrix,

$$(J^\mu\nu)_{\alpha\beta} = i (\delta^\mu_\alpha \delta^\nu_\beta - (\alpha \leftrightarrow \beta)).$$

The Hilbert space is built up from single (and multiple) particle states. At the same time, field theory is written using the language of field operators. We will begin with particle states first.
5.1.1 **Particles as different representations of the Lorentz Group**

In a quantum theory with certain symmetry, the states in Hilbert space must be in unitary representations of the symmetry group. The symmetry we consider here is Lorentz invariance, which tells us, for example, electrons traveling along different directions and at different speed can be describe by the same electron field. In a unitary representation, a Lorentz transformation $\Lambda$ is realized as a unitary matrix $U(\Lambda)$.

Different representations of a symmetry group can be labelled by a set of invariants. For Lorentz group, the most obvious invariant is mass. For a particle with momentum $k^\mu$, its mass $m^2 = k^\mu k_\mu$ is Lorentz invariant. Particles with different masses fall into different representations of the Lorentz group, and different representations with different masses obviously won’t transform into each other under Lorentz transformation. This representation is infinite dimensional. Indeed, $|k\rangle$ belongs to the representation with mass $m$, so does $|\Lambda k\rangle$, where $\Lambda$ is an arbitrary Lorentz transformation.

The particles with the same mass are not necessarily the same particles. There can be additional quantum numbers which can distinguish them. Quantum numbers such as electric charge commute with Lorentz transformation, and they can be included trivially. On the other hand, angular momentum transform non-trivially under rotation which is part of the Lorentz group. We now systematically classify representations of Lorentz group with different spin.

We follow the method of “induced representation”, or “little group”, pioneered by Wigner. It is a systematical way of ”building up” a full irreducible representation starting from a special member of the representation.

**Massive Particle**

We begin by choosing a reference vector, $n = (m, 0, 0, 0)$, i.e., the momentum of the particle in its rest frame. This is not a unique choice, but it is the most convenient for our discussion. The transformations of momentum which leave $n$ invariant is called the *little group* of $n$. We can denote a state as $|n, \alpha\rangle$, where the little group only acts on index $\alpha$. Obviously, the little group of $n$ is the rotational group, $SO(3)$, generated by $L^i$. The irreducible representations of the rotational group are labeled by angular momentum $S$, with $S(S + 1) = L^2$ and $L^3 = -S, ..., S$ ($2S + 1$ dimensional). For $R \in SO(3)$, we have

$$U(R)|n, \alpha\rangle = M_{\alpha\beta}(R)|n, \beta\rangle,$$

where $M$ is the spin $S$ representation of $SO(3)$. For generic momentum $k$ (with the same mass), we have a
Lorentz boost \( L(k) \) such that
\[
L(k) : n \to k, \text{ and } |k, \alpha \rangle = U(L(k))|n, \alpha \rangle.
\] (6)
The little group of \( k \), generated by \( L(k)RL^{-1}(k) \) (as we can easily check this leaves \( k \) invariant), is still \( SO(3) \).

Now, the general Lorentz transformation on a generic state \( |k, \alpha \rangle \) is
\[
U(\Lambda)|k, \alpha \rangle = U(L(\Lambda k))U(L^{-1}(\Lambda k) \cdot \Lambda \cdot L(k))|n, \alpha \rangle.
\] (7)

We have used \( \Lambda \cdot k = L(\Lambda k)L^{-1}(\Lambda k)AL(k) \cdot n \), and \( U(\Lambda')U(\Lambda'') = U(\Lambda'\Lambda'') \). \( U(L(\Lambda k)) \) is a Lorentz boost which \( L(\Lambda k) : n \to \Lambda k \). \( W(\Lambda, k) = L^{-1}(\Lambda k) \cdot \Lambda \cdot L(k) \) is a rotation which leaves \( n \) invariant. It is an element of the little group, and it is called “Wigner rotation”. Therefore,
\[
U(\Lambda)|k, \alpha \rangle = U(L(\Lambda k))M_{\alpha \beta}(W)|n, \beta \rangle = M_{\alpha \beta}(W)|\Lambda k, \beta \rangle
\] (8)

We have shown that any Lorentz transformation on generic state of massive particle, \( |k, \alpha \rangle \), can be decomposed into a boost and an \( SO(3) \) rotation which leaves the momentum invariant. In other words, any massive irreducible representation of the Lorentz group can be labelled by its mass and the spin of the particle.

Notice that this representation is infinite dimensional, as there are infinite number of \( k \)s which have the same mass, \( k^2 = m^2 \). Since the Lorentz group is non-compact, all of its unitary representations would need to be infinite dimensional. Therefore, this is as it should be.

**Massless Particle**

For massless particle, there is no rest frame. A convenient choice of reference vector is \( n = (1, 0, 0, 1) \). An obvious candidate of the little group is the 2-dimensional rotation in the x-y directions, \( SO(2) \). Therefore, the massless irreducible representations of the Lorentz group are labeled by mass and the irreducible representation of \( SO(2) \). The irreps of \( SO(2) \) are labeled by two numbers, \( \pm n/2 \), i.e., “spin up” and “spin down”. \( n \) is integer. They are the angular momentum along z-direction (the direction of the momentum of the particle). This is also called helicity. Therefore, any massless particle has two possible helicities, each of them belongs to a irreducible representation, and they do not mix under Lorentz transformations.

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1We are glossing over a subtlety here. The full little group is actually 2d Euclidean group which also include translations. We are focusing here on the \( SO(2) \) part, which is equivalent to taking the eigenvalues of the translation operators to be zero. The more general case has no clear physical interpretation and no known example in Nature.
5.1.2 Field operators transforming under the Lorentz Group

We would like to study the transformation law of field operators under the Lorentz group. We consider a general field $\Phi_\alpha(x)$, where $\alpha$ is some additional index which transform non-trivially under the Lorentz group. Scalar field is a trivial example. For more general field operators, under Lorentz transformation, we have

$$M^{-1}(\Lambda)\Phi_\alpha(x)M(\Lambda) = M_{\alpha\beta}(\Lambda)\Phi_\beta(\Lambda^{-1}x),$$

(9)

where the matrix $M_{\alpha\beta}(\Lambda)$ is a representation of the Lorentz group, which operates in the space of field labels.

We will focus on the case in which the number of fields is finite, since in reality, we always deal with finite number of particle species. Therefore, we are looking for finite dimensional representations of the Lorentz group\(^2\).

It turns out the finite dimensional representations of the Lorentz group can be classified using its $SU(2)$ subgroups. We define

$$J_+ = \frac{1}{2}(L + iK), \quad J_- = \frac{1}{2}(L - iK).$$

(10)

We can show that

$$[J_+, J_-] = 0, \quad [J^I_\pm, J^J_\pm] = i\epsilon^{IJK}J^K_\pm$$

(11)

This mean Lorentz algebra can be decomposed into two commuting $SU(2)$ sub-algebras, $SU(2)_+$ and $SU(2)_-$, generated by $J_+$ and $J_-$ respectively. The irreducible representations of $SU(2)$ are labeled by $j = n/2$ ($n = 1, 2, 3...$). They are $2j + 1$ dimensional with $J^2_\pm = -j, ..., j$. The representation of the Lorentz group can be constructed as direct products of the irreducible representations of the $SU(2)$ subgroups, labeled by $(j_+, j_-)$, with dimension $(2j_+ + 1)(2j_- + 1)$.

We note that $L^i = J^i_+ + J^i_-$. Therefore, the angular momentum of a direct product representation $(j_+, j_-)$ is $|j_+ - j_-|, ..., j_+ + j_-$. From this, we derive

$$(0, 0) : \text{ spin 0, scalar}$$

$$(1/2, 0) : \text{ spin 1/2, Weyl fermion, “left-handed”}$$

$$(0, 1/2) : \text{ spin 1/2, Weyl fermion, “right-handed”}$$

$$(1/2, 1/2) : \text{ spin 1, 4-vector}$$

(12)

\(^2\)We note such representations cannot be unitary. Lorentz group, containing boost, is not compact. On the other hand, the set of all finite dimensional unitary matrices, being $U(n)$, is compact and cannot contain Lorentz group as a subgroup.
5.2 Free Fermions

We now focus on spin-1/2 fermions. Recall that thinking about spin-1/2 in terms of $SO(3)$ rotations is sometimes not very convenient since it is double valued. It is useful to use a larger group which is locally (near identity) the same as $SO(3)$, and under which the rotation of spin-1/2 is single valued\(^3\). In the case of $SO(3)$ rotation (non-relativistic quantum mechanics), we use $SU(2)$. This is the origin of thinking about spinor as 2 component complex column vectors. Here in relativistic quantum field theory, we have Lorentz group $SO(1,3)$. The larger group can be identified as follows. Consider a four vector of $2 \times 2$ matrices

$$\sigma^\mu = (1, \sigma)$$

where $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ are the usual Pauli matrices. Now consider product

$$X = \sigma^\mu x_\mu = \begin{pmatrix} x_0 - x_3 & -x_1 + i x_2 \\ -x_1 + i x_2 & x_0 + x_3 \end{pmatrix}.$$ \(14\)

Notice that $X$ is Hermitian, and $\text{Det}(X) = x^\mu x_\mu$. Lorentz transformation on $x^\mu$ is defined as transformations which keeps $x^\mu$ real and $x^\mu x_\mu$ invariant. This is exactly as the requirement for linear transformations

$$X' = M X M^\dagger$$ \(15\)

which keeps $X$ Hermitian and $\text{Det}(X) = x^\mu x_\mu$ invariant. The set of $2 \times 2$ matrices $M$s which satisfy these requirements forms the $SL(2,C)$ group\(^4\).

\((\frac{1}{2},0)\) representation

\((\frac{1}{2},0)\) becomes the fundamental representation of $SL(2,C)$, i.e., 2 component complex vector. We denote the fermion field in this representation as

$$\begin{pmatrix} \frac{1}{2},0 \end{pmatrix}: \xi_\alpha.$$ \(16\)

Under Lorentz transformation,

$$\xi_\alpha \rightarrow M^\beta_\alpha \xi_\beta$$ \(17\)

$\alpha$ and $\beta$ are indices, $\alpha, \beta = 1, 2$. It is useful to think of $\xi_\alpha$ as a column vector. We have also used upper and lower indices (similar to Lorentz indices for a 4-vector) for later convenience. Notice that $SL(2,C)$ has an invariant tensor

$$\epsilon^{\alpha\beta\delta} M^\delta_\alpha M^\gamma_\beta = \epsilon^{\delta\gamma},$$ \(18\)

\(^3\)In fancier language, this is called a universal covering group of $SO(3)$

\(^4\)The $SL(2,C)$ is the universal covering group of $SO(1,3)$, and the actual isomorphism is $SO(1,3) \cong SL(2,C)/Z_2$. 

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where antisymmetric $\epsilon$ tensor is defined as $\epsilon^{12} = -\epsilon^{21} = 1$. It is similar to the fact that $g_{\mu\nu}$ is the invariant tensor of the Lorentz group. Just like $g_{\mu\nu}$ can be used to lower the indices and form invariants with 4-vectors, we can use $\epsilon$ to do the same with fermion field $\xi$.

$$\xi^\alpha = \epsilon^{\alpha\beta} \xi_\beta, \quad (19)$$

and under Lorentz transformation

$$\xi^\alpha \rightarrow \xi^\beta (M^{-1})^\alpha_\beta. \quad (20)$$

It is also easy to form invariants out of the invariant tensor.

$$\xi \eta \equiv \xi_\alpha \epsilon^{\alpha\beta} \eta^\beta. \quad (21)$$

Using the fact that fermion fields $\xi$ and $\eta$ are anti-commuting, we can show

$$\xi \eta \equiv \xi_\alpha \epsilon^{\alpha\beta} \eta^\beta = -\eta^\beta \xi_\alpha \epsilon^{\alpha\beta} = \eta^\beta \xi_\alpha \epsilon^{\beta\alpha} = \eta \xi. \quad (22)$$

A small Lorentz transformation with rotation $\theta$ and boost $\beta$ is generated by $1 - iL \cdot \theta - iK \cdot \beta + ...$. In $(\frac{1}{2}, 0)$, we have $L = \sigma/2$ and $K = -i\sigma/2$. Therefore, we have

$$M^\alpha_\beta = 1 - i\sigma/2 \cdot \theta - \sigma/2 \cdot \beta + ... \quad (23)$$

We can see the boost part is not unitary. This is, of course, consistent with the statement that there is no finite dimensional unitary representation for a non-compact group.

$(0, \frac{1}{2})$ representation

We will denote the fermions in this representation as

$$(0, \frac{1}{2}) : \xi^\dagger_\alpha \quad (24)$$

For now, this is just a definition. In particular, the use of dotted indices is purely a convention to remind us this is the $(0, \frac{1}{2})$ representation. The $^\dagger$ is a notation which originates from the fact that $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ are conjugate of each other, i.e.,

$$\xi^\dagger_\alpha = (\xi_\alpha)^\dagger. \quad (25)$$

Under Lorentz transformation, we have

$$\xi^\dagger_\alpha \rightarrow (M^*)^\alpha_\beta \xi^\dagger_\beta = (M^\dagger)^\beta_\alpha \xi^\dagger_\alpha = \xi^\dagger M^\dagger. \quad (26)$$
The invariant tensor with dotted indices are defined by \( \epsilon^{\dot{\alpha} \dot{\beta}} = (\epsilon^{\alpha \beta})^* \) (they are actually identical matrices in practice since \( \epsilon \) is real). We also have

\[
(\xi \eta)^\dagger = \eta^\dagger \xi^\dagger = \xi^\dagger \eta^\dagger = \xi^\dagger \eta^\dagger \epsilon_{\dot{\alpha} \dot{\beta}},
\]

where the first step should just be viewed as the definition of \( \dagger \) operation.

As we have already used above, with this indexing convention, the transformation matrix should be \( M_{\alpha \beta} \) and \( (M^\dagger)^{\dot{\alpha}}_{\dot{\beta}} \) for the \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\), respectively. Since we have already defined the transformation law of quantity \( X = \sigma^\mu x_\mu \) as \( X \rightarrow MXM^\dagger \), using the transformation law of \( x_\mu \) under Lorentz, we can see the transformation law of \( \sigma^\mu \) should be

\[
\sigma^\mu_{\alpha \dot{\beta}} \rightarrow M_{\alpha \alpha'} \sigma^\mu_{\alpha' \dot{\beta}} (M^\dagger)^{\dot{\beta}}_{\dot{\beta'}}.
\]

Note that the index structure of \( \sigma^\mu \) has been fixed. Similarly, we have index structure \( \bar{\sigma}^\mu_{\dot{\alpha} \beta} \).

Finally, we are in a position of using the index structures to form Lorentz invariants (which is the main motivation to invoke all these notation) by contracting upper and lower indices, dotted and un-dotted separately.

First, we have the following invariants.

\[
\xi \eta = \xi^\alpha \eta_\alpha, \quad \xi^\dagger \eta^\dagger = \xi^\dagger \eta^\dagger \epsilon_{\dot{\alpha} \dot{\beta}},
\]

These are Lorentz invariant bilinears which will become mass terms. We also have

\[
\xi^\dagger \bar{\sigma}^\mu \eta, \quad \xi \sigma^\mu \eta^\dagger.
\]

These are not Lorentz invariant yet. However, with all spinor indices contracted, these transforms as 4-vectors. They can be made Lorentz invariant by dot into another 4-vector. For example, \( \xi^\dagger \partial_\mu \bar{\sigma}^\mu \eta \) is Lorentz invariant, which will become the kinetic term.

**Lagrangian and Equation of Motion.**

We are now ready to write Lorentz invariant Lagrangian for Weyl fermions. For example, for a single Weyl fermion, we have

\[
\mathcal{L} = i \xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi - \frac{1}{2} m (\xi \xi^\dagger).
\]

Note that in the limit of \( m = 0 \), this Lagrangian has a \( U(1) \) symmetry \( \xi \rightarrow e^{i \theta} \xi \). We can call this symmetry “fermion number”, \( U(1)_F \). The mass term breaks such a symmetry. \( \xi \) in this theory is called a Majorana fermion, since it is its own anti-particle (there is no other fermion around).

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Quantization of this theory of a single fermion can proceed in a way very similar to the quantization of a real scalar field, with the main difference being the creation and annihilation operators are not anti-commuting. Feynman rules can be developed. In fact, in many modern applications, such as supersymmetry, working with 2-component fermions have advantages. However, our focus here would be laying a foundation for the discussion of QED, in which calculations can be somewhat simplified using 4 component formalism. The use of 4-component formalism also brings us closer to the standard textbook treatment of QED. Therefore, we will proceed to introduce the 4-component Dirac fermions. For application of 2-component technique (and many additional information), see H. Dreiner, H. Haber, and S. Martin, Physics Reports 494 (2010) 1-195. We have borrowed many equations from this unique reference.

We begin by considering two species of \(1/2, 0\) fermions

\[
L = \sum_{i=1,2} i\xi^{i\dagger} \bar{\sigma}^{\mu} \partial_{\mu} \xi_i - \frac{1}{2} m_i (\xi_i \xi_i + \xi_i^{\dagger} \xi_i^{\dagger}).
\]

where \(i\) labels fermion species. Quantization of this theory is a trivial generalization of the quantization of one species of fermion. However, simplification can occur if we focus on a special limit in which \(m_1 = m_2 = m\). In this limit, there is a enhanced \(SO(2)\) symmetry which rotate \(\xi_1\) and \(\xi_2\) into each other (and therefore also \(\xi_1^{\dagger}\) and \(\xi_2^{\dagger}\)). Following Noether procedure, we can calculate the conserved current of this symmetry to be

\[
J^\mu = i(\xi^{\dagger} \bar{\sigma}^\mu \xi_2 - \xi^{\dagger} \bar{\sigma}^\mu \xi_1).
\]

To identify the conserved charge better, we diagonalize the conserved current by

\[
\chi \equiv \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2), \quad \eta \equiv \frac{1}{\sqrt{2}}(\xi_1 - i\xi_2).
\]

We have

\[
J^\mu = \chi^{\dagger} \bar{\sigma}^\mu \chi - \eta^{\dagger} \bar{\sigma}^\mu \eta.
\]

In this base, the symmetry is a \(U(1)_Q\), where \(Q_\chi = 1\) and \(Q_\eta = -1\). The Lagrangian is

\[
L = i\chi^{\dagger} \bar{\sigma}^\mu \partial_{\mu} \chi + i\eta^{\dagger} \bar{\sigma}^\mu \partial_{\mu} \eta - m(\chi \eta + \chi^{\dagger} \eta^{\dagger}).
\]

In the limit \(m = 0\), there is an additional \(U(1)\) symmetry \(\xi \to e^{i\theta} \xi\) and \(\eta \to e^{i\theta} \eta\). This is called a chiral symmetry \(U(1)_{\text{chiral}}\).

It is natural to consider \(\chi\) and \(\eta\) parts of “one fermion”, since the mass term is off-diagonal. They carry opposite charge under \(U(1)_Q\), which can be interpreted as one of them being the anti-particle of the other. Now,
we proceed to re-express $\chi$ and $\eta$ in terms of a 4-component fermion field. In the 2-component formulation, $\sigma^\mu$ and $\bar{\sigma}^\mu$ play important roles. We begin by introducing their 4 component counter part, the $\gamma$-matrices. We begin with 4 $\gamma$-matrices denoted by $\gamma^\mu$, where $\mu = 0, 1, 2, 3$. We require the $\gamma$-matrices satisfy anti-commuting relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \quad (37)$$

One can show that Lorentz algebra in the $(1/2, 0) \oplus (0, 1/2)$ representation is generated by

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \quad (38)$$

We work in a base such that

$$\gamma^\mu = \left( \begin{array}{c} \sigma^\mu \\ \bar{\sigma}^\mu \end{array} \right), \quad \gamma^\mu\dagger = \gamma^0\gamma^\mu\gamma^0. \quad (39)$$

The boost and rotation operators are

$$S^{0i} = -\frac{i}{2} \left( \begin{array}{c} \sigma^i \\ -\bar{\sigma}^i \end{array} \right), \quad S^{ij} = \frac{1}{2}\epsilon^{ijk} \left( \begin{array}{c} \sigma^k \\ \sigma^k \end{array} \right). \quad (40)$$

A Dirac fermion, in $(1/2, 0) \oplus (0, 1/2)$ representation, can be assembled using 2-component fermions as

$$\psi = \left( \begin{array}{c} \chi_\alpha \\ \eta^{\dagger\dot{\alpha}} \end{array} \right). \quad (41)$$

We can also define

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \left( \begin{array}{c} -1 \\ 1 \end{array} \right). \quad (42)$$

We see that $\gamma^5$ is a diagonal operator whose diagonal elements corresponds to the left(-1) or right(+1) handedness of the 2-component part of the Dirac fermion. This is called chirality. In particular, the $U(1)_{\text{chiral}}$ transformation discussed earlier can be realized on the 4-compnent fermion as

$$\psi \rightarrow e^{-i\gamma^5\theta}\psi. \quad (43)$$

For forming Lorentz invariants, we also define

$$\bar{\psi} = \psi^\dagger\gamma^0 = (\eta^\alpha, \chi^{\dagger\dot{\alpha}}). \quad (44)$$

With these, we can write the 2 component Lagrangian, Eq. 36, as

$$\mathcal{L} = \frac{1}{2}i\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{2}m\bar{\psi}\psi. \quad (45)$$
For quantization of free fermion field in terms of momentum eigenstates, we first solve the wave function of free fermions. There are two solutions. We begin with the solution of the form \( u(p)e^{-ipx} \). The Dirac equation in this case is

\[
(\gamma^\mu p_\mu - m)u(p) = 0. \tag{46}
\]

The solution is of the form

\[
u(s)(p) = \begin{pmatrix} u_L \\ u_R \end{pmatrix} = \begin{pmatrix} \sqrt{p \cdot \sigma} x_s \\ \sqrt{p \cdot \bar{\sigma}} x_s \end{pmatrix}. \tag{47}\]

\( x_s \) are 2 component spinors, i.e., 2 component c-number column vectors. The 2 possible spinors, labeled by \( s = 1, 2 \), correspond to two polarizations of spin-1/2 particle along some particular axis. The square roots are just short handed way of writing

\[
\sqrt{p \cdot \sigma} = \frac{(E + m) - \sigma \cdot P}{\sqrt{2(E + m)}}, \quad \sqrt{p \cdot \bar{\sigma}} = \frac{(E + m) + \sigma \cdot P}{\sqrt{2(E + m)}}. \tag{48}\]

A particularly useful basis is called helicity, which uses the direction of motion as the spin axis. We denote the spinor in this basis \( x_\lambda \)

\[
\frac{1}{2} \sigma \cdot \hat{p} x_\lambda = \lambda x_\lambda, \quad \lambda = \pm \frac{1}{2}. \tag{49}\]

In the limit \( E \gg m \), we have

\[
u(p) = \sqrt{2E} \begin{pmatrix} (1/2 - \lambda)x_\lambda \\ (1/2 + \lambda)x_\lambda \end{pmatrix}. \tag{50}\]

Therefore, for helicity \( \lambda = \pm 1/2 \), only \( u_R \) or \( u_L \) is non-zero. In massless limit, helicity coincide with chirality.

Some useful relations:

\[
x_s^\dagger x_r = \delta_{sr}, \quad u^\dagger_r(p)u^s(p) = 2E\delta_{rs} \tag{51}\]

\[
\bar{u}^s(p) = u^{\dagger s}(p)\gamma^0, \quad \bar{u}^\dagger u^s(p) = 2m\delta_{rs}. \tag{52}\]

\[
\sum_s x_s x_s^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{53}\]

\[
\sum_s u^s(p)\bar{u}^s(p) = \gamma^\mu p_\mu + m = \not{p} + m \tag{54}\]

The other solution of Dirac equation is of the form

\[
v(p)e^{ipx}, \tag{55}\]
with
\[ v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} x_s \\ -\sqrt{p \cdot \sigma} x_s \end{pmatrix}. \] (56)

We have
\[ v^{tr}(p)v^s(p) = 2E\delta_{rs}, \quad \bar{v}^r v^s(p) = -2m\delta_{rs}. \] (57)

\[ \sum_s v^s(p)\bar{v}^s(p) = \gamma^\mu p_\mu - m = \not{p} - m. \] (58)

Now we are ready to write down the quantization of the free fermion field.

\[ \psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (a^s_p u^s(p)e^{-ipx} + b^{s\dagger}_p v^s(p)e^{ipx}). \] (59)

Therefore,
\[ \bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (a^{s\dagger}_p \bar{u}^s(p)e^{ipx} + b^s_p \bar{v}^s(p)e^{-ipx}). \] (60)

The fermionic creation and annihilation operators satisfy anti-commuting relations
\[ \{a^s_p, a^{r\dagger}_q\} = \{b^s_p, b^{r\dagger}_q\} = (2\pi)^3\delta^{(3)}(p - q)\delta^{rs}. \] (61)

It follows that the equal time anti-commutators of the fermionic fields are
\[ \{\psi_a(x), \psi^\dagger_b(y)\} = \{\psi^\dagger_a(x), \psi_b(y)\} = 0, \] (62), (63)

where a and b label 4 components of the Dirac fermion. The form of the quantized field, and the anti-commuting nature of the fields are fixed by Lorentz invariance and locality. It is another example of the spin-statistics theorem.

In the 4-component Dirac fermion case, the conserved current associated with \( U(1)_Q \) is \( J^\mu_Q = \bar{\psi}\gamma^\mu\psi \). The conserved the charge is
\[ Q = \int \frac{d^4p}{(2\pi)^4} \sum_s \left(a^{s\dagger}_p a^s_p - b^{s\dagger}_p b^s_p\right), \] (64)

with the clear interpretation that \( a^{s\dagger}_p \) and \( a^s_p \) are creation and annihilation operators for fermion with charge +1, while \( b^{s\dagger}_p \) and \( b^s_p \) are creation and annihilation operators for fermion with charge −1.

The fermion propagator is
\[ S_F(x - y) \equiv \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{ip^\mu + m}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}. \] (65)
We would like to emphasize again 4-component Dirac fermion is particularly useful for QED, with a charged massive fermion, the electron. With additional constraints, it can be used to deal with other cases, such as the Majorana fermion. However, in those cases, 2-component formalism is more natural.