Why Effective Field Theory (EFT)

- Key to our understanding of:
  1) Different physics laws on very different scales
  2) Renormalization in Quantum Field Theory.

- Necessary
  Our world is described by EFTs.

- Convenient & Powerful. Framework for calculation, with many applications.
1.1 Power counting, different kinds of operators.

Consider general Lagrangian of a real scalar

\[ L = \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{A}{4!} \phi^4 \\
+ \frac{C_1}{\Lambda^2} \phi^6 + \ldots \frac{C_n}{\Lambda^{2n}} \phi^{2n+2} + \ldots \\
+ \frac{d_1}{\Lambda^2} (\partial \phi) \phi^2 + \ldots \frac{d_n}{\Lambda^{2n}} (\partial \phi)^2 \phi^{2n} + \ldots \\
+ \text{more derivatives} \]

\( \Lambda: \text{cut-off (justified later)} \)

\( E, k < \Lambda \) consider low energy

\( \lambda, c_n, d_n < 1 \) i.e. perturbative.

\[ S = \int d^4x L \quad Z = \int D\phi e^{-iS} \]
Consider an experiment performed at energy (momentum) $E(k)$, i.e., with size $L \sim \frac{1}{E} \sim \frac{1}{k}$.

The relevant amplitude of field is $\phi_k$.

$$S \sim \frac{1}{k^2} \phi_k^2 + \frac{m^2}{k^2} \phi_k^2 + \frac{\lambda}{4!} \frac{1}{k^4} \phi_k^4 + \sum_n \left( c_n \left( \frac{k^2}{\Lambda^2} \right)^n \left( \phi_k \right)^{4+2n} + d_n \left( \frac{k^2}{\Lambda^2} \right)^n \left( \frac{\phi_k}{k} \right)^{2n+2} \right) + \ldots$$

1) requires $m < k$, otherwise, $\phi$ is not a relevant degree of freedom.

2) perturbativity $\rightarrow$ leading effect from kinetic term $\left( \frac{\phi_k}{k} \right)^2$

$$\int d\phi_k e^{-iS} \rightarrow \text{dominant contribution from } \phi_k \sim k.$$

3) $k > \Lambda$ infinite number of terms are important, $c_n \left( \frac{k^2}{\Lambda^2} \right)^n$ can be large, can be strongly coupled.

This Lagrangian is not useful for $k > \Lambda$. $\Lambda$ is a "cut-off".
4) For $E \sim k \ll \Lambda$

Only important terms are

$$\frac{1}{2} \partial^2 \phi^2 + \frac{1}{2} m^2 \phi^2 + \frac{A}{4!} \phi^4.$$  

That is, the renormalizable part of the Lagrangian.

"renormalizable" is a consequence of scale separation.

5) More on the cut-off

a. All known theory has a cut-off.

$\Lambda$ QCD for theory of mesons.
Lattice spacing for Landau-Ginsburg
$M_{pl}$ for all theories we know in Nature.

b. Viewed as a parameterization of unknown UV physics, beyond the validity of theory under consideration.
- A more general statement of power counting.

Consider a theory with a cut-off $\Lambda$

$$S_\Lambda = \int d^Dx \sum_i g_i O_i$$

$D$: space-time dimension

**Power counting:**

Energy dimension of $O_i$ is $\delta_i$

$$[O_i] = \delta_i$$

$S_\Lambda$ dimensionless $\rightarrow$ $[g_i] = D - \delta_i$

Define dimensionless coupling $\lambda_i$ as

$$g_i = \frac{\lambda_i}{\Lambda^{\delta_i - D}}$$
For a process at energy $E$

$$O_i \sim E^\delta_i \quad d^Dx \sim E^{-D}$$

$$S \sim q_i E^{\delta_i-D} = \lambda_i \left( \frac{E}{\Lambda} \right)^{\delta_i-D}$$

**Classification of operators**

<table>
<thead>
<tr>
<th>Operator dim</th>
<th>Importance as $E \to 0$</th>
<th>Classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_i &lt; D$</td>
<td>more important</td>
<td>relevant</td>
</tr>
<tr>
<td>$\delta &gt; D$</td>
<td>less important</td>
<td>irrelevant</td>
</tr>
<tr>
<td>$\delta = D$</td>
<td>same importance for all $E$</td>
<td>marginal</td>
</tr>
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$\rightarrow$ Low $E$, only relevant & marginal $\rightarrow$ renormalizable.

$\rightarrow$ We could use the process at $E$

a measurement of coupling

$$\lambda_i(E) = \lambda_i \left( \frac{E}{\Lambda} \right)^{\delta_i-D} \text{ "running coupling"}$$

$$S(E) = \frac{1}{2} \int d^Dx \lambda_i(E) \frac{1}{E^{\delta_i-D}} O_i$$
• A perturbative example

kinetic term \[ \int d^D x \ \frac{1}{2} (\partial \phi)^2 \]

sets \[ [\phi] \sim -1 + \frac{D}{2}. \]

For \( O_i \) made of \( M \phi_s \) and \( N \) derivatives

\[ [O_i] \sim M (-1 + \frac{D}{2}) + N \]

For \( D = 4 \):

\[ [\phi^2] \sim 2 \text{ relevant} \]
\[ [\phi^4] \sim 4 \text{ marginal} \]
\[ [\phi^6] \sim [ (\partial \phi)^2 \phi^2 ] \sim 6 \text{. irrelevant} \]

• Similar story for spin \( = \frac{1}{2}, 1, 2 \ldots \)
1.2 Renormalization Group (RG)

1.2.1. Wilsonian (RG)

Change of parameters as we change the cut-off.

\[ \Lambda \rightarrow \Lambda - d\Lambda \]

\[ \equiv b\Lambda \]

\[ \frac{d}{d\Lambda} \]

\[ f \]

\[ e^{iS_{b\Lambda}} = e^{iS_{\Lambda}} \]

\[ e^{iS_{b\Lambda}} = e^{iS_{\Lambda}} \]

\[ S_\Lambda(\lambda) \rightarrow S_{b\Lambda}(\lambda') \]

RG flow.

RG Equation (RGE)

\[ \frac{\partial S_\Lambda}{\partial \Lambda} = f(S_\Lambda) \]
Linearized RGE around small coupling

\[ \frac{1}{\Lambda} \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \end{pmatrix} \]

\[ \Lambda \frac{d}{d\Lambda} \lambda(\Lambda) = F \cdot \lambda(\Lambda) \]

\[ \lambda_i(\Lambda) = \lambda_i \left( \frac{E}{\Lambda} \right) \delta_i - D \]

\[ \rightarrow \text{F eigenvalues } \{ \delta_i - D \} \]
A perturbative example

\[ S_\Lambda = \int d^d x \left( \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \phi^4 + \cdots \right) d \rightarrow 4 \]

With decomposition \( \phi \rightarrow \phi + \hat{\phi} \)

\[ S_\Lambda = \int d^d x \left( \frac{1}{2} (\partial \phi + \partial \hat{\phi})^2 + \frac{1}{2} m^2 (\phi + \hat{\phi})^2 + \frac{1}{4!} (\phi + \hat{\phi})^4 + \cdots \right) \]

\[ = \int d^d x \left( L(\phi) + \frac{1}{2} (\partial \hat{\phi})^2 + \frac{1}{2} m^2 \hat{\phi}^2 \right. \]
\[ \left. + \lambda \left( \frac{1}{6} \phi^3 \hat{\phi} + \frac{1}{4} \phi^2 \hat{\phi}^2 + \frac{1}{6} \phi \hat{\phi}^3 + \frac{1}{4!} \hat{\phi}^4 \right) + \cdots \right] \]

\[ \phi \hat{\phi} \text{ terms (quadratic) gives no contribution.} \]

\[ e^{i S_{b\Lambda}} = \int d\hat{\phi} \ e^{i S_\Lambda} \]

\[ \hat{\phi}^{(k)} \hat{\phi}(\mathbf{p}) = \frac{1}{k^2} (2\pi)^d \delta^{(d)}(k + \mathbf{p}) \oplus k. \]
\[ \oplus k \rightarrow b\Lambda \leq |k| < \Lambda. \]

Vertices:

\[ \times \]

\[ \rightarrow \]
\[ \delta m^2 = \frac{\lambda}{2} \int_{b \Lambda < |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \]
\[ = \frac{\lambda}{(4\pi)^{d/2} \Gamma\left(\frac{d}{2}\right)} \frac{1 - b^{d-2}}{d-2} \Lambda^{d-2} \]

\[ \delta \lambda = -4! \frac{2}{2!} \left(\frac{\lambda}{4}\right)^2 \int_{b \Lambda < |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2}\right)^2 \]
\[ = \frac{-3 \lambda^2}{(4\pi)^{d/2} \Gamma\left(\frac{d}{2}\right)} \Lambda^{d-4} - (b\Lambda)^{d-4} \frac{d-4}{d-4} \]

\[ \lim_{d \rightarrow 4} \rightarrow -\frac{3}{16} \frac{\lambda^2}{\pi^2} \log \frac{1}{b} \]

\[ \delta m^2, \delta \lambda \text{ are the same as the counter terms in the BPHZ renormalization using cut-off regulators.} \]
• The EFT point of view.

– Every theory is an EFT defined with a cut-off. In this case, everything is finite.

– Renormalization, i.e., “computing loops”, is capturing the effect of UV physics (near the cut-off $\Lambda$) on the parameters of the theory.

– Schematically, parameters $m^2, \lambda$... in theory with cut-off $\Lambda$ are “bare parameters”. And $\delta m^2, \delta \lambda$... generated by integrating out UV physics, are “counter terms”.

– In practical computations, using cut-off and integrating out momentum shells are often inconvenient. We use dim-reg etc. But the physics is the same.
- From a "top-down" point of view, cut-off is from physical thresholds/scales, such as new resonances, phase transitions, etc. Yet, EFT by definition does not contain these d.o.f. Their effects are parameterized by the cut-off.

- Power counting / RGE arguments show that the UV effect can be packaged into a finite set of relevant and marginal operators, plus an infinite number of irrelevant operators.

- Irrelevant operators are not irrelevant. They can be very important. New discoveries were made through irrelevant operators.

  e.g. $\beta$-decay

  \[
  \begin{align*}
  n & \rightarrow p \\
  e & \frac{g^2}{m^2_W} \bar{p} \Gamma^\mu \nu n \bar{e} \Gamma^\mu \nu
  \end{align*}
  \]
e.g. gravitational interactions are all irrelevant.
More effects of integrating out UV physics.

Irrelevant operator

\[ \lambda^2 \phi^6 \]

\[ \lambda^2 \phi^2 \phi^4 \rightarrow \frac{\lambda^2}{\Lambda^2} \phi^2 \phi^2 \]

Lesson:

Irrelevant operators will be generated in general by integrating out UV physics.

Simply ignoring irrelevant operators is an assumption about UV physics.

In many cases, we have no right to make this assumption.
For example, the Standard Model.

- We excluded new physics (very roughly) below about a TeV
- However, operators such as

\[
\frac{H^+ \partial_\mu H^\dagger}{\Lambda^2} \quad H\text{: Higgs field}
\]

\( \Lambda \sim \text{TeV} \)

can still be there and have important effect.

Measuring irrelevant operators gives important information about UV theory.

\text{e.g.}\quad G_N \sim \frac{1}{M_{Pl}^2}

\( M_{Pl} \) (scale of quantum gravity) \( \sim 10^{19} \text{GeV} \)
- Effect of UV physics: naturalness

\[ m^2 \bigg|_{b\Lambda} = m^2 \bigg|_{\Lambda} + c \frac{\lambda}{16\pi^2} \Lambda^2 \quad c \sim O(1) \]

One cannot consistently hold mass to be zero at all scales. Suppose we have \( m^2 \bigg|_{\Lambda} = 0 \), then \( m^2 \bigg|_{b\Lambda} \sim c \frac{\lambda}{16\pi^2} \Lambda^2 \).

More generally, we expect scalar mass to be \( \sim c \frac{\lambda}{16\pi^2} \Lambda^2 \) (natural value).

If we want to have

\[ m^2 \bigg|_{\text{low energy}} \ll \Lambda^2 \]

Delicate cancellation (fine-tuning) is necessary.
Correction to marginal operator $\lambda \Phi^4$

$\delta \lambda = - \frac{3 \lambda^2}{16 \pi^2} \log \left( \frac{1}{b} \right) = - \frac{3 \lambda^2}{16 \pi^2} \log \left( \frac{\Lambda_{uv}}{\Lambda_{1R}} \right)$

* - sign, $\lambda$ smaller at lower energy.

* Mild (logarithmic) dependence on $\Lambda_{uv}$.

"Marginally irrelevant"

There are examples where the coupling is marginally relevant. The most famous example is QCD gauge coupling, which becomes strong at $\sim$ GeV, triggering confinement.
- Effects of UV physics: symmetry.

In our scalar example, operators $\phi^3, \phi^5$... are not generated by RG flow.

The simple reason for this is that there is a symmetry

$$\phi \leftrightarrow -\phi$$

• Another example:

Consider fermion with Yukawa interaction

$$y\phi \bar{\psi} \psi + \text{h.c.}$$

$\phi$

naively, linearly divergent.

However, it does not give a mass term

$$\text{no } \frac{y^2}{16\pi^2} \Lambda \bar{\phi} \psi$$

The reason is due to a symmetry

$$U(1)_{\text{chiral}} \left\{ \begin{array}{c}
\psi \rightarrow e^{i\gamma_5} \psi \\
\phi \rightarrow e^{-2i\alpha} \phi
\end{array} \right\} \Rightarrow \bar{\psi} \psi \rightarrow e^{2i\alpha} \bar{\psi} \psi$$

A term like $\Lambda \bar{\psi} \psi$ will break this.
So, if a fermion is massless to begin with, i.e., no $m_\Psi \Psi$ term, RG will not generate a mass.

If a fermion is massive, $m_\Psi \neq 0$, the RG will generate $\delta m \Psi \Psi$ with

$$\delta m = c \frac{y^2}{16\pi^2} m_\Psi \log \left( \frac{\Lambda}{m_\Psi} \right)$$

A crucial fact is that $\delta m \propto m_\Psi$

This is due to $m_\Psi$ is the ONLY parameter which breaks the U(1)$_{\text{chiral}}$ symmetry. So, $\delta m$, which is generated from RG and also breaks U(1)$_{\text{chiral}}$, must be proportional to $m_\Psi$.

This is a simple example of the powerful technique called spuriou analysis.
- More on symmetries
  
  Two kinds of symmetries

  1) Fundamental symmetry:
  
  Symmetries of the ultimate theory.

  2) Accidental symmetry.
  
  Symmetries whose breaking effects are less important at low energies "emergent".

  In practice, we consider all symmetries accidental.

  1) There are plausible arguments that all global symmetries are broken by quantum gravity effects.

  2) In practice, we simply cannot know if a symmetry is broken by small effects below the precision of exp measurement.
A typical scenario is an accidental symmetry is preserved by relevant and marginal operators, while it's broken by some irrelevant operators.

For example, consider a complex scalar $\phi$

$$L = \left| \partial \phi \right|^2 + m^2 \left| \phi \right|^2 + \frac{\lambda}{4} \left| \phi \right|^4$$

$$+ \left( \frac{c}{\Lambda^2} \phi^6 + \text{h.c.} \right) + \ldots$$

The renormalizable part preserves a $U(1)$ symmetry

$$\phi \rightarrow e^{i\alpha} \phi$$

However, dim-6 operator $\phi^6$ breaks $U(1)$.

$U(1)$ is accidental since the breaking effect is small for $E < \Lambda$, by our power counting argument.
Symmetry breaking

1) Explicit breaking.
   e.g. Fermion mass which breaks $U(1)_{\text{chiral}}$
   e.g. $\frac{c}{\lambda^2} \phi^6 + \text{h.c.}$ break $\phi \rightarrow e^{i\alpha} \phi$
   Small breaking $\rightarrow$ approximate symmetry
   Can use spurion analysis.

2) Spontaneous breaking

   $\rightarrow$ Goldstones.
Irrelevant operators and ability of making predictions.

Having irrelevant operators does not spoil calculability or predictive power. Instead, it’s a expansion in $\frac{E}{\Lambda}$. Operators with larger dimension $g_i$, “higher order”, give smaller (higher order in $\frac{E}{\Lambda}$) contributions.

For any given problem, we always just need an answer within certain precision. This is the best we can hope for anyways. Therefore, we only need to keep a finite number of irrelevant operators. Their couplings $c_i$, defined as

$$\frac{c_i}{\Lambda^{g_i-d}} O_i$$

are called Wilson coefficients.

Once Wilson coefficients are fixed (by experiments), we can make further predictions.
1.3 Philosophy of EFT (bottom up)

1) Theory is only valid up to a cut-off, such as new particle mass, phase transition scale, etc. Above the cut-off, a new theory.

2) Theory contains all relevant degrees of freedom below the cut-off.

3) Assume (accidental) symmetries for relevant and marginal operators. Write down all possible operators consistent with these symmetries. Including symmetry breaking terms for approximate symmetries.

4) Irrelevant operators

Assume which symmetries are preserved. Write down all operators allowed by these symmetries.

In practice, truncate the operators at certain order of $E/\lambda$. 
5) Free parameters in the resulting EFT fixed by measurement.

6) Cut-off scale:
   - Can be constrained if effect of irrelevant operator not observed.
     Examples: Standard Model precision measurements. (more later)
   - If observed $\Rightarrow$ new physics scale!
     Examples: $\beta$-decay $\rightarrow$ W-boson
     gravity $\rightarrow$ $M_{\text{Pl}}$
   - Theory sometimes predicts its own cut-off.
     Example: Spontaneous Symmetry Breaking and non-linear $\sigma$-model.