Conserved quantities.

Consider observable \( \hat{O} \). If it commutes with Hamiltonian \( \hat{H} \):

\[
[\hat{H}, \hat{O}] = 0
\]

We say \( \hat{O} \) is conserved.

Since \( [\hat{H}, \hat{O}] = 0 \), they share a complete set of common eigenstates, \( |E_i, \lambda_j\rangle \):

\[
\hat{H} |E_i, \lambda_j\rangle = E_i |E_i, \lambda_j\rangle, \quad \hat{O} |E_i, \lambda_j\rangle = \lambda_j |E_i, \lambda_j\rangle
\]

Now, we discuss why \( \hat{O} \) is called a conserved quantity. From Baker-Hausdorff formula, we have

\[
e^{\frac{\hat{H} t}{\hbar}} \hat{O} e^{-\frac{\hat{H} t}{\hbar}} = \hat{O} + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{it}{\hbar} \right)^n C_n
\]

where

\[
C_1 = [\hat{H}, \hat{O}]
\]

\[
C_2 = [\hat{H}, [\hat{H}, \hat{O}]]
\]

\cdots
In particular, for conserved quantities, we have
\[ e^{\frac{i}{\hbar} H t} \hat{O} e^{-\frac{i}{\hbar} H t} = \hat{O} \] \hspace{1cm} (1)

Now, consider a state at \( t=0 \), \( |\psi_0\rangle \). The expectation value of \( \hat{O} \) at \( t=0 \) is
\[ \langle \psi_0 | \hat{O} | \psi_0 \rangle. \]
At an arbitrary later time \( t \),
\[ |\psi_t\rangle = e^{-\frac{i}{\hbar} H t} |\psi_0\rangle. \]
The expectation value at that time is
\[ \langle \psi_t | \hat{O} | \psi_t \rangle = \langle \psi_0 | e^{\frac{i}{\hbar} H t} \hat{O} e^{-\frac{i}{\hbar} H t} | \psi_0 \rangle \]
using (1) \[ \rightarrow \]
\[ = \langle \psi_0 | \hat{O} | \psi_0 \rangle. \]
Hence, the expectation value of a conserved quantity \( \hat{O} \) is a constant, i.e., independent of time.
In particular, if $|\psi_0\rangle$ is an eigenstate of $\hat{0}$

$$\hat{0} |\psi_0\rangle = \lambda_i |\psi_0\rangle.$$

Then, $|\chi, t\rangle$ is also an eigenstate with the same eigenvalue. To see this,

$$\hat{0} |\chi, t\rangle = \hat{0} e^{-\frac{i}{\hbar} H t} |\psi_0\rangle$$

$$\overset{\text{\(\rightarrow\)}}{=} e^{-\frac{i}{\hbar} H t} \hat{0} |\psi_0\rangle$$

$$= \lambda_i e^{-\frac{i}{\hbar} H t} |\psi_0\rangle$$

$$= \lambda_i |\chi, t\rangle.$$
b) 2 spin-\(\frac{1}{2}\) particles.

i) \( H = a \mathbf{S}_{1z} + b \mathbf{S}_{2z}. \quad a \neq b. \)

Conserved: \( S_{1z}, S_{2z}, S_1^2, S_2^2, H. \)

Eigenstates: \( |\pm\rangle_1, |\pm\rangle_2. \)

ii) \( H = a (S_{1z} + S_{2z}). \)

Conserved: \( S_{1z}, S_{2z}, S_1^2, S_2^2, H. \)

\( S_z = S_{1z} + S_{2z}, \quad S^2 = (S_1 + S_2)^2. \)

Eigenstates: \( |\pm\rangle_1, |\pm\rangle_2 \)
\( \vdots \)
\( |1,1\rangle, |1,-1\rangle, |1,0\rangle, |0,0\rangle. \)

iii) \( H = c \mathbf{S}_1 \cdot \mathbf{S}_2. \)

Conserved: \( S_3, S^2, H. \)

Eigenstates: \( |1,1\rangle, |1,-1\rangle, |1,0\rangle, |0,0\rangle. \)
Examples of conserved quantities.

a) single particle system.

\[ H = a \vec{s} \cdot \vec{B} \]

\[ \vec{B} = B \hat{k} \]

\[ \rightarrow H = a |\vec{s}| \]

Conserved: \( s_z \), \( s^2 \), \( H \)

Eigenstates of \( H \) and \( s_z \): \( \pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \).

Now take \( \vec{B} = B \hat{n} \), \( \hat{n} \) is a unit vector pointing at arbitrary direction.

\[ H = aB \vec{s} \cdot \hat{n} \]

Conserved: \( \vec{s} \cdot \hat{n} \), \( s^2 \), \( H \)

Eigenstate: \( \pm |\hat{n}, + \rangle \).
iv) \( H = a s_{1z} + b s_{2z} + c s_1 \cdot s_2 \).

Conserved: \( S_z = s_{1z} + s_{2z} \).

Eigenstate: \( |1, 1\rangle, |1, -1\rangle \).

\[
|\Psi_3\rangle = \alpha |1, 0\rangle + \beta |0, 0\rangle
\]

\[
|\Psi_4\rangle = \alpha' |1, 0\rangle + \beta' |0, 0\rangle.
\]

where 

\( |\Psi_3\rangle, |\Psi_4\rangle \) are eigenstates of \( H \), obtained by diagonalizing the 2x2 matrix in \( |1, 0\rangle \) and \( |0, 0\rangle \) subspace.

And: 

\( S_z |\Psi_3\rangle = S_z |\Psi_4\rangle = 0 \).