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STRINGS ON ORBIFOLDS*

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String propagation on the quotient of a flat torus by a discrete group is considered We obtain an exactly soluble and more or less realistic method of string compactification

The most thoroughly understood example of superstring compactification is compactification on a flat torus [1]. This example is exactly soluble and obeys all of the equations of string theory. It is also clearly a significant example since it is one way to understand the construction of the heterotic string theory [2]. For compactification of physical dimensions from ten to four, the flat torus is not suitable since it leads in many ways to unappealing phenomenology. More realistic compactification schemes involve non-trivial supersymmetric non-linear sigma models, and the realistic ones seem to be rather complicated mathematically. The purpose of this paper is to describe a scheme of compactification of extra physical dimensions which is almost as simple as standard toroidal compactification and far more realistic.

The main phenomenological question involves the realization of spacetime and gauge symmetry breaking. In the string theory a way to achieve symmetry breaking is to twist the boundary conditions in such a way that there is no net charge corresponding to the broken symmetries on the string world sheet. Such a procedure has been used for supersymmetry [3] and gauge symmetry [4] breaking and it is natural to attempt to do so for Lorentz symmetries as well [5] The idea of using twisted boundary conditions to break Lorentz symmetries is that spacetime coordinates $x^{\mu}(\sigma)$ should not be periodic functions of σ but periodic up to a Lorentz transformation. Modular invariance puts severe constraints on such attempts, since a string sector with twisted boundary conditions in the σ-direction is related by modular transformations to sectors with various boundary conditions in the σ- and τ-directions. What will be described below is one consistent framework in which the twisted boundary conditions are related to the construction of a new manifold (or at least, a new orbifold, as we will describe later) The method of constructing new manifolds (or orbifolds) that we will employ is a classical geometrical method that we will implement directly in string theory.

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A classical example, which has been described in the physics literature [6], involves the K3 manifold. Consider the four-dimensional torus T with periodic coordinates x_i, $i=1\cdots4$, obeying $x_i \approx x_i+1$ Let g be the transformation $g(x_i)=-x_i$. Since $g^2=1$, g generates the group Z_2. If g acted freely (without fixed points), then the quotient space $X=T/Z_2$ would be a smooth, non-singular manifold. Actually, g does not act freely. There are sixteen fixed points - the points in which each coordinate x_i is an integer or half-integer. Because of this, X is not a smooth manifold but a so-called orbifold - a manifold with singularities that correspond to those obtained by dividing a smooth manifold by the non-free action of a discrete group. In this case, the singularities can be smoothed out or resolved by "blowing up" the fixed points. The resulting smooth manifold is a rather subtle and interesting one called the K3 surface The singularities encountered in the examples we will consider below can likewise be resolved, but this will not be our main concern. Our main concern will be to treat the string propagation directly on the singular orbifold. As we will discuss, this is exactly soluble. (In resolving singularities, one must introduce free parameters corresponding to the "size" of the blown-up fixed points. When these parameters are tiny, the exactly soluble problem of string propagation on the orbifold is an arbitrarily good approximation to propagation on the smooth manifold, if that is what is desired.)

The K3 manifold is in many ways unique. For instance, it is the only four-dimensional manifold of SU(2) holonomy. The construction that leads to K3 has an analogue in real dimension six(complex dimension three) that does not have the same degree of uniqueness. The simplest example was constructed by mathematicians and was described in [4], where it was called Z. More general examples have been constructed recently in [7]. Although the discussion readily generalizes to the other cases, we will illustrate our ideas by discussing string propagation on the Z manifold. To review the construction of the Z manifold, let z_i, $i=1, 2, 3$ be three complex variables. For each i, let T_i be the torus defined by $z_i \approx z_i+1 \approx z_i+e^{i\pi/3}$, and let α be the transformation $z_i \to z_i\, e^{2i\pi/3}$ acting on the product manifold $T=T_1\times T_2\times T_3$. Then α generates a group G isomorphic to Z_3. There are twenty seven fixed points of α, namely the points in which each z_i is an integer multiple of $\sqrt{\frac{1}{3}}\, e^{i\pi/6}$. Because of the existence of these twenty seven fixed points, the quotient space $Z=T/G$ is not a smooth manifold but an orbifold with twenty seven isolated singularities. We wish to describe string propagation on this orbifold.

To describe the quantum mechanics of a point particle propagating on Z, one approach would be to begin with the Hilbert space H_0 of wave functions for a particle propagating on T and then restrict ourselves to the Hilbert space H of G-invariant wave functions. The particle wave functions on Z can be identified with G-invariant wave functions on T Actually, in general some choices must be made to define what we mean by the action of G on H_0 If the particle under study has some internal quantum numbers, so that the wave function is actually a section of some vector bundle V over T, it is necessary to "lift" the action of G from T to V, and a choice of such a lifting will determine what we mean by the G-invariant subspace H. Differently put, in general we may wish to choose G to act on the internal quantum numbers as well as on the space-time manifold T, and how G acts on the internal quantum numbers will influence what is meant by the G-invariant subspace of Hilbert space

Following this logic for strings, the first step to describe string propagation on Z is to construct the Hilbert space H_0 of string states on T and project this onto the G-invariant subspace. As in the particle case, we must choose a G action on the internal degrees of freedom of the string, and in the case of the $E_8 \times E_8$ heterotic theory, this means that we must pick a homomorphism of G into $E_8 \times E_8$. Although other possibilities might be interesting, we will pick here a simple choice which is motivated by the idea of embedding the spin connection in the gauge group [4], and which also turns out to make sense in terms of string propagation on the orbifold Picking an $E_6 \times SU(3)$ subgroup of the first E_8, let β be the generator of the center of SU(3). Letting $\gamma = \alpha\beta$, and defining the action of γ on space-time fermions so that $\gamma^3 = 1$ (the other possibility would have been $\gamma^3 = (-1)^F$), γ generates a Z_3 group which we will also call G. Thus, one ingredient in constructing the Hilbert space that describes string propagation on Z is to construct the G-invariant subspace of the string Hilbert space for propagation on T

In contrast to the point-particle case, this is not the end of the story. If G acted freely, dividing T by the G action to obtain Z would introduce new sectors in the string propagation - winding sectors in which the string wraps around a non-contractible loop in Z Mathematically, these are sectors in which the coordinates $x^i(\sigma)$ of the string obey not $x^i(\sigma + 2\pi) = x^i(\sigma)$ but $x^i(\sigma + 2\pi) = gx^i(\sigma)$ for some $g \in G$. In the case at hand, G does not act freely, but we can hardly expect this extra complication to eliminate the need to consider the sectors (which, as we will see, no longer deserve to be called winding sectors) in which the string is closed only modulo a G transformation. We will call the new sectors twisted sectors Including the twisted sectors is necessary if x and gx are to be considered equivalent space-time points. Like the untwisted sector, the twisted ones must be projected onto G-invariant subspaces if we are interested in propagation on $T/G \approx Z$ (A precise argument for the need to include and project the twisted sectors will be given later on the basis of modular invariance)

Returning to the Z manifold, there are three sectors - the untwisted sector and the sectors twisted by γ to γ^{-1} We will discuss first the untwisted sector. To construct this sector, one simply formulates string propagation on the ordinary torus T and projects onto γ invariant states Massless modes on the torus are made by combining massless modes of the right-movers (which have the degrees of freedom of the open superstring) with massless modes of the left-movers (which carry $E_8 \times E_8$ as well as space-time bosonic degrees of freedom) A right-moving mode with eigenvalue λ of γ must be combined with a left-moving mode of eigenvalue λ^{-1} Here λ may be any cube root of unity (since $\gamma^3 = 1$). For right-movers the massless modes consist of a standard super-Maxwell multiplet whose relevant quantum numbers (λ and the four-dimensional helicity) are

$$\lambda = 1: \qquad (1, \tfrac{1}{2}) + (-\tfrac{1}{2}, -1)\,,$$

$$\lambda = e^{2i\pi/3}: \qquad (\tfrac{1}{2}, 0) + (\tfrac{1}{2}, 0) + (\tfrac{1}{2}, 0)\,,$$

$$\lambda = e^{-2i\pi/3}: \qquad (0, -\tfrac{1}{2}) + (0, -\tfrac{1}{2}) + (0, -\tfrac{1}{2})\,. \tag{1}$$

For left-movers, in addition to λ and the four-dimensional helicity, massless states must be labeled by the unbroken gauge quantum numbers $E_6 \times SU(3) \times E_8$, this being the subgroup of the gauge group that commutes with γ. The massless states are as follows ($E_6 \times SU(3) \times E_8$ quantum numbers are indicated in brackets and helicities in parentheses):

$$\lambda = 1:$$

$$([1,1,1], (1) + (-1)); ([78,1,1] + [1,8,1], (0)); ([1,1,248], (0))\,,$$

$$\lambda = e^{-2i\pi/3}:$$

$$([1,1,1], (0) + (0) + (0)); ([27,3,1], (0))\,,$$

$$\lambda = e^{2i\pi/3}:$$

$$([1,1,1], (0) + (0) + (0)), ([\overline{27}, \bar{3}, 1], (0)]\,. \tag{2}$$

Coupling together states from the above equations to make $\lambda = 1$, the allowed states are as follows. The gauge singlets are the supergravity multiplet $(2, \tfrac{3}{2})$ and $(-\tfrac{3}{2}, -2)$ and ten matter multiplets $(\tfrac{1}{2}, 0)$ and $(0, -\tfrac{1}{2})$. In addition there are massless gauge supermultiplets of unbroken $E_6 \times SU(3) \times E_8$. And there are massless chiral matter fields with gauge quantum numbers. The states with helicity $(\tfrac{1}{2}, 0)$ are three copies of $[27, 3]$ under $E_6 \times SU(3)$. Their antiparticles, of helicity $(0, -\tfrac{1}{2})$, transform as $[\overline{27}, \bar{3}]$. Viewed just in terms of E_6, there are nine chiral generations in the untwisted sector.

Let us consider the twisted sector corresponding to twisting by γ. There are some new features in this sector For the string to have a massless state, it must be capable of shrinking to zero size. To do so it must sit at one of the fixed points of γ In the case under study, the fixed points of γ are isolated, and a zero-size string sitting at one fixed point cannot be continuously deformed into a zero-size string sitting at another fixed point. Expansion about strings sitting at different fixed points leads to disjoint sectors in the string propagation. In the Z manifold, as there are 27 fixed points, we obtain 27 sectors corresponding to the expansion of the string about each of these fixed points. These sectors lead to isomorphic physics, as they are related by obvious symmetries (translations on the underlying torus). It is therefore adequate to concentrate on any one of the twenty seven sectors.

In the twisted sector, the quanta of oscillators are not all integral. To enumerate the physical degrees of freedom we will use the light cone gauge, with manifestly supersymmetric spacetime fermions. Let x_3, x_4 be the uncompactified transverse oscillators, and let x_i, $i=5, \ldots, 10$ be the compactified transverse oscillators. Then x_3 and x_4 are quantized with integral quanta of oscillation, but the other x_i are quantized with quanta that are integers plus or minus one third (plus one third for three of the x_i and minus one third for the other three). As we have chosen γ to respect space-time supersymmetry (since it lies in an SU(3) subgroup of the rotation group which leaves invariant one supersymmetry), the space-time fermions are quantized in the same way. In particular, only two of the space-time fermions (the ones that are quantized with integral oscillators) have zero modes. Quantization of these zero modes causes the ground state of the right-movers to be a doublet with $\lambda=1$ and helicity $(\frac{1}{2}, 0)$. The discussion of the left-movers in the twisted sector is more complicated. A relatively simple approach involves realizing an $O(16) \times O(16)$ subgroup of the gauge group linearly on thirty two fermions Actually, the first O(16) is broken to $O(10) \times SU(3)$ by the twisting. It is particularly easy to find and count the states that are spinors of this O(10) They come from a sector in which the ten fermions realizing this O(10) have integral quanta of oscillation and therefore have zero modes (and where the sixteen fermions of the other O(16) are antiperiodic). The six other fermions of the broken O(16) have quanta of oscillation that are not integral, three being shifted by $\frac{1}{3}$ and three by $-\frac{1}{3}$. Quantization of the ten zero modes gives a spinor of O(10) (of definite "chirality") which has no additional left-moving quantum numbers since there are no other zero modes to be quantized. Inclusion of the sector corresponding to a minus sign change in the boundary conditions of the sixteen fermions realizing the broken O(16) can be seen to fill this out into a 27 of E_6, and to generate as well massless E_6 singlets transforming as three triplets of SU(3). (These states all have $\lambda=1$.) The spectrum just described comes from expanding about one of the twenty seven fixed points of γ. All told, we get from the sector twisted by γ twenty seven positive chirality matter multiplets transforming as the [27, 1] of $E_6 \times SU(3)$ and eighty one such multiplets transforming as [1, 3] Happily, little further effort is required to analyze the states obtained by twisting with γ^{-1} They are just the antiparticles of the states just mentioned.

Let us collect these results and count the number of chiral multiplets obtained by combining all the pieces Under E_6, there are a total of thirty six "families" (the number of families is defined as the net number of positive chirality 27's minus negative chirality 27's). Nine positive chirality 27's (transforming as three triplets of the "flavor" group SU(3)) came from the untwisted quantization, while twenty seven positive chirality 27's come from the sector twisted by γ No positive chirality

27's come from the sector twisted by γ^{-1}, and no negative chirality 27's come from any sector. In addition, there are a large number of E_6 singlet massless chiral fields.

Let us compare these results with what one learns by studying the same manifold by field theoretic means [4]. In that framework, embedding the spin connection in the gauge group, the number of generations is predicted to be one-half the Euler character of the manifold. The Euler characteristic is 72 for the Z manifold, so the number of generations is predicted to be 36 – the same number we have just found. This suggests that the string propagation on the orbifold with α embedded in the gauge group in the manner we considered is a limiting case of string propagation on the smooth manifold with spin connection embedded in the gauge group. Some differences between our results here and results in [4] correspond to the fact that the unbroken SU(3) in our construction disappears if the fixed points are blown up and smoothed with spin connection embedded in the gauge group Therefore, the 36 chiral generations were not in [4] classified as representations of a family SU(3) Also, the number of massless E_6 singlets that appear in the field theoretic analysis, although large, is less than we have found here, presumably because some states that are kept massless by SU(3) but not by any topological considerations get mass if SU(3) is broken.

The procedure just given for counting the number of generations that appear in quantization on the Z orbifold was fairly explicit but also fairly long-winded We will now consider an efficient path integral method for computing the number of generations on an arbitrary orbifold. Thus, we start with an arbitrary smooth manifold M, with a discrete symmetry group G. In the case of the Z manifold, M was a flat torus, but for present purposes we can be more general. Let K be the orbifold M/G. If the spin connection of M (and the G action on tangent vectors of M) is embedded in the gauge group, then the number of generations is one half the Euler characteristic of K, so what we really want is an efficient method to calculate the Euler characteristic of an orbifold

In fact, the Euler characteristic of a manifold M is naturally viewed as the index $\mathrm{Tr}\,(-1)^F$ of the 1+1 dimensional supersymmetric non-linear sigma model [8]. (If the spin connection is embedded in the gauge group, this is the field theory that describes string propagation on M.) This can be computed as a path integral [9]:

$$\mathrm{Tr}\,(-1)^F = \int \mathrm{d}x^i(\sigma,\tau)\,\mathrm{d}\psi^i(\sigma,\tau)\exp\,[-I]\,, \tag{3}$$

where $\psi^i(\sigma,\tau)$ obey periodic boundary conditions in both sigma and tau, and I is the action of the supersymmetric non-linear sigma model.

To study the orbifold, one must introduce twisted boundary conditions. In general, one could try to pick any two elements h and g of G and require $x^i(\sigma+2\pi,\tau)=gx^i(\sigma,\tau)$, $x^i(\sigma,\tau+2\pi)=hx^i(\sigma,\tau)$. It turns out that we will only have need for the case in which g and h commute with each other Indeed, the boundary conditions just stated are more or less inconsistent otherwise.

Let $A(g, h)$ denote the value of the integral in (3) with boundary conditions twisted by g and h. We will describe how to compute the $A(g, h)$ and how to express the Euler characteristic of the orbifold in terms of them. Computing the $A(g, h)$ is particularly simple. Let $M(g, h)$ be the subspace of M consisting of points that are invariant under both g and h, and let $\chi(g, h)$ be the Euler characteristic of $M(g, h)$. Then as we will now argue, $A(g, h) = \chi(g, h)$. In fact, as $A(g, h)$ is a topological invariant, we may as well compute it by going to weak coupling (very large radius for M). The path integral is then dominated by paths of very nearly zero action, which means that the x^i must be (nearly) independent of sigma and tau This is compatible with the boundary conditions only if x^i is invariant under g, h, so x^i must lie in (or very near) $M(g, h)$ The bosonic and fermionic oscillations transverse to $M(g, h)$ give determinants which cancel in the usual supersymmetric fashion, so the path integral in (3) reduces to the path integral for a similar integral for a supersymmetric non-linear sigma model on $M(g, h)$. By the usual analysis of $\mathrm{Tr}\,(-1)^F$, the value of the integral is $\chi(g, h)$.

Now that we have calculated the $A(g, h)$, we want to use them to determine the Euler characteristic of the orbifold. Suppose that we want to discuss strings that obey untwisted boundary conditions in sigma, so $g = 1$. A winding by h in the tau direction then amounts to studying not $\mathrm{Tr}\,(-1)^F$ but $\mathrm{Tr}\, h(-1)^F$ (Our above arguments show that $\mathrm{Tr}\, h(-1)^F = \chi(1, h)$, an assertion which is known as the Lefschetz fixed-point theorem.) Projecting onto invariant states is carried out by the projection operator $Q = (1/N) \sum h$, where N is the number of elements in G and the sum in $\sum h$ runs over all elements in G. The contribution of strings untwisted in sigma to the Euler characteristic of the orbifold is thus $\mathrm{Tr}\, Q(-1)^F = (1/N) \sum \chi(1, h)$. (In the case of the Z manifold, this would give $\frac{1}{3}(0+27+27) = 18$, in agreement with the fact that we previously found nine generations from this sector.)

In field theory, we would stop here. In string theory, we cannot stop here; we must go on to discuss the contributions of sectors twisted in sigma as well as tau. The concrete reason for why we must do this is as follows. A modular transformation $(\sigma, \tau) \to (k\sigma + l\tau, m\sigma + n\tau)$, k, l, m, n integers, $kn - lm = 1$ will turn a sector twisted in tau only into one twisted both in sigma and tau. Modular invariance is of course needed for consistency of the theory of closed strings. At first sight one might think that modular invariance would force us to consider arbitrary pairs (g, h), but that is not so If g and h commute with each other, they are mapped into the commuting pair $(g^k h^l, g^m h^n)$, so modular invariance permits us to restrict attention to commuting pairs (g, h) This is just as well, since the twisted boundary conditions do not make too much sense unless g and h commute. These facts suggest the following formula for the Euler characteristic of K.

$$\chi(\mathrm{K}) = \frac{1}{N} \sum_{gh = hg} \chi(g, h) \,. \tag{4}$$

This formula is modular-invariant, receives the proper contribution from $g = 1$, and only involves commuting pairs. To see that it is also correct requires more thought. To construct the required Hilbert space, one begins with all possible sigma windings by group elements g, denote the corresponding Hilbert spaces as H_g. One then tries to make a G-invariant Hilbert space by inserting a projection operator. One expects the projection operator to be a symmetry of the Hilbert space, but in fact the individual H_g are not invariant under all of G. A given H_g only has a lesser symmetry group G_g consisting of elements that commute with g, which we may call the "little group" (or centralizer) of g An element h of G that does not commute with g turns H_g into $H_{h^{-1}gh}$. The way to form a G-invariant and modular-invariant Hilbert space is to sum over all possible H_g and project each H_g onto states invariant under its own little group - by including all possible tau windings that commute with g. This corresponds to the formula in (4) It is again easy to check this formula for the Z manifold, there are eight non-zero terms in (4), each contributing $(\frac{1}{3})\times 27=9$, so that $\chi(Z)=72$. As this agreement shows, (4) properly includes corrections to the Euler characteristic coming from blowing up the fixed points. It is remarkable that in this simple way string theory "knows" what happens when the singularities are resolved by blowing up the fixed points. Field theory does not lead in this way to the proper formula, since it omits the contributions of the twisted sectors.

The agreement between the string theory calculation of the Euler characteristic and the correct value is not an accident of the Z manifold but a general feature, at least for manifolds of SU(3) holonomy If one divides a manifold M by the action of a discrete group G, singularities are created if G does not act freely. The theory of how to resolve these singularities is rather complicated in general, but under favorable conditions there are simple prescriptions, which were summarized in [10]. In these cases, (4) gives a value of the Euler characteristic which agrees with that obtained in more standard ways. For instance, if G is Z_3 and N is the fixed point set of G in M, then the quotient $X=M/G$ has (after blowing up fixed points) the Euler characteristic [10] $\chi(X)=\frac{1}{3}(\chi(M)+8\chi(N))$, which is easily seen to agree with (4). It does not appear that eq. (4) is known to mathematicians, and as this formula makes sense in situations in which a standard procedure for resolving the singularities does not exist, (4) may be a clue to a more general procedure.

We have given the Z manifold as an example in which the string propagation is exactly soluble, but the considerations are more general The general idea is to begin with a flat torus T with symmetry group G, and study the quotient T/G. If G is embedded in a suitable way in E_8, one is studying an idealized limit of "embedding the spin connection in the gauge group." It is possible to study more general embeddings, but we have found that problems typically arise [11] from a mismatch of left- and right-moving energy levels. The simplest consistent examples occur when G is a subgroup of SU(3) leading to an unbroken $N=1$ supersymmetry in four dimensions. Again, one can consider more general examples; one typically encounters tachyons unless group elements not in SU(3) act freely [11] and even then there may be one-loop instabilities [3], but it may be that some examples work. Even though no smooth Ricci-flat manifolds with O(6) holonomy are known [12], we can still consider string theory on oribifolds with broken supersymmetry. We will discuss these and other matters related to the phenomenology of string theories on orbifolds more fully in [11].

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