ABSTRACT: This note summarizes the talk given on March 8th 2016 which was on introductory tensor network theory whose aim was to provide with some fundamentals for the following topic, the relation between tensor network and holography. This note basically follows the arguments in A Practical Introduction to Tensor Networks: Matrix Product States and Projected Entangled Pair States [1] and Quantum Information Meets Quantum Matter [2]
1 Tensor Network Basics

1.1 What is a tensor network

Generally speaking, tensor network is a kind of decomposition and graphical representation of a general tensor. In this language, a rank $k$ tensor is represented as an object with $k$ external legs. Depending on the dimension of that index, each leg can further be regarded as a bundle of tiny legs. In figure 1 we list some simple or frequently used examples of tensors, say scalar $s$, vector $v$ and rank 2 tensor $t_{ij}$.

![Figure 1](image)

Using this kind of representation, in principle we could realize all tensor manipulations in terms of drawing diagrams. For example, if we are given 2 rank 2 tensors $v_{ij}$ and $w_{kl}$ we connect them via combining one leg from each, graphically we are left with a new object with 2 legs left, i.e. a rank 2 tensor $t_{il}$ alone. Algebraically, it is the conventional tensor contraction (or matrix multiplication)

$$t_{il} = v_{ij}w_{kl}\delta_{jk}. \quad (1.1)$$

Therefore, in tensor network, tensor contractions can be done graphically by merely connecting legs of tensors. From this observation we could also see $\delta_{ij}$ is represented as a line (which of course has two legs.) This is depicted in figure 2

In quantum many body system one often encounter states of form

$$|\Psi\rangle = \sum_{\{i_k\}} \Psi_{i_1i_2...i_N}|i_1...i_N\rangle. \quad (1.2)$$
Given a specific basis $|i_1...i_N\rangle$, manipulating quantum states is essentially equivalent to manipulating the wave functions $\Psi_{i_1i_2...i_N}$ which are components of a rank $N$ tensor. We may first give some examples of graphical representation of various tensors that appear frequently in quantum many body systems. These are shown in figure 3 and 4.

What about the network? From the above examples there was no network like ingredient. The idea of network is to decompose a general big tensor, say $\Psi_{i_1...i_N}$ into contractions of little tensors. We have seen contractions by $\delta_{ij}$ are represented as lines connecting legs from tensors. Consequently, a huge tensor, under a decomposition, becomes a network. Generically speaking, an arbitrary network with net external legs $N$ is a rank $N$ tensor. However, depending on the structure of internal contractions and the numbers of parameters contained in building blocks, not every tensor network is useful and able to efficiently implement. In the following figure 5 are several celebrated examples of tensor networks, matrix product states (MPS), projected entangled pair states (PEPS), and multi-scale entanglement renormalization ansatz (MERA). It worth noting that in some literature MPS = PEPS and the PEPS here is called Tensor Product States. In this note I will regard PEPS as 2-dimensional many-body states.
MPS looks like the simplest tensor network we can imagine in 1D, yet it turns out to be very powerful, which is strongly related to the success of density matrix renormalization group (DMRG) algorithm in 1 dimension. Roughly speaking, when performing DMRG, if one postpones superblock combination procedure to the end of renormalization, one could see the whole structure is basically a MPS state representation. Later we will comment on the reason that MPS or DMRG could be so efficient in 1 dimension and its failure in higher dimensional space. Before diving into more sophisticated analysis, we can first count the degrees of freedom stored in a MPS. Suppose the dimension of the physical external leg $i_n$ is $p_n$, the dimensions of internal contraction bonds are bounded by $\chi$. The number of parameters to specify a $N$ site MPS is bounded by

$$\# \sim \mathcal{O}(N p \chi^2),$$

which scales linearly in $N$. On the other hand, we know the degrees of freedom of a general tensor is of order

$$\# \geq \min\{p_n\}^N.$$

It’s apparent that given $\chi$, MPS states are unlikely to spanned the whole Hilbert space of the target tensor. More generally, a practical tensor network usually contains parameters of order

$$\# = \mathcal{O}(\text{poly}(N)\text{poly}(\chi)).$$

In the following sections we will comment on the discrepancy in degrees of freedom between a general tensor and tensor networks states.

1.2 Why do we need a tensor network

In the previous subsection we introduced the notion of a tensor network as a kind of graphical representation of general tensors and one can hardly find the necessity of it. Indeed, given a general tensor,
a mere graphical representation doesn’t help much. However, if a general giant tensor, a big black box, can be illustrated in a network, according to different connecting schemes, states can be categorized by its entanglement pattern. Moreover, from the estimation at the end of last section we saw that the degrees of freedom of an applicable tensor network grow polynomially with system size, which is trackable.

This is one of the reasons that make tensor network representation useful. As we know it, the whole Hilbert space \( \mathcal{H} \) for an arbitrary many-body system is always way too large and takes time scales beyond the age of the universe to explore. Nonetheless, the states with certain entanglement pattern may merely occupy a small corner of the Hilbert space. If those are exactly what concern us, we need not to be able to sweep the complete \( \mathcal{H} \) for extracting physics. Instead, we could concentrate on states represented by the class of networks that acquire similar entanglement structure to obtain insightful information.

For example, fortunately, it has been shown that the ground states to local gapped many body Hamiltonians cannot be arbitrary in the sense that their entanglement entropy don’t scale extensively but obey the so called area law. Area law is of no doubt a special kind of entanglement pattern and therefore the set containing those ground states lie in a corner. We will show in the following sections explicitly that MPS states in 1 dimension obey this exotic property. Here we give a heuristic argument using PEPS in 2 dimensions. As shown in figure, the system is divided into inner part and outer part, where the inner part is of size \( 4L \). We can write a general state as

\[
|\Psi\rangle = \sum_{\alpha=1}^{\chi^{4L}} |\text{in}(\alpha)\rangle |\text{out}(\alpha)\rangle
\]

and therefore the deduced density matrix

\[
\rho_{\text{in}} = \text{tr}_{\mathcal{H}_{\text{out}}}[|\Psi\rangle\langle\Psi|] = \sum_{\alpha,\alpha'} \langle\text{out}(\alpha')|\text{out}(\alpha)\rangle |\text{in}(\alpha)\rangle\langle\text{in}(\alpha')|.
\]

This reduced density matrix has rank bounded by \( \chi^{4L} \) and the reduced density matrix is thus bounded by the rank of \( \rho_{\text{in}} \), say

\[
S(L) = -\text{tr}(\rho_{\text{in}} \log \rho_{\text{in}}) \leq 4L \log \chi.
\]

If \( \chi = 1 \), \( S(L) = 0 \). It’s anticipated since \( |\Psi\rangle \) is just a product state in this case. Changing the value of \( \chi \) modifies this relation in the logarithm and doesn’t change the scaling relation. This implies to change scaling relation one has to change the geometric pattern of the tensor network state.

1.3 How do we implement a tensor network

2 Matrix Product States

In this section we examine carefully the properties of matrix product states and its implications. To demonstrate these, we would like to have a formal expression of a matrix product state. It’s given by

\[
|\Psi\rangle = \sum_{i_1...i_N} \text{tr}[A_{i_1}^{[1]} A_{i_2}^{[2]} ... A_{i_N}^{[N]}]|i_1...i_N\rangle
\]

for a periodic system or

\[
|\Psi\rangle = \sum_{i_1...i_N} \langle l|[A_{i_1}^{[1]} A_{i_2}^{[2]} ... A_{i_N}^{[N]}]|r\rangle |i_1...i_N\rangle
\]

for a system with given boundary conditions \( \langle l \rangle \) and \( |r\rangle \). In the following are few examples.
1. \( \chi = 1 \) describes product states. For example, consider a two-level \((p_i = 2)\) many body system. \( A_1^{[i]} = \alpha \) and \( A_2^{[i]} = \beta \) with \( |\alpha|^2 + |\beta|^2 = 1. \)

\[
|\Psi\rangle = [\alpha |1\rangle + \beta |2\rangle][\alpha |1\rangle + \beta |2\rangle]. \tag{2.3}
\]

2. When \( \chi > 1 \), tensor product states are then usually entangled. An example with \( p_i = 2 \) and \( \chi = 2 \) is given by

\[
A_0^{[i]} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_1^{[i]} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.4}
\]

The state produced by these matrix coefficients is the GHZ (GreenbergerHorneZeilinger state) state

\[
|\text{GHZ}\rangle = |0\rangle^\otimes N + |1\rangle^\otimes N. \tag{2.5}
\]

3. Yet another example is the famous AKLT state which describes the ground state of anti-ferromagnetic spin-1 chain defined by the Hamiltonian

\[
H = \sum_i S_i \cdot S_{i+1} + \frac{1}{3} (S_i \cdot S_{i+1})^2. \tag{2.6}
\]

It can be described by a tensor network state \( p = 3 \) and \( \chi = 2. \)

\[
A_1 = \frac{1}{\sqrt{2}} (\sigma_x + i \sigma_y), \quad A_0 = -\sigma_z, \quad A_{-1} = - \frac{1}{\sqrt{2}} (\sigma_x - i \sigma_y). \tag{2.7}
\]

In practice, we rarely deal with a mere state. Instead, what we often manipulate is some matrix elements. Therefore, it’s convenient to introduce an object called the double tensor

\[
E = \sum_i A_i \otimes A_i^*, \quad (E_{\alpha \gamma, \beta \delta} = \sum_i A_{i, \alpha \beta} (A_{i, \gamma \delta})^*). \tag{2.8}
\]

Given \( A \)s, it’s straightforward to get \( E \)s. What about given \( E \)s? It turns out the result is not unique. Since the double tensor is a direct product of local matrices, if two set of matrices \( B \) and \( A \) are related by an unitary transformation

\[
A_i = \sum_j U_{ij} B_j, \tag{2.9}
\]

\[
E = \sum_{ij} U_{ij} B_j \otimes (U_{ik}^* B_k^*) = \sum_{ijk} U_{ij} U_{ik}^* B_j \otimes B_k = \sum_j B_j \otimes B_j. \tag{2.10}
\]

Reversely, if \( E \) can be decomposed into tensor products of \( A \)s or \( B \)s, then it’s true that \( A \) and \( B \) are related by an unitary transformation.

Even \( A \) cannot be uniquely determined by mere information of \( E \). There is an useful representation. \( E \) can be regarded as a matrix with grouped indices \((\alpha \beta)\) and \((\gamma \delta)\), which would be Hermitian and non-negative. We can diagonalize \( E \) to get eigenvalues \( \lambda_i \) and eigenvectors \( v_i \). Then we can identify

\[
A_{i, \alpha \beta} = \sqrt{\lambda_i} v_{i, \alpha \beta}.
\]

In the following let us examine two properties (i) a MPS has finite correlation length if the double
tensor has non-degenerate eigenspace for the largest eigenvalue and (ii) its entanglement entropy obeys area law.

To see the first fact, let us consider the correlation function of the single particle operator $O$ of a $N$-site system. Suppose site 1 and 2 are separated by length $L$. In terms of double tensors $E_s$,

$$
\langle O_1 O_2 \rangle - \langle O_1 \rangle \langle O_2 \rangle = \frac{\text{tr}[E^{N-L} E_1 E^L E_2]}{\text{tr}[E^N]} - \frac{\text{tr}[E^{N+1} E_1]}{\text{tr}[E^N]} \frac{\text{tr}[E^{N-1} E_2]}{\text{tr}[E^N]},
$$

(2.11)

where

$$
E[O] = \sum_{i,j} O_{ij} A_i \otimes A_j^*.
$$

(2.12)

Without loss of generality, we could set the scale such that the largest eigenvalue of $E$ to be 1 such that the norm of wave function $\text{tr}[E^N]$ is finite. In thermodynamic limit $N \rightarrow \infty$, $E \rightarrow P_1$, where $P_i$ is the projection operator of $i$th eigenspace and the two-point correlation function becomes

$$
\langle O_1 O_2 \rangle - \langle O_1 \rangle \langle O_2 \rangle = \frac{\text{tr}[P_1 E_1 (\sum \lambda P_i)^L E_2]}{\text{tr}[P_1]} - \frac{\text{tr}[P_1 E_1]}{\text{tr}[P_1]} \frac{\text{tr}[P_1 E_2]}{\text{tr}[P_1]}.
$$

(2.13)

From this expression we can see if $P_1$ is one-dimensional (non-degenerate), the two-point correlation function is dominated by $\lambda_2^L = e^{L \log \lambda_2}$ with $\lambda_2 < 1$. Hence, the correlation length is given by

$$
\xi = -\frac{1}{\log \lambda_2}.
$$

(2.14)

It this sense, MPS cannot describe phase transition states.

The information of entanglement entropy can also be extracted using double tensors. As we mention earlier, $E^{[i]}$ could be regarded as a matrix with grouped indices. Let us divided the system into $i = 1...k$ and $i = k+1...N$. Then we can define

$$
E^l = \prod_{i=1}^k E^{[i]}, \quad E^r = \prod_{i=k+1}^N E^{[i]}.
$$

(2.15)

Both $E^l$ and $E^r$ are $\chi^2 \times \chi^2$ Hermitian and non-negative matrices and hence have at most $\chi^2$ non-vanishing eigenvalues. By the approach mentioned above we can therefore find at most $\chi^2$ non-vanishing $A^l_i$ and $A^r_i$ in which physical information is stored. Consequently the physical degrees of freedom on both-hand sides are bounded by $\chi^2$ and

$$
S \leq 2 \log \chi.
$$

(2.16)

It’s the area law in one-dimension.

A remark here is that not every one-dimensional state obeying entanglement area law is a MPS. Fortunately, the inner bond dimension $\chi$ to approximate any such state scales polynomially with system size.

References
