SPACETIME FROM ENTANGLEMENT
- journal club notes -

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1 Outline

1. Introduction

   - Big picture: Want a quantum theory of gravity
   - Best understanding of quantum gravity so far arises through AdS/CFT correspondence
   - Remaining question: How/why does bulk spacetime emerge from CFT on boundary?
   - Suggestion: The connected bulk spacetime is intimately related to the entanglement of degrees of freedom in the CFT on boundary. We will explain what this means and provide evidence by considering two examples.
   - Open Question: Quantum gravity is hard, so it makes sense to focus on gravity in AdS space where there is the exact AdS/CFT correspondence. Nevertheless, we would like to ultimately understand quantum gravity in a spacetime that is asymptotically Minkowski. Understanding how the results presented in this talk generalize to Minkowski spacetime is currently unclear, and an important direction of future work. However, in this talk, we will concentrated on gravity in AdS – so keep in mind that we are not yet describing quantum gravity in our universe.
   - Standard Disclaimer: This is not my area of expertise, so the audience is encouraged to add comments, suggestions or questions to help clarify issues.

2. Brief review of entanglement, ρ_A and S_A with example

   - Consider a system with two spin 1/2 degrees of freedom. A general state in the Hilbert space looks like

   $$a|\uparrow\rangle|\uparrow\rangle + b|\uparrow\rangle|\downarrow\rangle + c|\downarrow\rangle|\uparrow\rangle + d|\downarrow\rangle|\downarrow\rangle \tag{1}$$

   - If the state can be written like

   $$(\alpha|\uparrow\rangle + \beta|\downarrow\rangle)(\gamma|\uparrow\rangle + \delta|\downarrow\rangle) \tag{2}$$

   the we say it is a product state, otherwise it is an entangled states. Most states in the Hilbert space have some level of entanglement.

   - An example of a product state is

   $$|\Psi\rangle_{prod} = \frac{1}{\sqrt{2}}|\downarrow\rangle|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle|\downarrow\rangle = |\downarrow\rangle(\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle) \tag{3}$$

   while an example of an entangled state is

   $$|\Psi\rangle_{entangled} = \frac{1}{\sqrt{2}}|\uparrow\rangle|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle|\downarrow\rangle \tag{4}$$

   Note that entanglement is purely a quantum mechanical effect – it follows from the very quantum mechanical notion that a particle can be in a superposition of states.

- It is desirable to have a more quantitative measure of the entanglement between spin 1 and spin 2. There are numerous quantitative measures, but one useful quantitative measure is the so-called entanglement entropy. Generally speaking, we divide the system into two parts, A and B. In this case

$$\rho_A = \mathrm{Tr}_B \rho \tag{5}$$

The entanglement entropy of system A is then defined to be

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A \tag{6}$$

If λ_i are the eigenvalues of ρ then this can be re-expressed as

$$S_A = -\sum_i \lambda_i \log(\lambda_i) \tag{7}$$

If you consider our two examples from before, we can calculate the reduced density matrix in each case

$$\rho_{A,prod} = |\downarrow\rangle\langle\downarrow| \tag{8}$$

In this case, the original state was a product state so even when you trace out the second spin you are still in a product state. For the entangled case this is not so

$$\rho_{A,entangled} = \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\uparrow\rangle\langle\uparrow| \tag{9}$$

Now if we evaluate the entanglement entropy for the product state we get zero, which makes sense because there was no entanglement in that state. On the other hand, for the entangled state

$$S_A = -\sum_i \lambda_i \log \lambda_i = \log(2) \tag{10}$$

which is actually the maximal amount of entanglement.

3. Example 2: A disentangling experiment

- Consider the example of a single CFT on S^d. The vacuum state will be dual to gravity on pure global AdS spacetime.
- Now consider two hemi-spheres of S^d, and label them A and B. As we have seen, a simple measure of the entanglement between regions A and B is the entanglement entropy

$$S(A) = -\mathrm{Tr}(\rho_A \mathrm{log} \rho_A), \quad \rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) \tag{11}$$

Because this is a local QFT, the Hilbert space can be decomposed as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The Ryu-Takayanagi

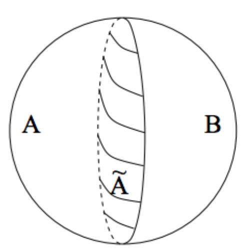

Figure 1:

formula provides a simple way to calculate the entanglement entropy. Let $\tilde{A}$ be the minimal surface whose boundary is equal to the boundary of A and B. Then the entanglement entropy is

$$S_A = \frac{\mathrm{Area}(\tilde{A})}{4G_N} \tag{12}$$

- Suppose the field theory on the boundary is regularized (e.g. on a lattice) so that S_A is finite. Imagine slowly varying away from the ground state to disentangle A and B. Because most of the entanglement comes from the boundary of A and B, this would likely involve applying some unitary operator to the boundary to transform it locally to a product state.
- What happens to the bulk geometry in this process? Well, because the entanglement entropy is proportional to Area($\tilde{A}$), if the two systems A and B are totally disentangled in this process, then the area must go to zero, implying that the dual spacetimes become completely separated.
- Additional evidence that the dual geometry separates when the entanglement goes to zero can be gleaned from considering the mutual information between any two subsystems $C \subset A$ and $D \subset B$. Recall from a previous talk that the mutual information is defined to be

$$I(C, D) = S(C) + S(D) - S(C \cup D) \tag{13}$$

  Also recall that it was shown that the mutual information is non-negative, and is equal to zero only if the entanglement between the two regions goes to zero (i.e. the density matrix of $C \cup D$ is the tensor product of density matrices for C and D.) As the entanglement between A and B goes to zero, the entanglement of any two regions must go to zero also.
- What is the geometric consequence of mutual information going to zero? First recall that for operators $\mathcal{O}_C$ and $\mathcal{O}_D$ acting only on regions C and D, the mutual information provides an upper bound for the following correlator.

$$I(C, D) \geq \frac{(\langle \mathcal{O}_C \mathcal{O}_D \rangle - \langle \mathcal{O}_C \rangle \langle \mathcal{O}_D \rangle)^2}{2|\mathcal{O}_C|^2 |\mathcal{O}|^2} \tag{14}$$

  Furthermore, in the AdS/CFT correspondence, certain two-point correlators of local operators provide a direct measure of the proper distance through the spacetime between the boundary points at which the operators are inserted.

$$\langle \mathcal{O}_C(x_C) \mathcal{O}_D(x_D) \rangle \; e^{-mL} \tag{15}$$

  where m is the mass of the excitations in the dual spacetime description and L is the proper distance of the shortest geodesic between x_C and x_D in the spacetime geometry.
- We conclude from these considerations that as regions A and B are disentangled, the proper distance through the bulk geometry between any two points in regions A and B, diverges.

4. Example 2: Eternal black hole in AdS

- First consider two (non-interacting) copies of a CFT, and consider the state which is a product state of the two copies: $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$. What bulk geometry is this state dual to? Since there are no interactions and no entanglement, it must be dual to two disconnected spacetimes.
- Next, consider the state

$$|\Psi(\beta)\rangle = \sum_i e^{\frac{-\beta E_i}{2}} |E_i, 1\rangle \otimes |E_i, 2\rangle \tag{16}$$

  where $|E_i, j\rangle$ is the i'th energy eigenstate of CFT j.
- Question: What spacetime is this dual to? The literal interpretation is a quantum superposition of disconnected spacetimes. However, it has been argued that it is dual to the connected eternal AdS black hole spacetime as illustrated in figure 2
- Evidence for this claim:
  -Presence of horizons can be associated with lack of interactions between CFT's
  -Two asymptotically AdS spacetimes in balckhole geometry suggest its dual to two copies of CFT
  -Each observer sees a Schwarzchild geometry which corresponds to a thermal state. If we form the density matrix corresponding to the state $|\Psi(\beta)\rangle$ and trace over the degrees of freedom of one of the CFT's we get

$$\text{Tr}_2[|\Psi(\beta\rangle\langle\Psi(\beta)|] = \sum_i e^{-\beta E_i} |E_i, 1\rangle\langle E_i| = \rho_1 \tag{17}$$

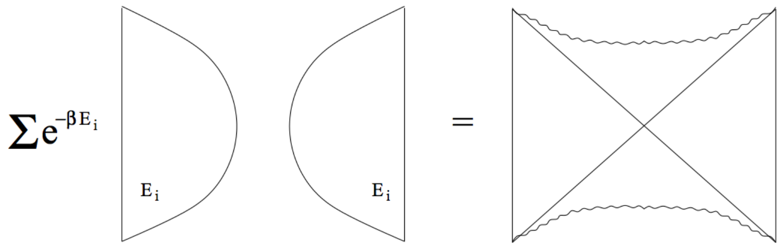

Figure 2:

5. First law of entanglement entropy

   - Having demonstrated how changing the entanglement on the boundary can affect the geometry in the bulk, we would like to take this further and see what we can learn about allowed geometries by using known properties of entanglement entropy.
   - Deriving Einstein's equations from entropy equations – What is most remarkable about this derivation is that we are starting with a purely quantum mechanical statement about a field theory and using that to derive the (linearized) equations for classical gravity. So, surprisingly, the incompatibility of quantum mechanics and gravity has been turned on its head, and known facts about gravity have been derived starting from quantum mechanics.
   - First we derive the first law of entanglement entropy. Recall the definition of entanglement entropy

     $$S_A = -\mathrm{Tr}\rho_A \log \rho_A \tag{18}$$

     Because the reduced density matrix is both Hermitian and positive semi-definite, it can be expressed as

     $$\rho_A = \frac{e^{-H_A}}{\mathrm{Tr}e^{-H_A}} \tag{19}$$

     which defines the Hermitian operator H_A, which is called the modular Hamiltonian (note that we saw an example of this in last week's talk, where H_A was the generator of boosts). The name comes from, I believe, the fact that for a canonical ensemble in thermal equilibrium the density matrix is $\rho = \frac{1}{Z}e^{-\beta H}$.
   - Now consider an infinitesimal variation of $\rho_A \to \rho_A + \delta\rho_A$, the resulting variation of the entanglement entropy is

     $$\begin{aligned} \delta S_A &= -\mathrm{Tr}(\delta\rho_A \log \rho_A) - \mathrm{Tr}(\rho_A \rho_A^{-1} \delta\rho_A) && (20)\\ &= \mathrm{Tr}(\delta\rho_A H_A) - \mathrm{Tr}(\delta\rho_A) && (21)\\ &= \mathrm{Tr}(\delta\rho_A H_A) && (22) \end{aligned}$$

     where in the last step we used the fact that $\mathrm{Tr}\rho_A = 1$, so its trace variation vanishes by definition. Written slightly differently, we have

     $$\delta S_A = \delta\langle H_A \rangle \tag{23}$$

   - In the case of a state in thermal: $\rho = \frac{1}{Z}e^{-\beta H}$, the modular Hamiltonian is equal to βH so this relation implies

     $$\delta S = \frac{1}{T}\delta\langle H \rangle \tag{24}$$

     which is an exact quantum version of the first law of thermodynamics.
   - Derivation of vacuum Einstein's equations

     Recall from last week's talk that we derived the following result for a general QFT

     $$\rho_A = e^{-2\pi H_\eta} \tag{25}$$

where H_η is the generator of boosts and A is the right Rindler wedge. Now, if we suppose this QFT is actually a CFT, then we can use a conformal transformation to map this region onto a ball shaped region, and this allows us to derive for a d-dimensional Minkowski space, and for a ball of radius R centered on a point x_0 for a particular time slice $t = t_0$ the modular Hamiltonian takes the form

$$H_B = 2\pi \int_B \frac{R^2 - |x - x_0|^2}{2R} T_{tt}(t_0, x) \tag{26}$$

where this result was derived in a paper by Casini, Huerta and Myers. The next step is to consider the limit of a very small ball and use this first-law to derive a relationship between $\delta T_{\mu,\nu}$ and δS.

$$\lim_{R\to 0} \delta E_B = 2\pi\delta\langle T_{tt}(x_0)\rangle \int d^{d-1}x \frac{R^2 - x^2}{2R} = \frac{2\pi R^d \Omega_{d-2}}{d^2 - 1} \delta\langle T_{tt}(x_0)\rangle \tag{27}$$

Or, to turn this around, we have

$$\delta\langle T_{tt}(x_0)\rangle = \frac{d^2 - 1}{2\pi\Omega_{d-2}} \lim_{R\to 0} (\frac{1}{R_d} \delta S_B(R, x_0)) \tag{28}$$

The next step is to use Ryu-Takayanagi to equate $\delta S_B = \frac{\delta \text{Area}(\tilde{B})}{4G_N}$. We will parametrize perturbations to the metric as

$$ds^2 = \frac{l^2}{z^2}(dz^2 + g_{\mu\nu} dx^\mu dx^\nu) \tag{29}$$

with

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{30}$$

Thus,

$$\delta S_B = \frac{\delta \text{Area}(\tilde{B})}{4G_N} = \frac{Rl^{d-3}}{8G_N} \int d^{d-1}x z^{2-d} (\delta^{ij} - \frac{1}{R^2}(x^i - x_0^i)(x^j - x_0^j)) h_{ij}(z, t_0, x) \tag{31}$$

Performing the integral we arrive at, in the limit $R \to 0$

$$\delta\langle T_{tt}(x_0)\rangle = \frac{dl^{d-3}}{16\pi G_N} h^i{}_i \tag{32}$$

This result can be generalized to an arbitrary Lorentz frame, and if we use the tracelessness and conservation of the stress-energy tensor we can arrive at a simple form

$$\delta\langle T_{\mu\nu}(x_0)\rangle = \frac{dl^{d-3}}{16\pi G_N} h_{\mu\nu} \tag{33}$$

Ok, this is great because it provides a relationship between perturbations of the stress-energy tensor in the CFT to perturbations of the metric. This is the usual result for the linearized holographics stress tensor in Einstein gravity in AdS_{d+1}.

- The next step is to use this relation for general balls. So, generally, we have

$$\delta S = \frac{\delta \text{Area}(\tilde{B})}{4G_N} = \delta E_B = 2\pi \int_B \frac{R^2 - |x - x_0|^2}{2R} \delta\langle T_{tt}(t_0, x)\rangle \tag{34}$$

$$\frac{Rl^{d-3}}{8G_N} \int_{\tilde{B}} d^{d-1}x z^{2-d} (\delta^{ij} - \frac{1}{R^2}(x^i - x_0^i)(x^j - x_0^j)) h_{ij}(z, t_0, x) = 2\pi \int_B \frac{R^2 - |x - x_0|^2}{2R} \frac{dl^{d-3}}{16\pi G_N} h^i{}_i \tag{35}$$

This is the $\delta S = \delta E$ constraint translated into gravitational language. It is a non-local constraint between the metric perturbation on the minimal bounding surface and the metric perturbation on the boundary of AdS. We have one such constraint for each choice of ball and each frame of reference.

- The final step is to convert this non-local constraint to a local constraint that implies Einstein's equations. To do this we make use of the Iyer-Wald formalism. Suppose there exists a d-1 form χ such that

$$\int_B \chi = \delta E_B, \quad \int_{\tilde{B}} \delta S_B \tag{36}$$

and

$$d\chi = -2\xi_B^a \delta E^g_{ab} \epsilon^b \tag{37}$$

where ξ_B is a particular Killing vector which vanishes on $\tilde{B}$, ϵ^b is the volume form and δE^g_{ab} are the linearized gravitational equations of motion. Then

$$0 = \delta S_B - \delta E_B = \int_{\tilde{B}} \chi - \int_B \chi = \int_{\partial\Sigma} \chi = \int_\Sigma d\chi = -2 \int_\Sigma \xi_B^t \delta E^g_{tt} \epsilon^t \tag{38}$$

From this one can show that $\delta E^g_{tt} = 0$ everywhere. Furthermore, imposing these constraints for a frame of reference for an observer with an arbitrary 4-velocity, we can prove the stronger result

$$\delta E^g_{\mu\nu} = 0 \tag{39}$$

Using the initial value formulation, one can also show that $\delta E^g_{z\mu} = 0$ and $\delta E^g_{zz} = 0$.

References

[1] Van Raamsdonk, Mark. "BUILDING UP SPACE?TIME WITH QUANTUM ENTANGLEMENT." International Journal of Modern Physics D 19.14 (2010): 2429-2435.

[2] Maldacena, Juan. "Eternal black holes in anti-de Sitter." Journal of High Energy Physics 2003.04 (2003): 021.