Conformal Bootstrap

CFT's in d Eucl. dims

bootstrap: study implications of sym/consistency conditions on corr. fns. of local ops
\[ \langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle \]

We'll study four-pt. fns. of scalars \( \rightarrow \) four pt. fns. of other SO(d) reps are being worked out/open.

- Conformal inv. implies:
\[ \langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle = \frac{x_{12}^{\Delta_{12}/2} x_{34}^{\Delta_{34}/2}}{x_{14}^{\Delta_{14}/2}} G(u,v) \]

\[ G(u,v) : \text{arbitrary fn. of the conf. inv. cross-sections} \]
\[ u = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{34}^2} \quad v = \frac{x_{14}^2 x_{34}^2}{x_{14}^2 x_{24}^2} \]

- OPE's relate n pt. fns. to n-1 pt. fns. \( \rightarrow \) Note that 3-pt. fns. are completely fixed in a CFT by conformal inv. (for scalars up to a const.)

\[ \phi_1(x_1) \phi_2(x_2) = \sum_{O: \text{primary}} \lambda_{12O} C(x_{12}, \partial_{x_1})^{\ell_1 \cdots \ell_2} Q_{l_1 \cdots l_2}(x_2) \quad \text{where } O's \text{ are primary tensors} \]

\[ \text{Conf wt: } \Delta \quad \text{Spin: } l \quad \rightarrow \text{sym. traceless} \]

\[ \text{can make that replacement in correlation fns. as long as} \]
\[ x_{12}, x_{34} \]

\[ \text{other field insertions} \]

\[ \langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle = \sum_0 \lambda_{12O} \lambda_{34O} W_0(x_1 \cdots x_4) \]

\[ \text{where} \quad W_0(x_1 \cdots x_4) = C(x_{12}, \partial_{x_1})^{\ell_1 \cdots \ell_2} C(x_{34}, \partial_{x_3})^{\ell_3 \cdots \ell_4} \langle \phi_{l_1} \cdots \phi_{l_4}(x_2) \phi_{l_3} \cdots \phi_{l_4}(x_4) \rangle \]

\[ = \frac{x_{12}^{\Delta_{12}/2} x_{34}^{\Delta_{34}/2}}{x_{14}^{\Delta_{14}/2}} G_0(u,v) \]

\[ G_0(u,v) : \text{conformal block} \]
1. **Direct method (hard)** Find $C(x_1, x_2)_{l_1, l_2}$'s first:

For $\text{Conf} \cdot \text{inv.} \cdot \text{fixes}$
\[
\langle \phi_1 \phi_2 \sigma^{(1)} \rangle \quad \text{require consistency with} \quad \phi_1 \phi_2 \sim \Sigma \quad \text{OPE}
\]
and $\langle \sigma^{(1)} \sigma^{(1)} \rangle$

\[\text{(See ref. 11/00/040 for } l=1,2 \text{ cases)}\]

Then compute $W_0$'s using $C(...,C(..., <\sigma^{(1)}) ...$; very complicated even for $l=1$.

2. **Casimir diff. eqn:**

\[\frac{1}{2} M_{AB} M^{AB} : \text{a Casimir for } SO(d+1,1) \]

separates the contr. of $\sigma$ and its descendants.

\[\langle \sigma^{(1)} \rangle \to \langle \text{D}_{\sigma_0}(u, v) \rangle \to \frac{1}{2} C_{d+1} \sigma_0(u, v) \]

\[\text{use OPE and that } \sigma \text{ is an eigenstate} \to \frac{1}{2} C_{d+1} \sigma_0(u, v) = \Delta^p \text{ (for } d=1,2 \text{, etc.)} \]

**Conformal blocks satisfy a second order PDE**

\[\text{PDE: } \text{D}_{\sigma_0}(u, v) = \frac{1}{2} C_{d+1} \sigma_0(u, v) \]

\[\text{can be solved in terms of hypergeom. fns. for } d \text{ even}, \quad u = \frac{e}{2}, \quad v = (1-t)(1-\frac{e}{2}) \rightarrow \text{complex coord. in } 2D\]

\[= \text{series solns - recursion relations in general } d \]

\[L_{\sigma_0} \text{ fixed pt. for } u = v \rightarrow L \text{ pick soln. with correct boundary beh.}\]

3. **Shadow formalism** (idea dates back to '70s, see 1204.3849)

lifting from $d$ dim $\rightarrow (d+1,1)$ dim, where conformal inv. $SO(d+1,1)$ is geometric.

'almost' possible to just write down a FNC.

\[\int d^d x \langle \phi_1 \phi_2 \sigma(x) \rangle \langle \hat{\sigma}(\hat{x}) \phi_3 \phi_4 \rangle \quad \text{shadow proj.} \]

\[L \rightarrow \text{shadow op } \hat{\sigma}(\hat{x}) = \int d^d y \frac{1}{(-2x.y)^{d-2}} \sigma(\hat{y}) \rightarrow \text{formally a primary with } \sigma \text{ conf. wt } d-1 \Delta \]

**Note:** It's non-local.

In practice (e.g., used in 1406.4858 studying 3D Ising model):

\[\rightarrow \text{first introduce 'radial' coords: } r e^{i \theta} = \frac{x}{1+1/2} \quad \cos \theta = t \quad \gamma = \frac{d-2}{2} \]

\[\sigma(x, y) \rightarrow \frac{1}{2} z \text{ in } x+y \]

In these coords, the Casimir eqn. has 'spherical sym,' so its solns are

\[\mathcal{G}_{\Delta, \gamma}(r, \gamma) = \sum_{n=0}^{\infty} \sum_{j=0}^{\Delta-\gamma} B_n j \frac{j!}{(2j)!} C_j \left( \frac{\gamma}{2} \right) \]

\[\rightarrow \text{ dependent parity.} \]

**Note:** $B_n$ are rational fns. of $\Delta$. 

**In 3D:**

\[\mathcal{P}_{\ell} \left( \cos \theta \right) \quad \text{in 3D} \]
What do $\Delta$ poles mean?

A pole at $\Delta = \Delta_0$ $\leftrightarrow$ $|\omega> = P^{\Delta_0} |0> \quad \text{in the conf.}$ (remember that using conformal block expansion)

mult. of $|0>$ becomes null (is essentially using a completeness reln.)

$\Rightarrow$ every state in the subrep. generated by $|\omega>$ is also null.

So expect $g_{\Delta, \Delta_0} \sim \frac{(\text{const.})_n \Delta - \Delta_0}{\Delta - \Delta'} g_{\Delta_0, \Delta'}$ as $\Delta \to \Delta_0$

This is the main idea of Zamolodchikov recursion relations.

$\Rightarrow$ in practice these numbers can be pulled from Casimir eqn. soln. (ref. 1.1.7)

$\supset \text{not done}$

Can be helpful in generalizing to higher spin.

($\text{C;n};, \Delta; A, \lambda$'s in theory should be computable from conformal rep. theory)

Imposing Crossing Sym.: ___

There is a region in which both 1-2 and 2-3 OPE's are convergent.

Both should give the same result

Consider $\langle \phi \phi \phi \phi \rangle$ correlator:

$\sum_{\phi \phi \phi \phi} \lambda_{\phi \phi \phi \phi}^2 \text{g}_{\Delta, \phi}(u,v) - (u\leftrightarrow v) = 0$

Introduce $F_{\Delta, \phi}(u, v) = \lambda_{\phi \phi \phi \phi}^2 \text{g}_{\Delta, \phi}(u, v) - (u\leftrightarrow v)$

$\sum_{\phi \phi \phi \phi} \lambda_{\phi \phi \phi \phi}^2 F_{\Delta, \phi}(u, v) = 0$

$\Rightarrow \lambda$'s are real in a unitary theory

$\lambda^2 \geq 0$.

Suppose we found a linear functional $\alpha$

such that $\alpha(F_{\Delta, \phi}) = 1$

for $\phi \in$ in the spectrum (satisfying unitarity, we can also add assumptions about gaps)

$\Rightarrow$ This eqn. can not be satisfied $\Rightarrow$ this spectrum is ruled out just by $\langle \phi \phi \phi \phi \rangle$ correlator

In practice two problems:

(1) The space of linear functionals $\alpha$ is infinite dim. (infinitely many $u,v$ in the common conv. region)

(2) The number of constraints $\alpha(F_{\Delta, \phi})$ is infinite.

(1) is easy to solve: take a finite dim. subspace of $\alpha$'s

$L$ still can rigorously exclude spectra, may not give the most stringent constraints though.

like checking the first few terms in Taylor expansion of two func. $f_1$ and $f_2$ to see if they are equal.
In practice, take linear combinations of derivatives around the sym. point \( \varepsilon = \delta = \frac{1}{2} \) \( (r=r_\infty = 3.25 \beta = 0.17, \eta = 1) \)

\[
\alpha : F(r, \eta) \longrightarrow \sum_{m \in \Delta} a_{mn} \frac{\partial^m}{\partial z^m} F(r, \eta) \text{ polynomial in } \Delta \text{ numbers } \rightarrow \text{ there is an } r_\infty; \text{ suppressing factor here}
\]

Note that \( \partial^m z^m g_{\Delta, \eta} (r, \eta) = r_\infty \left( q_{\Delta, \eta} (\Delta) + \sum \frac{q_{\Delta, \eta}}{\Delta - \Delta_i} \right) \)

(2) is more problematic, firstly we don't know \((\Delta, \eta)\) values in the spectrum.

- Even if we did we can not just truncate to a finite number of them.
- A linear fit or satisfying the conditions for our choice of \((\Delta, \eta)\) may be,
- Failing to satisfy the positivity conditions we ignored leading us to wrongly
- Eliminate a spectrum.

We need to make approximations in a controllable fashion:

- Increasing \( \Delta \) (or \( l \) by unitarity based) suppresses \( z^m \partial^m g_{\Delta, \eta} \) by powers of \( r_\infty \to 0.17 \),

  \( \Rightarrow \text{ restrict } l \text{ to } 0, 1, \ldots, L \text{ for some large } L \)

  \( \Rightarrow \text{ conf wt. } \Delta \text{ can vary continuously } \rightarrow \text{ argue large } \Delta \text{ suppression, discretize } \Delta \text{ to approximate } (\text{early approaches used this}) \)

\( \downarrow \)

A second way: (also useful in generalizing to mixed correlators)

Because of the \( r_\infty \) suppression we can truncate the sum over \( \sum_{\Delta \in \Delta} \) and get an approximation

\[
\partial^m z^m g_{\Delta, \eta} \approx r_\infty \frac{\partial^m}{\partial z^m} \sum_{\Delta} \frac{P^{mn}(\Delta)}{\pi' (\Delta - \Delta_i)} \text{ polynomial}
\]

\( \Delta \geq \text{ positive (a's are below unitarity bound) } \rightarrow \text{ can factor out in the constraints} \)

Defining \( x = \Delta - \Delta_{\text{min}} \rightarrow \alpha F_{\Delta, \eta} > 0 \rightarrow \sum_{m n} a_{mn} \tilde{P}^{mn}(x) > 0 \text{ for all } x > 0 \text{ } \alpha (F_{\Delta, \eta}) > 0 \)

\( \Rightarrow \text{ This is a 'polynomial matrix program' problem} \)

\( \Rightarrow \text{ Optimization with polynomial constraints} \)

We can reduce it to a SDP (semidefinite programing) problem (with ordinary inequalities)

\( \text{Essentially by the following result:} \)

Hilbert: A polynomial \( p(x) \) is nonneg. on \( [0, \infty) \) iff \( p(x) = f(x) + x g(x) \) where \( f, g \) are

\( \text{sums of squares of polynomials} \)

\( \text{positive semidef} \) so \( \sum_{m n} \tilde{P}^{mn}(x) > 0 \text{ for all } x > 0 \)

\( \Rightarrow p(x) = \mathbb{R}^T A \mathbb{R} + x (\mathbb{R}^T B \mathbb{R}) \text{ for } A, B \succeq 0 \)

\( \Rightarrow \mathbb{R} = (\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_d}) \)
Applications and Prospects

- 3D Ising Model (assume $\mathbb{Z}_2$ sym. and that there are only two relevant ops: $\mathcal{O}$, $\mathcal{E}$ scalars)

- Conformal blocks for higher spin external states ~3 analogs of $P_{\mathcal{O}}^m(n)$ ?

- Room for improvement in numerical algorithms (they are lacking precision)

- 2D CFTs with $c>1$, susy, CY3 applications? Using Virasoro blocks...

- 5D QCD in conformal window

- 6D (2,0) SCFT ~ same work being done