N = 2 SCFT and Elliptic Genus

- N=2 SCFT and spectral flow
- Elliptic genus: Def and calculation for K3 surface
- relation with Jacobi form
- Appendix: BPS representation

N = 2 superconformal extension of free boson theory could be started by a complex free boson $\phi(z, \bar{z}) = \frac{i}{\sqrt{2}} \left( X^0(z, \bar{z}) + i X^3(z, \bar{z}) \right)$ and a complex free fermion $\psi(z, \bar{z}) = \frac{i}{\sqrt{2}} \left( \psi^0(z, \bar{z}) + i \psi^3(z, \bar{z}) \right)$

Similar to $N=1$ case, let focus on the holomorphic part

\[ \partial \bar{\phi}(z) = \frac{i}{2} \left( \partial X^0(z) + i \partial X^3(z) \right) \quad \psi(z) = \frac{i}{\sqrt{2}} \left( \psi^0(z) + i \psi^3(z) \right) \]

The total central charge $c = 2(\frac{1}{2} + 1) = 3$

There is a field $h = 1$, which plays the role of current

\[ j(z) = -i \bar{\phi}(z) \]

$\bar{\phi}$ and $\psi$ are complex conjugate to $\phi$ and $\psi$.

We can construct two world sheet supercurrents (correspond to 2 supercharges)

\[ G_+^+(z) = i \sqrt{2} \bar{\phi} \psi(z) \quad G^-_-(z) = i \sqrt{2} \bar{\phi} \psi(z) \]

Express the $N=2$ SCFT in terms of Laurent modes:

\[ [L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m,n,0} \]

\[ [L_m, J_n] = -n J_{m+n} \]

\[ [L_m, G^+_r] = (\frac{m}{2} - r) G^+_{m+r} \]

\[ [J_m, J_n] = \frac{c}{3} m \delta_{m+n,0} \]

\[ [J_m, G^+_r] = \pm G^+_{m+r} \]

\[ [G^+_r, G^-_s] = 2 L_{r+s} + (r-s) J_{m+s} + \frac{c}{3} (r^3 - s^3) \delta_{m+s,0} \]

\[ \{ G^+_r, G^+_s \} = \{ G^-_r, G^-_s \} = 0 \]

We observe the Cartan subalgebra of $N=2$ super Virasoro algebra is generated by $L_0$ and $J_0$. States in Hilbert space are labeled by $h$ and $\bar{h}$.

A notable feature of $N=2$ super Virasoro algebra is spectral flow, which is a
Continuous class of automorphism.
\[
\begin{align*}
L_n &\rightarrow L'_n = L_n + \eta J_n + \frac{\eta}{2} \bar{\eta} D_n, \\
J_n &\rightarrow J'_n = J_n + \frac{\eta}{2} \bar{\eta} D_n, \\
G_t &\rightarrow G^t = G^t \eta
\end{align*}
\Rightarrow \begin{cases}
\lambda = k - \frac{\eta}{2} + \frac{\bar{\eta} k}{2} \\
8\eta = 8 - \frac{\eta}{2} \bar{\eta}
\end{cases}
\]

The spectral flow interpolates between NS and R sectors. For chiral primary fields,
\[
\begin{align*}
\eta = \frac{9}{2} \Rightarrow & \quad \text{NS, } \eta = \frac{1}{2} \quad \Rightarrow \quad h_{\frac{1}{2}} = \frac{e}{24}, \quad g_{-\frac{1}{2}} = 8 + \frac{e}{6} \\
\eta = -\frac{9}{2} \Rightarrow & \quad \text{NS, } \eta = -\frac{1}{2} \quad \Rightarrow \quad h_{-\frac{1}{2}} = \frac{e}{24}, \quad g_{\frac{1}{2}} = -8 + \frac{e}{6}
\end{align*}
\]

The only invariant operator w.r.t spectral flow is \(\frac{3}{2} \eta C_0 = J_0^2\).

\(N=2\) SCA has a simple extension to \(N=4\) SCA, or equivalently \(N=4\) SCFT has an \(N=2\) subalgebra, with \(J_0^3\) reducing to \(J_0 \in N=2\) SCA.

Def: elliptic genus \(Z_{ek}(z; \bar{z}) = \text{Tr}_{R^8} (-1)^F q^{L_0 - \frac{c}{24}} e^{-2\pi i z J_0^3}\).

The right moving part gives a constant in the presence of BPS states. Our aim reduce to \(\text{Tr}_{W} (-1)^F q^{L_0 - \frac{c}{24}} e^{-2\pi i \frac{1}{2}}\).

The method monopole story begins with the elliptic genus for \(K3\).

\(Z_{ek}(K3)(z; \bar{z})\) is given in EOT’s paper
\[
Z_{ek}(K3)(z; \bar{z}) = 8 \left[ \left( \frac{\theta_1(z; \bar{z})}{\theta_3(z; \bar{z})} \right)^2 + \left( \frac{\theta_3(z; \bar{z})}{\theta_5(z; \bar{z})} \right)^2 + \left( \frac{\theta_5(z; \bar{z})}{\theta_1(z; \bar{z})} \right)^2 \right]
\]

Since the elliptic genus is independent on the modulus of \(K3\), the calculation could be performed in one of the orbifold limit, such as \(T^4/Z_2\). Similar to the partition function of orbifold
\[
Z_{orb} = \frac{1}{16} \sum_{g \text{ even}} g \frac{9 g}{4}
\]

\[
\text{Tr}_{W} (-1)^F q^{L_0 - \frac{c}{24}} e^{-2\pi i \frac{1}{2}} = \text{Tr}_{W} \left[ \frac{1+g}{2} (-1)^F q^{L_0 - \frac{c}{24}} + \frac{1+g}{2} (-1)^F q^{L_0 - \frac{c}{24}} \right]
\]

Here \(g\) acts on \(X\) to be \(X \rightarrow -X\).
For the space-time manifold $T^4/\mathbb{Z}_2$, we have 2 complex bosons and 2 complex fermions.

In (10), $\text{Tr}_{\text{even}} \frac{1}{2} (-1)^F \phi_{\alpha}^\alpha \chi^{1/4} \chi^3$ is simply the elliptic genus for $T^4$, and vanishes due to apparent supersymmetry. While the last 3 pieces gives $2 \left( \Theta_1(z;i) \right)^2 + \left( \Theta_2(z;i) \right)^2 - \left( \Theta_3(z;i) \right)^2 = \left( \Theta_4(z;i) \right)^2$

Combining $\text{Tr} (1) \frac{\omega}{\beta} = 4$, we recover EOT's result.

Geometrically, elliptic genus is a weak Jacobi form of weight 0 and index $-\frac{d}{2}$, $d = \dim_{\mathbb{R}} M$ for Calabi-Yau manifold $M$.

If the function $\Phi(z;i) : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ transform under $\text{SL}(2,\mathbb{Z}) \times \mathbb{Z}^2$:

$$\Phi \left( \frac{a \tau + b}{c \tau + d}, \frac{z + \lambda}{c \tau + d} \right) = \exp \left( 2 \pi i t \frac{c z^2}{c \tau + d} \right) \Phi(z;i)$$

$$\Phi \left( \tau, \frac{z + \lambda \tau + \mu}{c \tau + d} \right) = \exp \left( - \pi i t \frac{c (\mu^2 + \lambda z)}{c \tau + d} \right) \Phi(z;i) \quad \lambda, \mu \in \mathbb{Z}$$

and have furthermore the expansion $\Phi(z;i) = \sum_{n \in \mathbb{Z}} f(n) \exp(\pi i n z^2)$.

The spectral flow symmetry implies the Fourier expansion of elliptic genus takes form:

$$Z(z;i) = \sum_{n \in \mathbb{Z}} a_n \exp(\pi i n z^2)$$

$$\frac{1}{2} \cdot 3d = 4nt \to t = \frac{1}{2^n}$$

Statement (8) verified!

With property of Jacobi form, we may guess the expression $\Phi_{24} = \left( \frac{\Theta_2(z;i)}{\Theta_1(z;i)} \right)^2$

$$Z(T^4)(z;i) = \gamma(T^4) = 0 \quad \Rightarrow \quad Z(T^4)(z;i) = 0$$

$$Z(k^3)(z;i) = \gamma(k^3) = 24 \quad \Rightarrow \quad Z(k^3)(z;i) = 2 \Phi_{24}(z;i)$$

$$h_{1,0}^1 \begin{pmatrix} h_{1,2}^1 & h_{1,1}^1 & h_{1,0}^1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

$$h_{1,0}^1 \begin{pmatrix} h_{1,1}^1 & h_{1,2}^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$h_{1,0}^1 \begin{pmatrix} h_{1,1}^1 & h_{1,0}^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$h_{1,0}^1 \begin{pmatrix} h_{1,1}^1 & h_{1,0}^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$
More generally, \[ Z(M)_{(r;i;z)} = \int_M \text{ch}(E_{g,y}) \text{Td}(M) \]

Euler characteristic \( \chi(M) = Z_{(r;0)} \)

Hermitian signature \( \sigma(M) = Z_{(r; \frac{1}{2})} \)

\( \hat{A} \)-rain genus \( \hat{A} = \frac{q}{h} Z_{\left( r; \frac{1+i}{2} \right)} \)

\[ \chi = \frac{1}{2 \pi} \sum \lambda_i \eta_{\lambda} = \sum (-1)^{2i+6} h_{\lambda, \lambda} \]

Argument: In elliptic genus, the right moving sector is frozen to supersymmetric ground states (bps) while in left moving sector, all states in Hilbert space contribute.

In Kawai et al., \( Z_{(r;i;z)} \) was recovered from \( \chi, \sigma, \hat{A} \), in which \( r \) took special values, \( \frac{1}{24} \left( \frac{1}{\eta} \right)^2 \left( \frac{64 g^2}{\eta^8} \right) - 2 \frac{g}{\eta^4} \left( \frac{g}{\eta} \right)^2 \).

(15) is identical to (5) by a relation among \( \frac{1}{24} \left( \frac{1}{\eta} \right)^2 \left( \frac{64 g^2}{\eta^8} \right) - 2 \frac{g}{\eta^4} \left( \frac{g}{\eta} \right)^2 \), \( \chi = 14 \).

Now let's turn to the start of Mathieu Moonshine - EOT expand \( Z_{(r;i;z)} \)

\[ Z_{(r;i;z)} = 24 \text{ch}^R_{h=4, l=0} (r;i;z) + \sum_{l} \left( \frac{\theta_{(r;i;z)}^2}{h - \eta^4} \right) \]

character formula for BPS rep \( \Rightarrow \) non-BPS rep

Non-BPS rep splits into a sum of BPS reps at unitary bound \( h = 4 \)

\[ \frac{1}{\eta^3} \theta_{(r;i;z)} = 2 \text{ch}^R_{h=4, l=0} (r;i;z) + \text{ch}^R_{h=4, l=0} (r;i;z) \]

Expansion function \( \Sigma(\eta) = -2 \frac{\theta_2^2}{\eta^3} \left( 1 - \sum_{n=1}^{\infty} A_n \frac{\eta^n}{\theta_2} \right) \)

\[ Z_{(r;i;z)} = \text{ch}^R_{h=4, l=0} (r;i;z) - 2 \text{ch}^R_{h=4, l=0} (r;i;z) + \sum_{n=1}^{\infty} A_n \frac{\eta^n}{\theta_2} \frac{\theta_{(r;i;z)}^2}{\eta^4} \]

(17)
\[ \begin{array}{cccccccc}
\eta & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
A_\eta & 45 & 231 & 770 & 2277 & 5716 & 13915 & 30543 & 65550 \\
\end{array} \]

As are equal to dims of irrep of \( \text{M} \).

and \( A_\eta = 3520 + 10395 \) sum of dims

\[ A_7 = 10395 + 5796 + 5444 + 5313 + 2024 + 1771 \]

Similar for \( A_\eta \) (\( \eta > 8 \))

This observation reminds us of Monstrous Moonshine,

\[ J(q) = \frac{1}{q} + 744q + 21493760q^2 \]

The coefficients could be decomposed into sum of dim of irrep of Monster group

Mock modular forms

One class of important functions after Eq. (9)

\[ \hat{\mu}_{\tau}(\tau; z) = \frac{8}{\eta^2} \eta(\tau; z), \quad \mu_{\tau}(\tau; z) = -\frac{i e^{\pi i z}}{\eta(\tau; z)} \sum_{n=0}^{\infty} -\frac{q^{2(n+1)}}{1 - q^n e^{2\pi i z}} \]

\[ \sum_{\tau} = -\frac{1}{2} \left[ \mu_{\tau}(z = \frac{1}{2}) + \mu_{\tau}(z = \frac{1+i}{2}) + \mu_{\tau}(z = \frac{1+i}{2}) \right] \]

\( \mu_{\tau}(z) \) has the form of Lerch sum, and in fact it is a mock modular form.

The modularity of \( \mu_{\tau}(z) \) is not good enough however

\[ \hat{\mu}_{\tau}(\tau; z) \equiv \mu_{\tau}(z; \tau) - \frac{i}{2} R(0; \tau) \]

with

\[ R(0; \tau) = \sum_{n \in \mathbb{Z}} \left( 1 - e^{-\pi i (n+1)^2} \right)^{n+1} \]

\[ E(z) \equiv \int_{-z}^{z} e^{\pi u} du \text{ error function} \]

Now the transformation formulae are

\[ \hat{\mu}_{\tau}(z; \tau) = -\sqrt{i} \hat{\mu}_{\tau}(\frac{1}{\tau}; \frac{z}{\tau}) \]

\[ \hat{\mu}_{\tau}(\tau; z) = e^{\pi i} \hat{\mu}_{\tau}(\tau; z) \]

\[ \hat{\mu}_{\tau}(\tau; z+1) = \hat{\mu}_{\tau}(\tau; z+1) = \hat{\mu}_{\tau}(\tau; z) \]

\( R(0; \tau) \) is non-holomorphic and could be expressed as the integral

\[ i R(0; \tau) = \int_{-i}^{i} \left[ \frac{1}{1 - i(\tau z)} \frac{1}{(\eta(z))^2} \right] dz \text{ identified as shadow} \]

More generally

\[ \hat{h}(\tau) = h(\tau) + \left( \frac{1}{2} \right)^{1/2} \int_{-i}^{i} (\tau z) e^{\pi i (z-\tau)} \frac{1}{(\eta(z))^2} dz \text{ shadow} \]
BPS rep.

BPS rep. describe massless states in compactified string theory.

Non-BPS states have continuous values of $h$ and correspond to massive excitations.

$N=4$ SCFT contains $T(z)$, 4 supercurrents, a triplet of affine currents in $SU(2)$.

$$\text{ch}_{K\chi, \lambda}(z; \bar{z}) = \text{Tr} (\partial \bar{z} J^3 \partial^2 \bar{z}^{3 \chi - \frac{\lambda}{24}})$$

Characters in NS and R sectors are related by spectral flow.

Def.: In NS sector of Hilbert space of $N=2$ SCFT,

$$G^+_{\frac{1}{2}} |h, \bar{h}> = 0 \quad \text{(chiral)} \quad G^-_{\frac{1}{2}} |h, \bar{h}> = 0 \quad \text{(anti-chiral)}$$

$N=2$ super primary states are defined by $G^+_{h+\frac{1}{2}} |h, \bar{h}> = 0 = G^-_{h-\frac{1}{2}} |h, \bar{h}>$

Proposition: A state $|h, \bar{h}>$ is chiral primary iff $h = \frac{\bar{h}}{2}$.

Usually, for $N=2$ supersymmetry, there are 4 components arranged in a superfield,

$$| \phi>, \partial \phi, \partial^2 \phi, \partial^3 \phi >$$

However, in the case of chiral field, $G^+_{\frac{1}{2}} |h, \bar{h}> = 0$, so $N=2$ super primary consists only of 2 components $| h, \bar{h}>$ and $G^-_{\frac{1}{2}} | h, \bar{h}>$.

In general, if a super algebra allows for non-trivial central charges, there exists so-called BPS multiplet which are shorter than the average length of a supermultiplet.

long rep $\Rightarrow$ short rep