# N = 2 SCFT And Elliptic Genus / Frank Lin

- N=2 SCFT and spectral flow
- Elliptic genus Def and calculation for K3 surface
- relation with Jacobi form
- Appendix : BPS representation

N=2 superconformal extension of free boson theory could be started by a complex free boson $\Phi(z,\bar{z}) = \frac{1}{\sqrt{2}}\left(X^{(1)}(z,\bar{z}) + iX^{(2)}(z,\bar{z})\right)$ and a complex free fermion $\Psi(z,\bar{z}) = \frac{1}{\sqrt{2}}\left(\psi^{(1)}(z,\bar{z}) + i\psi^{(2)}(z,\bar{z})\right)$ ①

Similar to N=1 case, Let focus on the holomorphic part

$$i\partial\Phi(z) = \frac{i}{\sqrt{2}}\left(\partial X^{(1)}(z) + i\partial X^{(2)}(z)\right) \qquad \psi(z) = \frac{1}{\sqrt{2}}\left(\psi^{1}(z) + i\psi^{2}(z)\right) \quad ②$$

The total central charge $c = 2\left(\frac{1}{2}+1\right) = 3$

There is a field of $h=1$, which plays the role of current

$$j(z) = -:\psi\bar{\psi}:(z)$$

$\bar{\Phi}$ and $\bar{\psi}$ are complex conjugate to $\Phi$ and $\psi$

We can construct 2 world sheet supercurrents (correspond to 2 supercharges)

$$G^{+}(z) = i\sqrt{2}\,\partial\bar{\Phi}\,\psi(z) \qquad G^{-}(z) = i\sqrt{2}\,\partial\Phi\,\bar{\psi}(z) \quad ③$$

Express the N=2 SCA in terms of Laurent modes :

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}$$

$$[L_m, j_n] = -n j_{m+n}$$

$$[L_m, G_r^{\pm}] = \left(\frac{m}{2}-r\right)G^{\pm}_{m+r} \quad ④$$

$$[j_m, j_n] = \frac{c}{3}m\,\delta_{m+n,0}$$

$$[j_m, G_r^{\pm}] = \pm G^{\pm}_{m+r}$$

$$\{G_r^{+}, G_s^{-}\} = 2L_{r+s} + (r-s)j_{r+s} + \frac{c}{3}\left(r^2-\frac{1}{4}\right)\delta_{r+s,0}$$

$$\{G_r^{+}, G_s^{+}\} = \{G_r^{-}, G_s^{-}\} = 0$$

We observe the Cartan subalgebra of N=2 super Virasoro algebra is generated by $L_0$ and $j_0$. States in Hilbert space are labeled by $h$ and $q$.

A notable feature of N=2 super Virasoro algebra is spectral flow, which is a continuous class of automorphism.

$$\begin{cases} L_n \longrightarrow L_n' = L_n + \eta j_n + \frac{\eta^2}{6} c\, \delta_{n,0} \\ j_n \longrightarrow j_n' = j_n + \frac{c}{3}\eta\, \delta_{n,0} \\ G_r^{\pm} \longrightarrow G_r^{\pm\,\prime} = G_{r\pm\eta}^{\pm} \end{cases} \Rightarrow \begin{cases} h_\eta = h - \eta q + \frac{c}{6}\eta^2 \\ q_\eta = q - \frac{c}{3}\eta \end{cases} \qquad (5)$$

The spectral flow interpolate between NS and R sectors. For chiral primary fields,

$$\left| h_0 = \tfrac{q_0}{2}, q_0 \right\rangle_{NS} \xrightarrow{\eta = \frac{1}{2}} \left| h_{\frac{1}{2}} = \tfrac{c}{24}, q_{\frac{1}{2}} = q_0 - \tfrac{c}{6} \right\rangle_R$$

$$\left| h_0 = -\tfrac{q_0}{2}, -q_0 \right\rangle_{NS} \xrightarrow{\eta = -\frac{1}{2}} \left| h_{-\frac{1}{2}} = \tfrac{c}{24}, q_{-\frac{1}{2}} = -q_0 + \tfrac{c}{6} \right\rangle_R \qquad \text{ground states} \qquad (6)$$

The only invariant operator w.r.t spectral flow is $\frac{2}{3}c L_0 - J_0^2$.

$N=2$ SCA has a simple extension to $N=4$ SCA, or equivalently $N=4$ SCFT has an $N=2$ subalgebra, with $J_0^3$ reducing to $J_0$ in $N=2$ SCA.

Def: elliptic genus $Z_{ell}(\tau; z) = \mathrm{Tr}_{R\times R} (-1)^{F_L + F_R} q^{L_0} \bar{q}^{\bar{L}_0} e^{4\pi i z J_0^3}$. (7)

The right moving part gives a const. in the presence of BPS states. Our aim reduce to $\mathrm{Tr}_R (-1)^{F_L} q^{L_0} y^{2J_0^3}$ $\quad (y = e^{2\pi i z})$

The mathieu moonshine story begins with the elliptic genus for K3.

$Z_{ell}(K3)(\tau; z)$ is given in EOT's paper

$$Z_{ell}(K3)(\tau; z) = 8\left[\left(\frac{\theta_2(\tau;z)}{\theta_2(\tau;0)}\right)^2 + \left(\frac{\theta_3(\tau;z)}{\theta_3(\tau;0)}\right)^2 + \left(\frac{\theta_4(\tau;z)}{\theta_4(\tau;0)}\right)^2\right] \qquad (8)$$

Since the elliptic genus is independent on the moduli of K3, the calculation could be performed in one of the orbifold limit, such as $T^4/\mathbb{Z}_2$. Similar to the partition function of orbifold,

$$Z_{T/G} = \frac{1}{|G|}\sum_{g,h\in G} g\,\underset{h}{\square} \qquad (9)$$

$$\mathrm{Tr}_R (-1)^{F_L} q^{L_0} y^{2J_0^3} = \mathrm{Tr}_{untw} \frac{1+g}{2}(-1)^{F_L} q^{L_0} y^{2J_0^3} + \mathrm{Tr}_{tw} \frac{1+g}{2}(-1)^{F_L} q^{L_0} y^{2J_0^3} \qquad (10)$$

Here $g$ acts on $X$ to be $X \longrightarrow -X$

For the space-time manifold $T^4/\mathbb{Z}_2$, we have 2 complex bosons and 2 complex fermions

In (10) $\mathrm{Tr}_{untw} \frac{1}{2}(-1)^{F_L} q^{L_0} y^{2J_0^3}$ is simply the elliptic genus for $T^4$, and vanishes due to apparent supersymmetry. While the last 3 pieces gives $2\left(\frac{\theta_2(\tau;z)}{\theta_2(\tau;0)}\right)^2$, $2\left(\frac{\theta_4(\tau;z)}{\theta_4(\tau;0)}\right)^2$ and $2\left(\frac{\theta_3(\tau;z)}{\theta_3(\tau;0)}\right)^2$.

Combining $\mathrm{Tr}_R (-1)^{F_R} \bar{q}^{\bar{L}_0} = 4$, we recover EOT's result.

Geometrically, elliptic genus is a weak Jacobi form of weight 0 and index $-\frac{d}{2}$. (*)
$d = \dim_{\mathbb{C}} M$ for Calabi-Yau manifold $M$.

If the function $\phi(\tau, z): H \times \mathbb{C} \to \mathbb{C}$ transform under $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$.

$$\phi\left(\frac{a\tau+b}{c\tau+d}; \frac{z}{c\tau+d}\right) = (c\tau+d)^k \exp\left(2\pi i t \frac{cz^2}{c\tau+d}\right)\phi(\tau;z)$$

$$\cancel{\phi(\tau; z+\lambda\tau+\mu) = \exp\left(-2\pi i t(\lambda^2\tau+2\lambda z)\right)\phi(\tau;z) \qquad \lambda,\mu\in\mathbb{Z}}$$

and have furthermore the expansion $\phi(\tau;z) = \sum_{n\geq 0,\, l+t\in\mathbb{Z}} f(4tn-l^2)\, q^n y^l$ (11)
with $f(4tn-l^2)=0$ for $n<0$ weak Jacobi form

The spectral flow symmetry implies the Fourier expansion of elliptic genus takes form

$$Z(\tau;z) = \sum_{n,l} q^n y^l\, a\left(\frac{2}{3}cn - l^2\right) \qquad \text{Argument 2} \qquad (12)$$

$\frac{2}{3}n \cdot 3d = 4nt \rightarrow t = \frac{d}{2}$ Statement (*) verified!

With property of Jacobi form, we may guess the expression (8)
$\dim_{\mathbb{C}}(T^4) = \dim_{\mathbb{C}}(K3) = 2$, we expect their elliptic genus to be weak Jacobi form of weight 0 and index 1. While this sort of weak Jacobi form is unique $\phi_{0,1} = 4\sum_{i=2,3,4}\left(\frac{\theta_i(\tau;z)}{\theta_i(\tau;0)}\right)^2$

$$\begin{aligned} Z(T^4)(\tau;0) &= \chi(T^4) = 0 \\ Z(K3)(\tau;0) &= \chi(K3) = 24 \end{aligned} \Rightarrow \begin{cases} Z(T^4)(\tau;z) = 0 \\ Z(K3)(\tau;z) = 2\phi_{0,1}(\tau;z) \end{cases} \qquad (13)$$

$$\begin{array}{ccccc} & & h_{2,2} & & \\ & h_{2,1} & & h_{1,2} & \\ h_{2,0} & & h_{1,1} & & h_{0,2} \\ & h_{1,0} & & h_{0,1} & \\ & & h_{0,0} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

More generally, $Z(M)(\tau;z) = \int_M ch(E_{q,y})\, Td(M)$

Euler characteristic $\chi(M) = Z(\tau;0)$

Hirzebruch signature $\sigma(M) = Z(\tau;\tfrac{1}{2})$ $+O(q)$ (14)

$\hat{A}$-roof genus $\hat{A} = q^{\hat{c}/6}\, Z(\tau; \frac{1+\tau}{2})$ $+O(q)$

$\chi = \sum_n (-1)^n b_n = \sum_{p,q} (-1)^{p+q} h^{p,q}$

Argument: In elliptic genus, the right moving sector is frozen to supersymmetric ground states (BPS); while in left moving sector, all states in Hilbert space contribute.

In Kawai et al., $Z(\tau;z)$ was recovered from $\chi, \sigma, \hat{A}$, in which $z$ took special values.

$$Z(\tau;z) = 24\left(\frac{\theta_3(\tau;z)}{\theta_3}\right)^2 - 2\frac{\theta_4^4 - \theta_2^4}{\eta^4}\left(\frac{\theta_1(\tau;z)}{\eta}\right)^2 \quad (15)$$

(15) is identical to (8) by a relation among $\theta_j(\tau;z)$'s in Whittacker and Watson.

$$\hat{A} = 2\frac{\theta_2^4 - \theta_4^4}{\eta^4}\left(\frac{\theta_3}{\eta}\right)^2, \quad \sigma = 2\frac{\theta_3^4 + \theta_4^4}{\eta^4}\left(\frac{\theta_2}{\eta}\right)^2, \quad \chi = 24$$

Now Let's turn to the start of Mathieu Moonshine, EOT expand $Z_{ell}$

$$Z_{ell}(K3)(\tau;z) = 24\, ch^{\hat{R}}_{h=\frac{1}{4},l=0}(\tau;z) + \Sigma(\tau)\boxed{\frac{\theta_1(\tau;z)^2}{\eta^3}} \quad (16)$$

character formula for BPS rep — non-BPS rep

Non-BPS rep splits into a sum of BPS reps at unitary bound $h = \frac{1}{4}$

$$q^{-\frac{1}{8}}\frac{\theta_1(\tau;z)^2}{\eta^3} = 2\, ch^{\hat{R}}_{h=\frac{1}{4},l=0}(\tau;z) + ch^{\hat{R}}_{h=\frac{1}{4},l=\frac{1}{2}}(\tau;z) \quad (17)$$

Expansion function $\Sigma(\tau) = -2q^{-\frac{1}{8}}\left(1 - \sum_{n=1}^{\infty} A_n q^n\right)$ (18)

$$\Rightarrow Z_{ell}(K3)(\tau;z) = 20\, ch^{\hat{R}}_{h=\frac{1}{4},l=0}(\tau;z) - 2\, ch^{\hat{R}}_{h=\frac{1}{4},l=\frac{1}{2}}(\tau;z) + 2\sum_{n=1}^{\infty} A_n q^{n-\frac{1}{8}}\frac{\theta_1(\tau;z)^2}{\eta(\tau)^3} \quad (19)$$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $A_n$ | 45 | 231 | 770 | 2277 | 5796 | 13915 | 30843 | 65550 |

$A_1, \cdots A_5$ are equal to dim's of irrep of $M_{24}$.

and $A_6 = 3520 + 10395$ sum of dim's

$A_7 = 10395 + 5796 + 5544 + 5313 + 2024 + 1771$

Similar for $A_n$ $(n \geqslant 8)$

This observation reminds us of Monstrous Moonshine,

$J(q) = \frac{1}{q} + 196884q + 21493760q^2$

the coefficients could be decomposed into sum of dim of irrep of Monster group.

Mock modular forms

One class of important functions enter Eq (19)

$$h^{\hat{R}}_{h=\frac{1}{4},\,l=0}(\tau;z) = \frac{\theta_1(\tau;z)^2}{\eta^3}\mu(\tau;z), \quad \mu(\tau;z) = \frac{-ie^{\pi i z}}{\theta_1(\tau;z)}\sum_{n=-\infty}^{+\infty}(-1)^n\frac{q^{\frac{n}{2}(n+1)}e^{2\pi i n z}}{1-q^n e^{2\pi i z}}$$

$$\Sigma(\tau) = -8\left[\mu\left(\tau;z=\tfrac{1}{2}\right) + \mu\left(\tau;z=\tfrac{1+\tau}{2}\right) + \mu\left(\tau;z=\tfrac{\tau}{2}\right)\right] \quad (20)$$

$\mu(\tau;z)$ has the form of Lerch sum, and infact it is a mock modular form.

The modularity of $\mu(\tau;z)$ is not good enough, however.

$$\hat{\mu}(\tau;z) \equiv \mu(\tau;z) - \frac{1}{2}R(0;\tau)$$

with $R(0,\tau) = \sum_{n\in\mathbb{Z}}(-1)^n\left[\mathrm{sgn}(n+\tfrac{1}{2}) - E\left((n+\tfrac{1}{2})\sqrt{2\mathrm{Im}\tau}\right)\right]q^{-\frac{1}{2}(n+\frac{1}{2})^2}$

$E(z) \equiv 2\int_0^z e^{-\pi u^2}du$ error function

Now the transformation formulae are

$$\hat{\mu}(\tau;z) = -\sqrt{\frac{i}{\tau}}\,\hat{\mu}\left(\frac{-1}{\tau},\frac{z}{\tau}\right), \quad \hat{\mu}(\tau+1;z) = e^{-\frac{\pi}{4}i}\hat{\mu}(\tau;z)$$

$$\hat{\mu}(\tau;z+1) = \hat{\mu}(\tau;z+\tau) = \hat{\mu}(\tau;z)$$

$R(0;\tau)$ is non-holomorphic, and could be expressed as the integral

$iR(0;\tau) = \int_{-\bar{\tau}}^{i\infty}\sqrt{-i(z+\tau)}\,[\eta(z)]^3\,dz$ identified as shadow

More generally, $\hat{h}(\tau) = h(\tau) + \left(\frac{i}{2}\right)^{k-1}\int_{-\bar{\tau}}^{i\infty}(z+\tau)^{-k}\,\overline{g(-\bar{z})}\,dz$ shadow

$$\begin{tikzcd}[column sep=small] & G^+_{-\frac{1}{2}}|h,q\rangle \arrow[dr] & \\ |h,q\rangle \arrow[ur] \arrow[dr] & & G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} |h,q\rangle \\ & G^-_{-\frac{1}{2}}|h,q\rangle \arrow[ur] & \end{tikzcd}$$

super charge

BPS rep describe massless states in compactified string theory
non-BPS states has continuous values of $h$ and correspond to massive excitations.
$N=4$ SCA contains $T(z)$, 4 supercurrents, a triplet of affine currents in $SU(2)_k$

$$ch_{k,h,l}(\tau;z) = \mathrm{Tr}\left(e^{4\pi i z J_0^3}\, q^{L_0-\frac{c}{24}}\right)$$

characters in NS and R sectors are related by spectral flow.

Def: In NS sector of Hilbert space of $N=2$ SCFT,

$G^+_{-\frac{1}{2}}|h,q\rangle = 0$ (chiral) $\qquad$ $G^-_{-\frac{1}{2}}|h,q\rangle = 0$ (anti-chiral)

$N=2$ super primary states are defined by $G^+_{n+\frac{1}{2}}|h,q\rangle = 0 = G^-_{n+\frac{1}{2}}|h,q\rangle$

Proposition: A state $|h,q\rangle$ is chiral primary iff $h=\frac{q}{2}$.

Usually, for $N=2$ supersymmetry, there are 4 components arranged a superfield,

$$|\phi\rangle,\ Q_A|\phi\rangle,\ Q_B|\phi\rangle,\ Q_AQ_B|\phi\rangle$$

However, in the case of chiral field, $G^+_{-\frac{1}{2}}|h,q\rangle = 0$, so $N=2$ super primary consists only of 2 components $|h,q\rangle$ and $G^-_{-\frac{1}{2}}|h,q\rangle$.
In general, if a super algebra allows for non-trivial central charges, there exists so-called BPS multiplet which are shorter than the average length of a supermultiplet.

long rep $\Rightarrow$ short rep