Inks:

QFT on Curved Spacetime

- Plan
  - Lightning review of QFT on flat spacetime for real free KQ field.
  - General construction for QFTs on curved spacetime (free)
    - expose ambiguity in approach: notion of "positive frequency"
    - ambiguity in notion of particles
      (in flat spacetime, we could get around this since inertial observers all agree on what's positive frequency & what isn't)
      (in curved space, inertial observers not available to you)
    - ambiguity remains in flat space:
  - Application: uniformly accelerated observers in flat spacetime
    - observe thermal bath of particles
      \[ \rho_x = \text{Tr}_L [ \rho_{\text{vac}} e^{iH_{\text{rel}}} ] = e^{-\frac{\text{H}_{\text{rel}}}{\beta}} \]
      where H_{\text{rel}} is generator of boosts
  - Don't have time to get to, but this fact has many applications
    - Casini, Huerta, Myers 1102.0340
    - Nozaki, Numasawa, Pezzizatti, Takaogi 1809.071
Quantizing Free Real Scalar in Flat Space

- Recall: Expand the field operator in Fourier modes
  \[ \phi(x) = \sum_{k} (\phi_{k} e^{ikx} + \phi^{*}_{k} e^{-ikx}) \]
  \[ \Rightarrow \phi_{k} = m^{2} \phi \]
  - CCR = \{ [\phi_{k}, \phi^{*}_{l}] = 2\Omega_{kl} \}

- By virtue of this, have a Fock space decomposition of the many body Hilbert space \( \mathcal{H} \)
  \[ \mathcal{H} = \bigoplus_{\mathbf{N}_{k}} \mathcal{H}_{\mathbf{N}_{k}} \]
  Complete set of commuting, Hermitian operators

- Unique vacuum: \( \mathbf{N}_{k} = 0 \Rightarrow \mathcal{H}_{0} \)
  State with "no particles"

Ambiguities:
- The one choice from which everything followed
- Made a choice of basis for the solution space (complexified)
- Many possible choices:
  \[ \phi(x) = \sum_{k} (\phi_{k} e^{ikx} + \phi^{*}_{k} e^{-ikx}) \]

- New Fock space decomposition arising from \( \mathbf{N}_{k} = \delta_{k} \mathbf{b}^{*} \mathbf{b} \)
  - A new vacuum \( \mathbf{b} | 0 \rangle^{\mathbf{b}} = 0 \)
  - In general, will include \( \mathbf{a} \)'s \( + \mathbf{b} \)'s

(Will generally happen when new notion of positive freqs "thefts" involves both positive \& negative freq's in the construction)
QFT on Curved Space-time

- real, free KG field (all considerations will generalize to arbitrary free theories)
  - $\Phi$ anti-symmetric, non-degenerate form on $E$ (the solution space).
    - symplectic form
    \[
    \Omega(\phi_1, \phi_2) = \int_E \left( \phi_1 \partial \phi_2 - \phi_2 \partial \phi_1 \right)
    \]
  - $\Omega$ is anti-hermitian to $\Sigma$, $\Omega(\phi, \phi^*) = -\Omega(\phi^*, \phi)$
  - well-defined on $H$ by virtue of the ECM.
    \[
    \Omega_n(\phi_1, \phi_2) - \Omega_n(\phi_2, \phi_1) = \int_E \left( \phi_1 \partial \phi_2 - \phi_2 \partial \phi_1 \right)
    \]
  - $\Omega_n(\phi_i, \phi_i^*) = 0$

- want to construct a Hilbert space of classical solutions
- start with $E$ and define
  \[
  \langle \phi_1, \phi_2 \rangle = \int G(\phi_2^*, \phi_1)
  \]
- properties:
  (i) $\langle \cdot, \cdot \rangle$ is non-degenerate since $\Omega$ is.
  (ii) $\langle \phi_1^*, \phi_2 \rangle = \langle \phi_1, \phi_2^* \rangle$ a hermitian form.
  (iii) $\langle \phi, \phi \rangle^* = -\langle \phi, \phi \rangle$

- problem: not positive definite $\langle \phi, \phi \rangle > 0 = \langle \phi^*, \phi^* \rangle < 0$

- can always choose an orthonormal basis $\{ \phi_i, \phi_i^* \}$
  \[
  \langle \phi_i, \phi_j \rangle = -\langle \phi_j^*, \phi_i \rangle = \delta_{ij}
  \]
- $H = \text{Span} \{ \phi_i \}$ (the coulumb completion $E$)
  - single particle Hilbert space $\{ H, \langle \cdot, \cdot \rangle \}$

Now proceed as previous story
- $\phi = \xi(\text{a} \phi^* + \text{b} \phi)$ vacuum state $\text{c} \phi = 0$
- If made another choice of basis $\{ \phi_i, \phi_i^* \}$ would have
  \[
  \phi = \xi(\text{b} \phi_i^* + \text{b}^* \phi_i) \quad \text{b} \phi_i = 0
  \]
How are these choices related by taking inner product with basis vectors:

\[
\langle \psi, \phi \rangle \quad a_i = \sum_j \langle \psi | b_j \psi \rangle \langle b_j | \phi \rangle
\]

\[
\langle b_i | \phi \rangle \quad b_i = \frac{1}{2} (c_i | c_i \rangle + d_i | d_i \rangle)
\]

where

\[
\begin{align*}
c_{ij} &= \langle e_j | c_i \rangle \\
d_{ij} &= \langle e_j | d_i \rangle \\
c_{ij} &= \langle e_i | c_j \rangle \\
d_{ij} &= \langle e_i | d_j \rangle
\end{align*}
\]

are called Bogoliubov coefficients.

Finding these is simply down to evaluating an integral, what is relationship between the two vacua in terms of them?

\[
\langle \psi, 0 \rangle_a = \langle \phi, 0 \rangle_a = \frac{1}{2} |b_{ij}|^2
\]

This is fact mentioned before, if positive or negative freq., modes mixed \( \langle \psi, 0 \rangle \neq 0 \), then a vacuum has particles from the \( b \) perspective.

(\text{Can be more explicit})

- Can find explicit formula for a vacuum in terms of \( b \) vacuum.

Expand \( \langle 0 | \psi \rangle_a = \sum_{n=0}^{\infty} a_{n}^\dagger b_n^\dagger \cdots b_1^\dagger b_0^\dagger \langle 0 | \psi \rangle_b \)

(\text{\textup{n}-particle wavefunction})

Solve \( a_i | 0 \rangle_a = 0 \) \( \Rightarrow (b_i \cdots b_0 \cdots | 0 \rangle) \langle 0 | \psi \rangle_b = 0 \) where \( \langle 0 | \psi \rangle_b = \sum_{n=0}^{\infty} |n \rangle \langle n | \psi \rangle_b \)

New next vacuum:

\[
\langle 0 | \psi \rangle_{a+b} = \sum_{n=0}^{\infty} (|n \rangle \langle n | \psi \rangle_b - \frac{1}{2} \sum_{k=1}^{n} c_{ik} d_{kj}) b_k^\dagger \cdots b_1^\dagger b_0^\dagger | 0 \rangle
\]

Recursion relation for \( \text{\textup{n}-particle wavefunction} \) in terms of \( \text{\textup{n-1}-particle wavefunction} \)

- particle one:

\[
\eta_{n-1} = 0, \quad \phi_{n} = \frac{1}{(2n)!} \sum_{n=0}^{\infty} (c_i | d_i \rangle \eta_i \cdots \eta_0 \rangle \langle \psi | \phi \rangle_0 \psi \rangle_b
\]

(Exactly solves for \( | 0 \rangle_a \) in terms of \( | 0 \rangle_b \) + the Bogoliubov coefficients.)
Rudler Spacetime

(Now lets apply this formalism to a problem)

- seek particle content of L & R according to uniformly accelerated observers

Consider a patch of Minkowski covered by

\[ \begin{pmatrix} t \\ x \end{pmatrix} = \pm \begin{pmatrix} \alpha e^{\pm \kappa t} \sinh \kappa x \\ \alpha e^{\pm \kappa t} \cosh \kappa x \end{pmatrix}, \quad -\infty < x, y, z < \infty \]

Note: \( t^2 - x^2 = \alpha^2 e^{2\kappa t} \) so curves of const \( \kappa \)

are timelike hyperbolas

\[ \text{for such curves } \sqrt{t^2 - x^2} = \alpha e^{\kappa t} \]

(\( \text{so as } \kappa \to \infty \) observers approach inertial, \( \text{as } \kappa \to -\infty \), accel. diverges.)

-say this is coord system adopted by observer of accel. \( \alpha \)

- not existence

of horizon, observers in \( R \) have no access to info in \( L \) &

vice versa

- but both \( R \& L \) are globally hyperbolic spacetimes in own right

- well posed initial value formulation on each wedge, can consider classical mech in \( L \) or \( R \) & construct corresponding hilbert space

For simplicity, take \( m=0 \) case.

\[ \text{ECM reads } (\gamma^2 - \kappa^2) \Phi = 0 \]

positive freq sols in right & left accel. wedges

\[ \phi_k = \begin{cases} \frac{1}{2} e^{\gamma (t+x)} & \text{in } R \\ \frac{1}{2} e^{-\gamma (t-x)} & \text{in } L \end{cases} \]

\( \omega = 1 \kappa \)

Expand field

\[ \phi = \frac{1}{2} \left( a_k e^{i (t+x)} + a_{-k} e^{-i (t-x)} \right) \]
\[ \phi = \int \frac{dk}{2\pi} \left( b_k^* e^{ikx} + a_k b_k e^{ikx} + \text{left} \right) \]

...in principle it remains to evaluate some inner products to get:

\[ 10^n a \text{ in terms of } 10^n b. \]

A much easier way: can prove

\[ a_k + e^{i\omega a} (a_k)^* \] is purely positive (Fourier) frequency

\[ \left( b_k e^{i\omega a} b_k^* \right) 10^n a = 0 \]

...can read off the \( \omega \) matrix

\[ \omega_{ij} = \begin{cases} \frac{e^{-i\omega a}}{\sqrt{c_i}} & \text{when one mode is in } \mathbf{K} \text{ and one in } \mathbf{L} \\ 0 & \text{otherwise} \end{cases} \]

Gives

\[ 10^n \rho = \frac{1}{2} \sum_{\omega} e^{-i\omega a} \ln \tau_{\omega \rightarrow 1} \ln \tau_{\omega \rightarrow 0} \]

where \( \ln \tau_{\omega} = \frac{1}{\sqrt{c_i}} \int_{\omega}^{\infty} e^{-i\omega a} \ln \tau_{\omega} \) etc...

Maximally extended state.

Can now retrieve modular Hamiltonian.

\[ \rho = \text{Tr}_L 10^n \rho_0 = \frac{1}{2} \sum_{\omega} e^{-i\omega a} \ln \tau_{\omega \rightarrow 1} \ln \tau_{\omega \rightarrow 0} = e^{-i\omega a} \rho_{\omega} \]

where \( \rho_{\omega} \) is the generator of boosts.

\[ H_\omega = \int \frac{dk}{2\pi} \omega_k b_k b_k^* \]

Actually, this is a particular case of a more general fact from axiomatic QFT. The Bisognano-Wichmann theorem

\[ \rho = e^{-i\omega_0 H_0} \]

\( H_0 \) generator of boosts.