2+1 Dimensional Gravity and Chern-Simons Theory

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This is a short talk for CMT-HEP journal club. I will first discuss the relation between three dimensional gravity and CHern-Simons theory. One should find that Einstein gravity on three dimensional spacetime manifold, classically, has a Chern-Simons discription. Then, I review the cannonical quantization of Chern-Simons theory which provides some useful information about quantum Hilbert space of three dimensional quantum qravity.

1 Three Dimensional Gravity is Topological

Three dimensional gravity has no local degree of freedom. This can be seen from Einstein-Hilbert action on a three dimensional manifold M.

$$I = \frac{1}{16\pi G}\int_M d^3x\sqrt{-g}(R-2\Lambda) \tag{1.1}$$

Extremizing this action with respect to the metric $g_{\mu\nu}$ yields the Einstein equation.

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{1.2}$$

As a result, any solution of classical equation of motion has constant curvature. This means that on three dimensional spacetimes there are no local propagating degree of freedom, that is, no gravitational wave. We can also understand this by simply counting. A symmetric tensor $g_{\mu\nu}$ in D dimensional spacetime has $D(D+1)/2$ independent components. We can remove D freedom by using diffeomorphism invariance and another D freedom by the fact that D components of the metric appear in the action with no time derivative. Thus we have

$$\frac{D(D+1)}{2} - D - D = \frac{D(D-3)}{2} \tag{1.3}$$

degree of freedom remaining. In three dimension, this counting leads to zero degree of freedom. At the first glance, this result is very frustrating. However, as we will see later, this theory is actually interesting since it can has some topological structure. Even through every solution of spacetime is locally equivalent to each other, they can have different global properties.

2 Relation to Gauge Theory

Einstein's gravity is a gauge theory in the sense that we can interpret diffeomorphism as gauge transformation. As a result, its fundamental variables should transform under some gauge group. However, Einstein Hilbert action is not of the same form as gauge theory. For example, Hilbert action takes the following form in four dimension,

$$I = \int e\wedge e\wedge(d\omega+\omega^2) \tag{2.1}$$

Here, instead of metric, I use vierbein e^a_j and spin connection ω^a_{jc} as fundamental variables. This description should be equivalent to using metric as fundamental varibles as long as vierbein is non-degenerate and we can transform metric and vierbein by

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}; \quad \eta_{ab} = e^\mu_a e^\nu_b g_{\mu\nu} \tag{2.2}$$

Classically, we assume this is always true. In quantum theory, this is also true when we consider small quantum fluctuation around classical solution. Small fluctuation should not take variables too far away from classical solution, thus vierbein should remain invertable. However, this may be violated

when we consider non-perturbative effect. If we interpret e and ω as gauge field, Hilbert action should have a gauge theory counterpart, $\int A \wedge A \wedge (dA + A^2)$. However, there is no such action in gauge theory.
In three dimension, situation becomes different. Three dimensional Einstein-Hilbert action is

$$I = \frac{1}{2} \int \epsilon^{ijk} \epsilon_{abc} e_i^a (\partial_j \omega_k^{bc} - \partial_k \omega_j^{be} + [\omega_j \; \omega_k]^{bc}) = \int e \wedge (d\omega + \omega^2) \tag{2.3}$$

Again we can regard e and ω as gauge field and turn this action into $\int A \wedge (dA + A^2)$. This is Chern-Simons three form. We may now ask whether gravity in three dimension has a Chern-Simons interpretation or not. If so, what is its gauge group?
It turns out that three dimensional pure gravity, without cosmological constant, is equivalent to Chern-Simons theory with gauge group ISO(2,1). Let's recall some fact about group ISO(2,1) and construct its Chern-Simons action step by step. ISO(2,1) has six generators, P^a for translation and J^{ab} for Lorentz generators. $(a, b = 1, 2, 3)$. In three dimension, we can write $J^a = \frac{1}{2}\epsilon^{abc} J_{bc}$. The commutation relation is

$$[J_a, \; J_b] = \epsilon_{abc} J^c; \quad [J_a, \; P_a] = \epsilon_{abc} P^c; \quad [J_a, \; P_b] = 0 \tag{2.4}$$

The invariant quadratic form can be found to be

$$tr J_a P_b = \delta_{ab}; \quad tr J_a J_b = tr P_a P_b = 0 \tag{2.5}$$

Gauge field is a one form which takes value in ISO(2,1), that is

$$A_j = e_j^a P_a + \omega_j^a J_a \tag{2.6}$$

Here I attach e to translation and ω to rotation. If ISO(2,1) is the correct gauge group, an infinitesimal gauge transformation should gnerate coordinate trasformation. Gauge transformation with a variable $\alpha = \rho^a P_a + \tau_a J^a$ is $\delta A_j = -D_j \alpha$. The minus sign is a convention and covariant derivative D_j is

$$D_j \phi = \partial_j \phi + [A_j \; \phi] \tag{2.7}$$

Substituting equation (2.6) into the gauge transformation gives

$$\begin{aligned} \delta e_j^a &= -\partial_j \rho^a - \epsilon^{abc} e_{jb} \tau_c - \epsilon^{abc} \omega_{ib} \rho_c \\ \delta \omega_j^a &= -\partial_j \tau^a - \epsilon^{abc} \omega_{ib} \tau_c \end{aligned} \tag{2.8}$$

This transformation laws don't coincide with the usual coordinate transformation law. It is not surprising that the second term of each lines can be identified with the local Lorentz transformation since τ_c is attached to Lorentz generator J^c in the gauge transformation. The problem is the other three terms. Hopefully we should be able to identify them as diffeomorphism. Let's focus on the transformation of e first. Under a diffeomorphism gnerated by a vector field $-v^j$, the stanard transformation $\tilde{\delta} e$ and $\tilde{\delta} \omega$ are Lie derivatives.

$$\begin{aligned} \tilde{\delta} e_j^a &= \mathcal{L}_{-v} e_j^a = -v^k (\partial_k e_j^a - \partial_j e_k^a) - \partial_j (v^k e_k^a) \\ \tilde{\delta} \omega_j^a &= \mathcal{L}_{-v} \omega_j^a = -v^k (\partial_k \omega_j^a - \partial_j \omega_k^a) - \partial_j (v^k \omega_k^a) \end{aligned} \tag{2.9}$$

If we set $\rho^a = v^k e_k^a$, the difference between δe and $\tilde{\delta} e$ becomes

$$\tilde{\delta} e_j^a - \delta e_j^a = -v^k (D_k e_j^a - D_j e_k^a) \tag{2.10}$$

However, $D_k e_j^a - D_j e_k^a$ vanishes by equation of motion. As a result, we find that the two transformation laws coninde with each other on shell.
Now we are ready to construct a Chern-Simons action. To do that, we can first construct a gauge theory of the form $\int tr F \wedge F$ in four dimension and identify Chern-Simons action as the surface term of this action. Curvature tensor F is

$$F_{ij} = [D_i, \; D_j] = P_a \left[\partial_i e_j^a - \partial_j e_i^a + \epsilon^{abc} (\omega_{ib} e_{jc} + e_{ib} \omega_{fc}) \right] + J_a \left[\partial_i \omega_j^a - \partial_j \omega_i^a + \epsilon^{abc} \omega_{ib} \omega_{jc} \right] \tag{2.11}$$

Action on a four dimensional manifold Y is

$$\int_Y trF \wedge F = \int_Y \epsilon^{ijkl}(\partial_i e_j^a - \partial_j e_i^a + \epsilon^{abc}(\omega_{ib} e_{jc} + e_{ib}\omega_{fc}))(\partial_k \omega_{al} - \partial_l \omega_{ak} + \epsilon_{abc}\omega_k^b \omega_l^c) \tag{2.12}$$

This term is a total derivative. Therefore, if the four dimensional manifold Y has for its boundary a three dimensional manifold M, the theory on the boundary is described by Chern-Simons action, i.e.

$$I_{CS} = \int_M \epsilon^{ijk} \left(e_{ia}(\partial_j \omega_k^a - \partial_k \omega_j^a + \epsilon_{abc}\omega_j^b \omega_k^c) \right) \tag{2.13}$$

This is Einstein-Hilbert action in three dimension and is automatically invariant under gauge transformation. Now, we find a Chern-Simons description of three dimensional gravity.

3 Inclusion of Cosmological Constant

We can generalize our discussion by including a cosmological constant term.

$$I = I_{CS} + \int_M \frac{\lambda}{3} \epsilon^{ijk} \epsilon_{abc} e_i^a e_j^b e_k^c \tag{3.1}$$

Spacetime is not flat but still locally homogeneous with constant curvature. For positive or negative cosmological constant, covering space is no longer Minkowski space but de Sitter or anti-de Sitter space, respectively. Instead of ISO(2,1), these two spaces have for their symmetry SO(3,1) and SO(2,2). Therefore, it is natural to guess that the gauge group related to three dimensional gravity with cosmological constant is these two group instead of ISO(2,1). I will consider negative cosmological constant only and verify above conjuture. Commutation relation is

$$[J_a,\ J_b] = \epsilon_{abc} J^c; \quad [J_a,\ P_b] = \epsilon_{abc} P^c; \quad [P_a,\ P_b] = \lambda \epsilon_{abc} J^c \tag{3.2}$$

This algebra is simplied by introducing

$$J_a^{\pm} = \frac{1}{2}\left(J_a \pm \frac{P_a}{\sqrt{\lambda}} \right) \tag{3.3}$$

Commutation relation becomes

$$[J_a^+,\ J_b^+] = \epsilon_{abc} J_c^+; \quad [J_a^-,\ J_b^-] = \epsilon_{abc} J_c^-; \quad [J_a^+,\ J_b^-] = 0 \tag{3.4}$$

This is not surprising since SO(2,2) can be separated to be two copy of $SL(2,R)$, SO(2,2) $\simeq SL(2,R) \times SL(2,R)$. We can follow all the procedure again. The connection is

$$A_j^{a\pm} = \omega_j^a \pm \sqrt{\lambda} e_j^a \tag{3.5}$$

Covariant derivative is

$$D_j = \partial_j + J_a^+ A_j^{a+} + J_a^- A_j^{a-} \tag{3.6}$$

Chern-Simons action is

$$I^{\pm} = \int_M \epsilon^{ijk} \left(2A_i^{a\pm} \partial_j A_k^{a\pm} + \frac{2}{3} \epsilon_{abc} A_i^{a\pm} A_j^{b\pm} A_k^{c\pm} \right) \tag{3.7}$$

One can check that the desired action of gravity with cosmological constant term is $(I^+ - I^-)/4\sqrt{\lambda}$. There is another possible action $(I^+ + I^-)/2$. This action is also invariant under SO(2,2) gauge transformation. For a general Chern-Simons theory with group $SO(2,2)$, any linear combination of these two actions are classically equivalent, but not in quantum theory.

4 Quantization

We have learned gravity in three dimensional spacetime is equivalent to a Chern-Simons theory. Therefore, quantization of gravity becomes a problem of quantizating Chern-Simons theory, which has been studied for decays. I want to review the canonical quatization of non-abelian Chern-Simons theory here. Let's begin with abelian case. The first step is to identify the time direction on the 3-manifold M. Locally we can decompose M as $\Sigma \times \mathbb{R}$ with Σ a closed Reimann surface and $\mathbb{R}$ the time direction. A natural gauge choice is Coulomb gauge $A^0 = 0$. Under this gauge condition, Chern-Simons action is quadratic in A.

$$I = \frac{k}{4\pi} \int dt \int \epsilon^{ij} tr A_i \frac{d}{dt} A_j \tag{4.1}$$

From this, we can read the Poisson bracket.

$$\{A_i^a(x),\ A_j^b(y)\} = \frac{4\pi}{k} \epsilon_{ij} \delta^{ab} \delta^2(x-y) \tag{4.2}$$

The quadratic form above look simple, however, this theory is actually non-linear. In addition to Poisson bracket, we should impose some constraint equations resulting from integrating non-dynamical field, A^0. The equation is Gauss's law constraints, $\delta\mathcal{L}/\delta A^0$.

$$\epsilon^{ij} F_{ij}^a = 0 \tag{4.3}$$

This equation implies curvature on Σ is flat. In non-abelian gauge group, this constraint is non-linear. Now the problem is how to quantize Chern-Simons theory with this constraint.

4.1 Classical Phase Space

Let's consider how to construct classical phase space with constraint equation. In general, classical phase space without constraint is a infinite dimensional space $\mathcal{M}_0$, so is its quantum theory. To impose constraint, there are two steps classically. First, we declare that physical state of conanical variables q^i are values for which satisfies constraint equation $G(q^i) = 0$ and we only consider these physical solutions. The second step is to interpret constraints as the generators of certain transformation on phase space.

$$\delta q^i = \sum \epsilon^I \{G^I,\ q^i\} \tag{4.4}$$

with G^I being a set of constraint equations and ϵ^I being infinitesimal parameter. This transformation can be regarded as action of a certain group $\mathcal{G}$. We should declare further that two physical solutions is equivalent if they differ by an element of $\mathcal{G}$ generated by equation (4.4). That is, the physical phase space of a constraint system is the space of solutions of constraint equation modulo the action of group $\mathcal{G}$, the constraint group.

Let me elaborate this by a simple example. Consider a particle moving in two dimensional space confined on a unit circle. The unconstraint phase space $\mathcal{M}_0$ is space of coordinate, (x, y, p_x, p_y). Since the motion is confined on a unit circle, the constrain equation is $G(x, y) = x^2 + y^2 - 1 = 0$. To begin with, we can restrict ourselves to points such that $x^2 + y^2 - 1 = 0$. However, this is only one half the story. Momentum p_x, p_y of the particle are not arbitrary, they must tangent to the unit circle. This is why we have to modulo out the constraint group. Action generated by $G(x, y)$ is

$$\delta p_x = \epsilon\{G,\ p_x\} = 2\epsilon x; \qquad \delta p_y = \epsilon\{G,\ p_y\} = 2\epsilon y \tag{4.5}$$

We should idnetify two points in the phase space if they differ by the above transformation. That is

$$(x, y, p_x, p_y) \sim (x, y, p_x + \epsilon x, p_y + \epsilon y) \tag{4.6}$$

From above equation, it is obvious that the infinitesimal transformation generates the radial components of mementum. Therefore, the radial component of momentum is not physical and can be gauged away, leaving the tangential components.

In Chern-Simons Theory, imposing constraint equation (4.3) restrict the phase space to be flat connection of surface Σ since the equation implies vaniching curvature. The only gauge-invariant observables that do not vanish must be global observable becaause locally the connection is pure gauge. From equation (4.4), the action of constraint group is

$$\delta A_k^b(y) = \lambda_a \{\epsilon^{ij} F_{ij}^a(x),\ A_k^b(y)\} = \frac{8\pi}{k}\lambda_a (D_k)^{ab}\delta^2(x-y) \tag{4.7}$$

with λ_a being some parameters. This is nothing but the gauge transformation. That is, physical states are irrelavent of gauge transformation. To summary, we find

$\mathcal{M}$ is space of flat connection modulo gauge transformation on surface Σ.

As mentioned before, the only gauge invariant observables are global observables, which is Wilson loop.

$$U_j^i(s_n, s_m) = P\exp\left(-\int_{s_m}^{s_n} ds \frac{dx^\mu(s)}{ds} A_\mu^a T_{aj}^i\right) \tag{4.8}$$

This is also called the gauge holonomy matrix, or holonomy for short. By expanding the path order exponential and using Stoke's theorem, holonomy of an infinitesimal curve is

$$U_j^i \sim \delta_j^i - \int F_{\mu\nu}^a T_j^{ia} \Sigma_{\mu\nu} \tag{4.9}$$

Therefore, if geometry doesn't have non-trivial cycle and the curvature is flat, there are no observables. But, if geometry has some non-trivail cycle, we can no longer use Stoke's theorem in this case, this may give some non-trivial holonomy. As a result, dimension of phase space becomes counting how many non-trivial cycles on surface Σ. This may be seen explicitly by homomorphisms. Wilson loop is the same as assigning an element in $\mathcal{G}$ to every element in $\pi_1(\Sigma)$. However, we have to mod out an overall conjugation action on the Wilson loop. Thus, the phase space is

$$\mathcal{M} = Hom(\pi_1(\Sigma) \to G)/conj \tag{4.10}$$

For genus larger than one, Reimann surface, the dimension of $\mathcal{M}$ is

$$dim(\mathcal{M}) = (2g-2)dim(G) \tag{4.11}$$

where the $2g$ comes from the $2g$ cycles and the -2 comes from the overall conjugation as well as the relation among the $2g$ cycles. Since we have one quantum state per unit volume in classical phase space, it follows that quantum Hilbert space is finite dimensional!

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